Signal Processing on Databases

Jeremy Kepner

Lecture 8: Kronecker graphs, data generation, and performance

LINCOLN LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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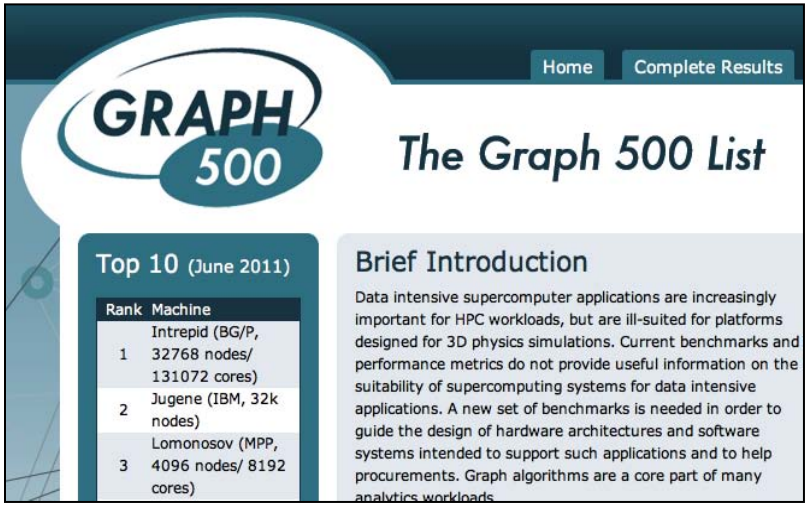
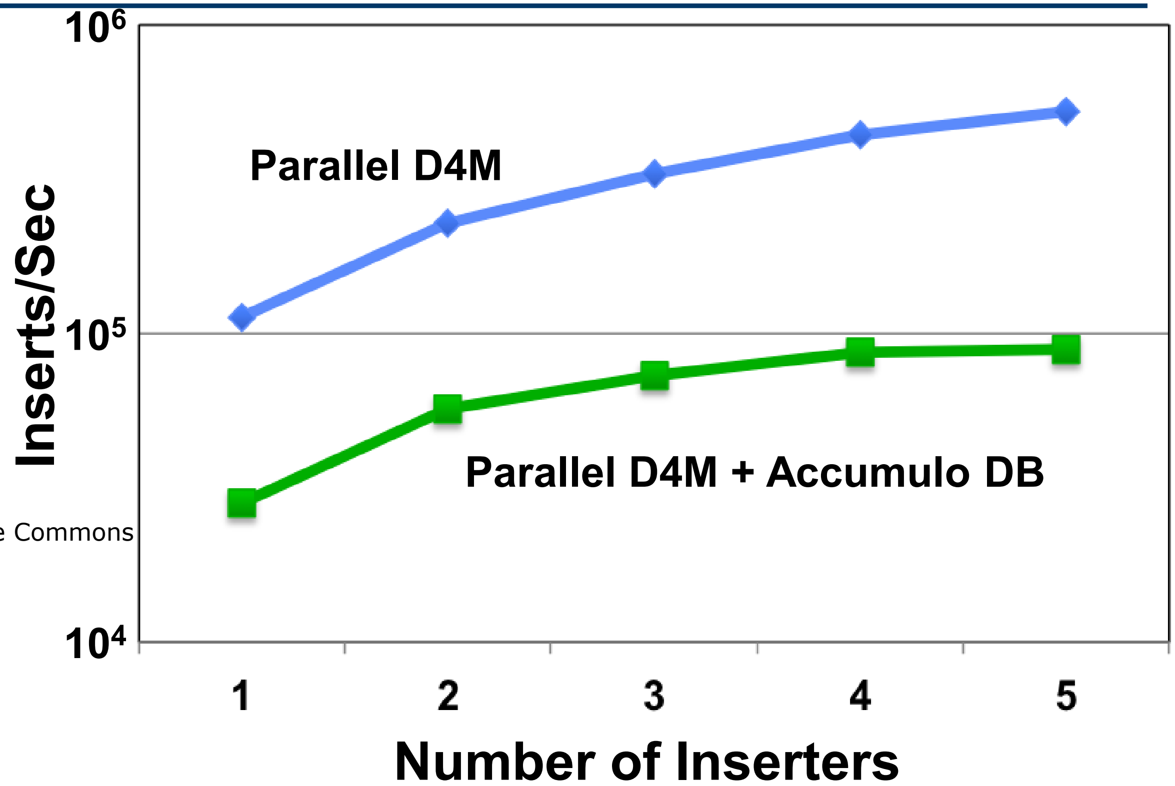
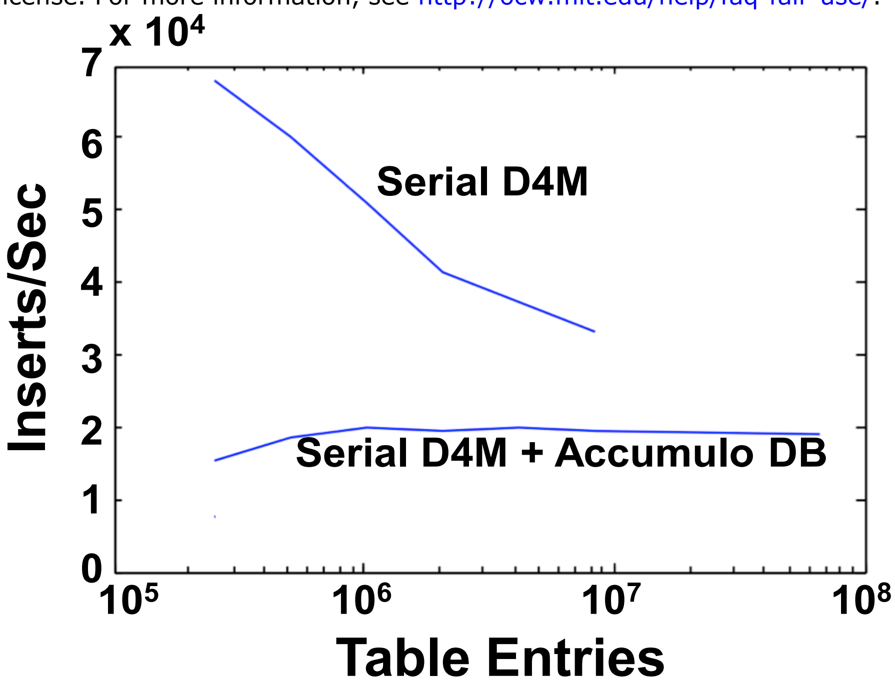
Outline

