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Lecture Notes for 8.225 / STS.042, "Physics in the 20th Century": $E = mc^2$

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Introduction

Albert Einstein submitted his paper, "On the electrodynamics of moving bodies," to the *Annalen der Physik* in June 1905; this paper laid the groundwork for what would become known as the special theory of relativity.[1] Soon after completing the paper, Einstein wrote to his friend Conrad Habicht, a fellow member of his informal "Olympia Academy": "a consequence of the work on electrodynamics has suddenly occurred to me, namely, that the principle of relativity in conjunction with Maxwell's fundamental equations requires that the mass of a body is a direct measure of its energy content — that light transfers mass. [...] This thought is both amusing and attractive; but whether or not the good Lord laughs at me concerning this notion and has led me around by the nose — that I cannot know."[2] Einstein submitted a short paper to the *Annalen der Physik* in September 1905 with the title, "Does the inertia of a body depend upon its energy content?"[3] It was in this short paper — a mere 3 pages! — that Einstein derived the now-famous equation, $E = mc^2$.

Einstein returned many times over the years to this equation, offering a variety of simpler ways to re-derive the result. In these short notes we will consider one of his later derivations. We will also briefly consider the concept of "relativistic momentum," in terms of which we find the more general expression of Einstein's equation, $E = \gamma mc^2$, where γ is the usual factor we have encountered several times: $\gamma \equiv 1/\sqrt{1 - (v/c)^2}$.

[1] A. Einstein, "Zur Elektrodynamik bewegter Körper," *Annalen der Physik* **17** (1905): 891-921. An English translation is available in John Stachel, ed., *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics* (Princeton: Princeton University Press, 2005 [1998]), 123-160.

[2] Albert Einstein to Conrad Habicht, undated (ca. summer 1905), as translated and quoted in Arthur I. Miller, *Albert Einstein's Special Theory of Relativity* (Reading, MA: Addison-Wesley, 1981), p. 353.

[3] A. Einstein "Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?," *Annalen der Physik* **18** (1905): 639-41. An English translation is available in Stachel, *Einstein's Miraculous Year*, 161-164.

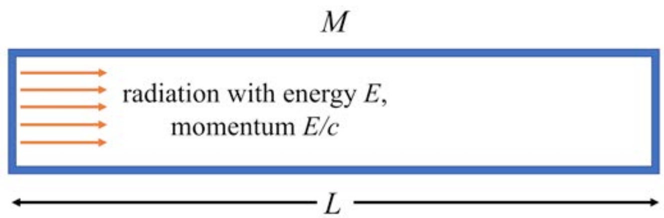

Figure 1: A box of mass M and length L in which a burst of radiation, carrying energy E, travels from left to right.

The Energy of a Moving Box

Consider a box of mass M and length L floating in space and at rest with respect to us. Suddenly a burst of radiation, with energy E, is emitted from the left end of the box and travels toward the right, as in Fig. 1. From Maxwell's equations, we know that this radiation carries momentum of magnitude $|\mathbf{p}_{\text{radiation}}| = E/c$, where, as usual, c is the speed of light. Prior to the release of the radiation the system had zero total momentum, and there are no external forces acting on the box, so even after the release of the radiation the system's total momentum must remain zero. This means that the box must *recoil* in the opposite direction from the direction of motion of the radiation, to preserve $\mathbf{p}_{\text{total}} = \mathbf{p}_{\text{box}} + \mathbf{p}_{\text{radiation}} = 0$. The velocity of the box's recoil $\mathbf{v}$ is fixed by the conservation of momentum. If we consider a very massive box we can expect the recoil velocity to remain small, $|\mathbf{v}| \ll c$. Requiring $\mathbf{p}_{\text{total}} = 0$ then yields

$$M\mathbf{v} + \frac{E}{c}\hat{\mathbf{x}} = 0 \Longrightarrow \mathbf{v} = -\frac{E}{Mc}\hat{\mathbf{x}}\,. \tag{1}$$

(Here $\hat{\mathbf{x}}$ is a unit vector pointing along the x axis.) If the radiation travels in the $+\hat{\mathbf{x}}$ direction, then the box will recoil in the $-\hat{\mathbf{x}}$ direction, with speed $v = |\mathbf{v}| = E/(Mc)$. After the light travels the length of the box it will hit the other end of the box. Since the radiation carries momentum $|\mathbf{p}_{\text{radiation}}| = E/c$, it will deliver an impulse to the right side of the box that will halt the box's recoil, bringing the box to rest again.