- Introduction
  - Graph500
  - Kronecker Graphs
- $B^{\otimes K}$ Graphs
- $(B+I)^{\otimes K}$ Graphs
- Performance
- Summary

Graph500 Benchmark Performance



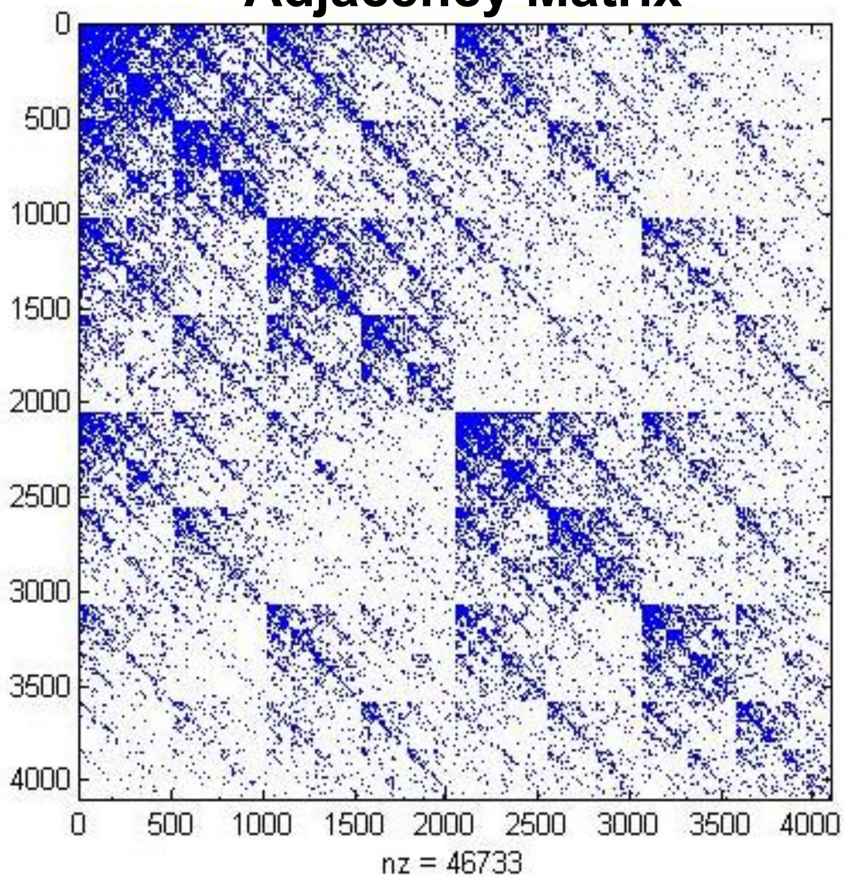




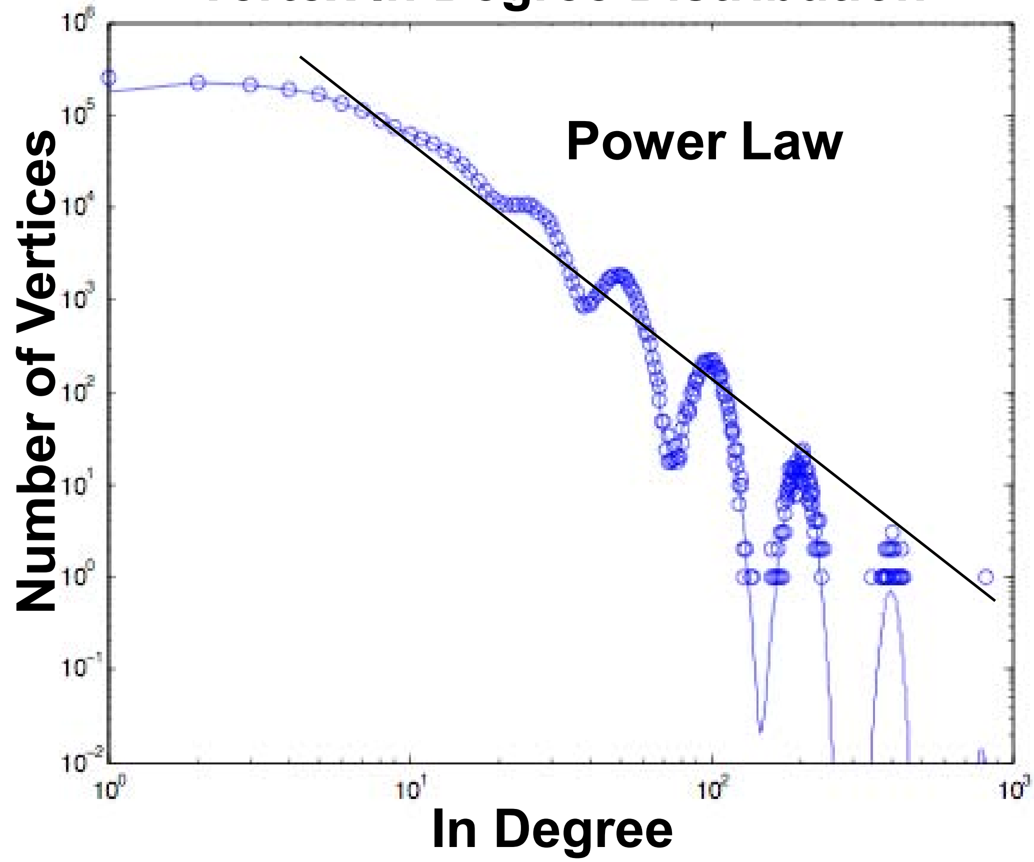
- Graph500 generates power law data
- D4M (in memory) + Accumulo (storage) provides scalable high performance

Power Law Modeling of Kronecker Graphs

Adjacency Matrix



Vertex In Degree Distribution



- Real world data (internet, social networks, …) has connections on all scales (i.e power law)
- Can be modeled with Kronecker Graphs: $G^{\otimes k} = G^{\otimes k-1} \otimes G$
  - Where "$\otimes$" denotes the Kronecker product of two matrices