We have assumed $v \ll c$, so we can neglect the motion of the box when calculating the duration Δt, the time-of-flight for the radiation to cross the length of the box. In that limit, $\Delta t = L/c$. Between the emission and absorption of the radiation within the box, the box will have moved a distance Δx to the left, as in Fig. 2:

$$\Delta x = v\Delta t = \left(-\frac{E}{Mc}\right)\left(\frac{L}{c}\right) = -\frac{EL}{Mc^2}\,, \tag{2}$$

where the minus sign reminds us that the box has moved in the $-\hat{\mathbf{x}}$ direction during its recoil motion.

Throughout this entire process, the box has been subject to no external forces. Hence the

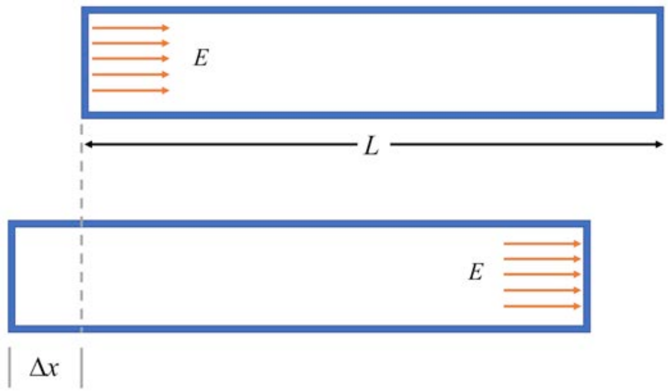

Figure 2: After the radiation traverses the length of the box it will strike the right side of the box, bringing the box's recoil motion to a halt. During the time that the radiation travels within the box, the box moves a distance Δx to the left.

center of mass of the system has not moved. In general, the center of mass for a collection of N objects is given by

$$\sum_{i=1}^{N} m_i \left(\mathbf{r}_i - \mathbf{R}\right) = 0\,, \tag{3}$$

where $\mathbf{R}$ is the position of the center of mass, and the N objects have masses m_i and positions $\mathbf{r}_i$. We may adopt coordinates such that initially $\mathbf{R} = 0$, that is, the center of mass is initially located at the origin. In the absence of external forces, $d\mathbf{R}/dt = 0$. In our case, the massive box has clearly changed its position over time, recoiling a distance Δx to the left. In order for the system's center of mass to remain unchanged, therefore, some *mass equivalent* must have traveled with the radiation the distance L from the left side of the box to the right. Let us call this mass equivalent m. Then Eq. (3) becomes

$$mL + M\Delta x = 0 \Longrightarrow m = -\frac{M}{L}\Delta x\,. \tag{4}$$

In other words, the mass equivalent m associated with the radiation that moved a length L to the right has compensated for the motion of the massive box M a distance Δx to the left. Combining Eqs. (2) and (4), we find

$$m = \left(-\frac{M}{L}\right)\left(-\frac{EL}{Mc^2}\right) = \frac{E}{c^2}\,, \tag{5}$$

or, more famously,

$$E = mc^2\,. \tag{6}$$

We arrived at Eq. (6) by neglecting quantities of order $(v/c)^2$. If we had calculated the effects to arbitrary order in (v/c), we would have found the result Einstein originally derived:

$$E = \gamma mc^2\,, \tag{7}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} . \tag{8}$$

Relativistic Momentum

In our class discussion of Hermann Minkowski's geometrical interpretation of Einstein's 1905 work on the electrodynamics of moving bodies, we will see that Minkowski found an *invariant* quantity on which observers in *any* inertial reference frame will agree, even though (as Einstein had demonstrated) the observers will disagree about measurements of length and of time. For finite durations Δt and displacements $\Delta \mathbf{x}$, Minkowski showed that the quantity s^2 remains invariant across inertial reference frames:

$$\begin{aligned} s^2 &\equiv c^2\left(\Delta t\right)^2 - \left(\Delta \mathbf{x}\right)^2 \\ &= c^2\left(\Delta t'\right)^2 - \left(\Delta \mathbf{x}'\right)^2 . \end{aligned} \tag{9}$$

The quantity s is often called the "spacetime interval." We may define the *proper time* $\Delta\tau$ for an object as the time measured by a clock that is moving with that object:

$$c\Delta\tau \equiv s . \tag{10}$$

For a clock sitting at rest, $\Delta \mathbf{x} = 0$ and the proper time simply equals the coordinate time measured in that object's reference frame, $\Delta\tau = \Delta t$. However, since *any* inertial observer will agree on the spacetime interval s^2 — and since the speed of light c is a universal constant, on which all inertial observers will also agree — any inertial observer will *also* agree on the proper time $\Delta\tau$, even when they disagree about coordinate-dependent durations Δt.

Just as we may consider an infinitesimal displacement in time or space, dt or $d\mathbf{x}$, we may also consider an infinitesimal spacetime interval:

$$\begin{aligned} ds^2 &= c^2 dt^2 - d\mathbf{x}^2 \\ &= c^2 dt^2\left[1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right] \\ &= c^2 dt^2\left[1 - \left(\frac{v^2}{c^2}\right)\right] \\ &= \frac{c^2}{\gamma^2} dt^2 , \end{aligned} \tag{11}$$

where I have used $\mathbf{v} = d\mathbf{x}/dt$, $v = |\mathbf{v}|$, and the definition of γ in Eq. (8). Using Eqs. (10) and (11), we may find an expression relating the infinitesimal proper time $d\tau = ds/c$ to the coordinate time dt in a given inertial reference frame:

$$\frac{dt}{d\tau} = \gamma , \tag{12}$$

which is just another way of expressing the *time dilation* of one observer's measurement of the rate at which time passes on a moving clock.

In Newtonian mechanics (that is, when we ignore special relativity), we define the momentum of an object of mass m to be

$$\mathbf{p}_{\rm nonrel} \equiv m\frac{d\mathbf{x}}{dt} \,. \tag{13}$$

Since all inertial observers will agree on the *proper time* associated with a moving object, even if they disagree on the coordinate times associated with various reference frames, we may generalize Eq. (13) and write the *relativistic momentum* for an object of mass m as

$$\begin{aligned} \mathbf{p}_{\rm rel} &\equiv m\frac{d\mathbf{x}}{d\tau} \\ &= m\frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} \\ &= \gamma m\mathbf{v} \,, \end{aligned} \tag{14}$$

upon using $\mathbf{v} = d\mathbf{x}/dt$ and Eq. (12).

Now we can see that the general expression in Eq. (7), $E = \gamma mc^2$, is equivalent to taking into account both the "rest energy" of an object — a nonzero quantity for any massive object, even if the object is sitting at rest in a given reference frame — as well as its kinetic energy, due to its motion. Let us *posit* that

$$E^2 = m^2c^4 + |\mathbf{p}_{\rm rel}|^2c^2 \,. \tag{15}$$

(As a quick check, note that light has vanishing rest mass, $m = 0$, and hence Eq. (15) implies that $E = |\mathbf{p}_{\rm radiation}|c$, which is consistent with Maxwell's equations and was central to the argument that Einstein used in his discussion of the momentum carried by the radiation within the box, which we considered above.) Upon using Eq. (14), we find

$$\begin{aligned} E^2 &= m^2c^4 + \gamma^2m^2v^2c^2 \\ &= m^2c^4\left[1 + \gamma^2\frac{v^2}{c^2}\right] \\ &= m^2c^4\left[1 + \frac{v^2/c^2}{[1-(v^2/c^2)]}\right] \\ &= \frac{m^2c^4}{[1-(v^2/c^2)]}\left[1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right] \\ &= \gamma^2m^2c^4 \,, \end{aligned} \tag{16}$$

or

$$E = \gamma mc^2 \,. \tag{17}$$

An object with mass m at rest (with $v = 0$) will have a "rest energy" $E = mc^2$. An object with mass m that is moving with some velocity $\mathbf{v}$ will have a relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$ and energy $E = \gamma mc^2$.