Outline

- Introduction
- $B^{\otimes K}$ Graphs
  - Definitions
  - Bipartite Graphs
  - Degree Distribution
- $(B+I)^{\otimes K}$ Graphs
- Performance
- Summary

Kronecker Products and Graph

Kronecker Product

- Let B be a N_BxN_B matrix
- Let C be a N_CxN_C matrix
- Then the Kronecker product of B and C will produce a N_BN_CxN_BN_C matrix A:

$$A = B \otimes C = \begin{pmatrix} b_{1,1}C & b_{1,2}C & \ldots & b_{1,M_B}C \\ b_{2,1}C & b_{2,2}C & \ldots & b_{2,M_B}C \\ \vdots & \vdots & & \vdots \\ b_{N_B,1}C & b_{N_B,2}C & \ldots & b_{N_B,M_B}C \end{pmatrix}$$

Kronecker Graph (Leskovec 2005 & Chakrabati 2004)

- Let G be a NxN adjacency matrix
- Kronecker exponent to the power k is:

$$G^{\otimes k} = G^{\otimes k-1} \otimes G$$

Types of Kronecker Graphs

Explicit

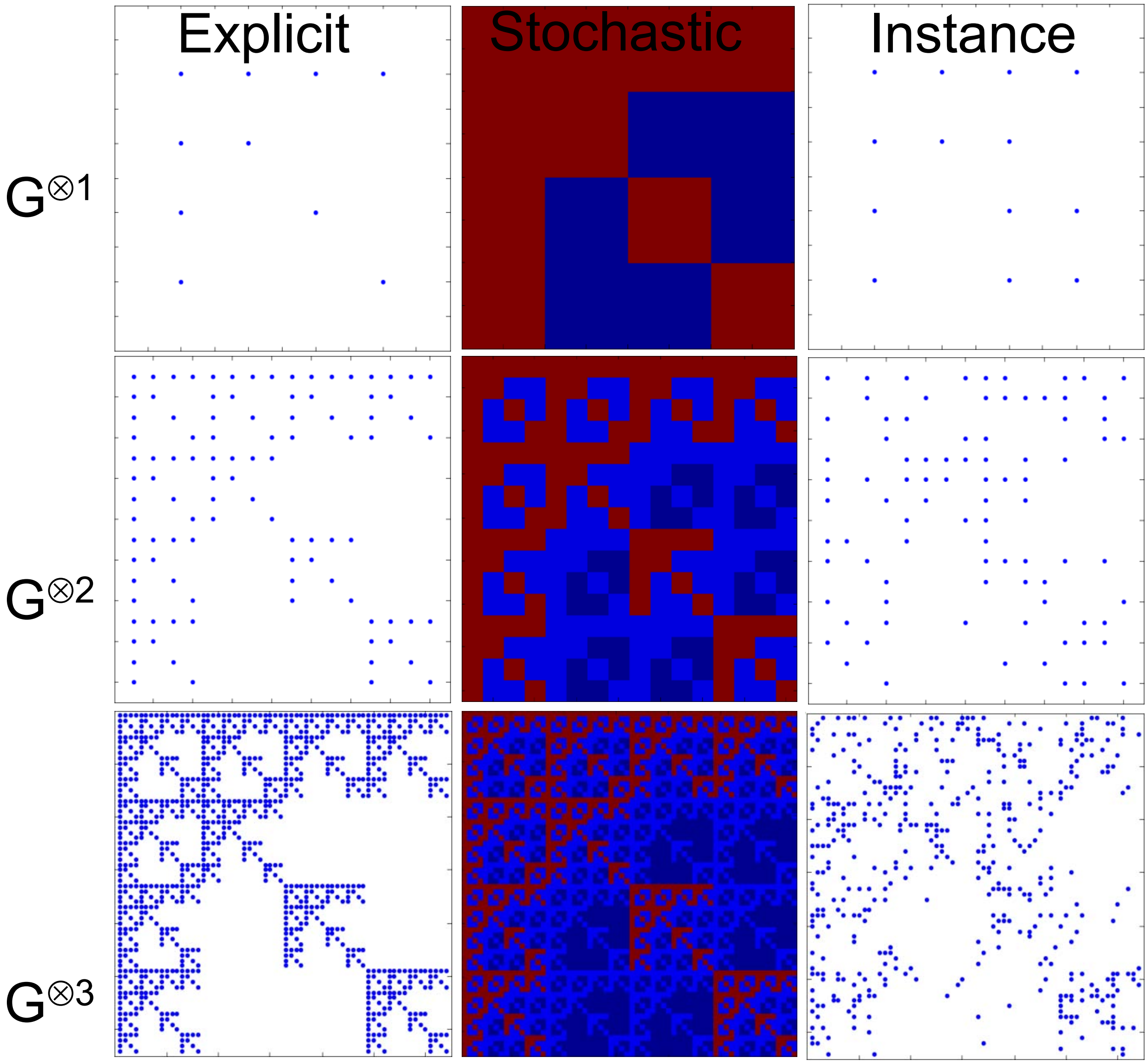
- **G only 1 and 0s**

Stochastic

- **G contains probabilities**

Instance

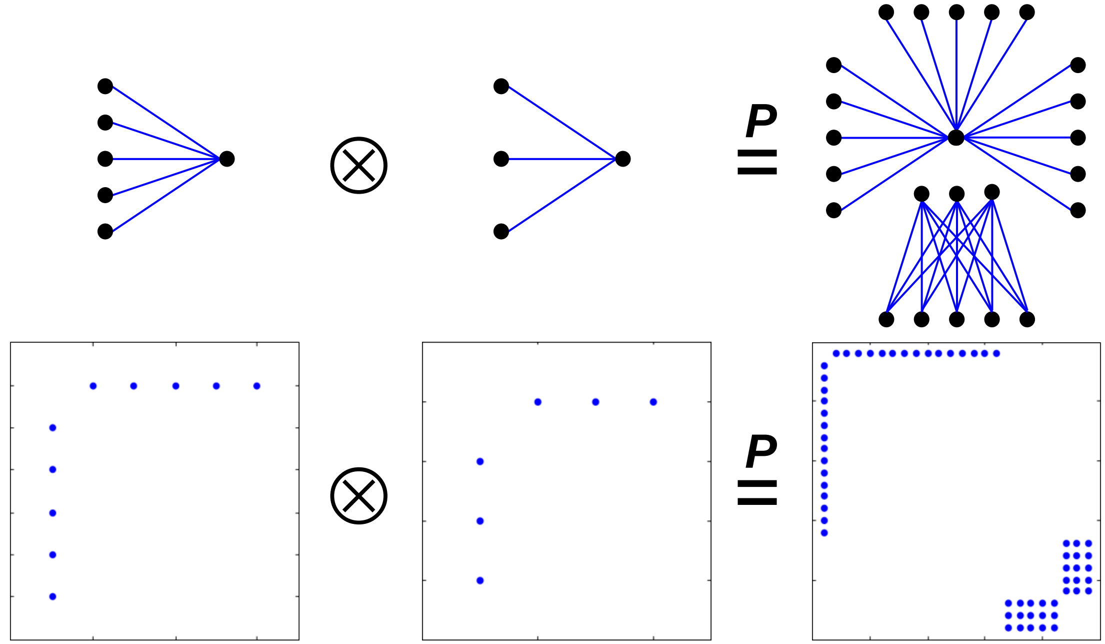
- **A set of M points (edges) drawn from a stochastic**



Kronecker Product of a Bipartite Graph

$\overset{P}{=}$

Equal with the right permutation



$$B(5,1) \otimes B(3,1) \overset{P}{=} B(15,1) \cup B(3,5)$$

- **Fundamental result [Weischel 1962] is that the Kronecker product of two complete bipartite graphs is two complete bipartite graphs**
- **More generally**

$$B(n_1, m_1) \otimes B(n_2, m_2) \overset{P}{=} B(n_1 n_2, m_1 m_2) \cup B(n_2 m_1, n_1 m_2)$$

Degree Distribution of Bipartite Kronecker Graphs

- **Kronecker exponent of a bipartite graph produces many independent bipartite graphs**

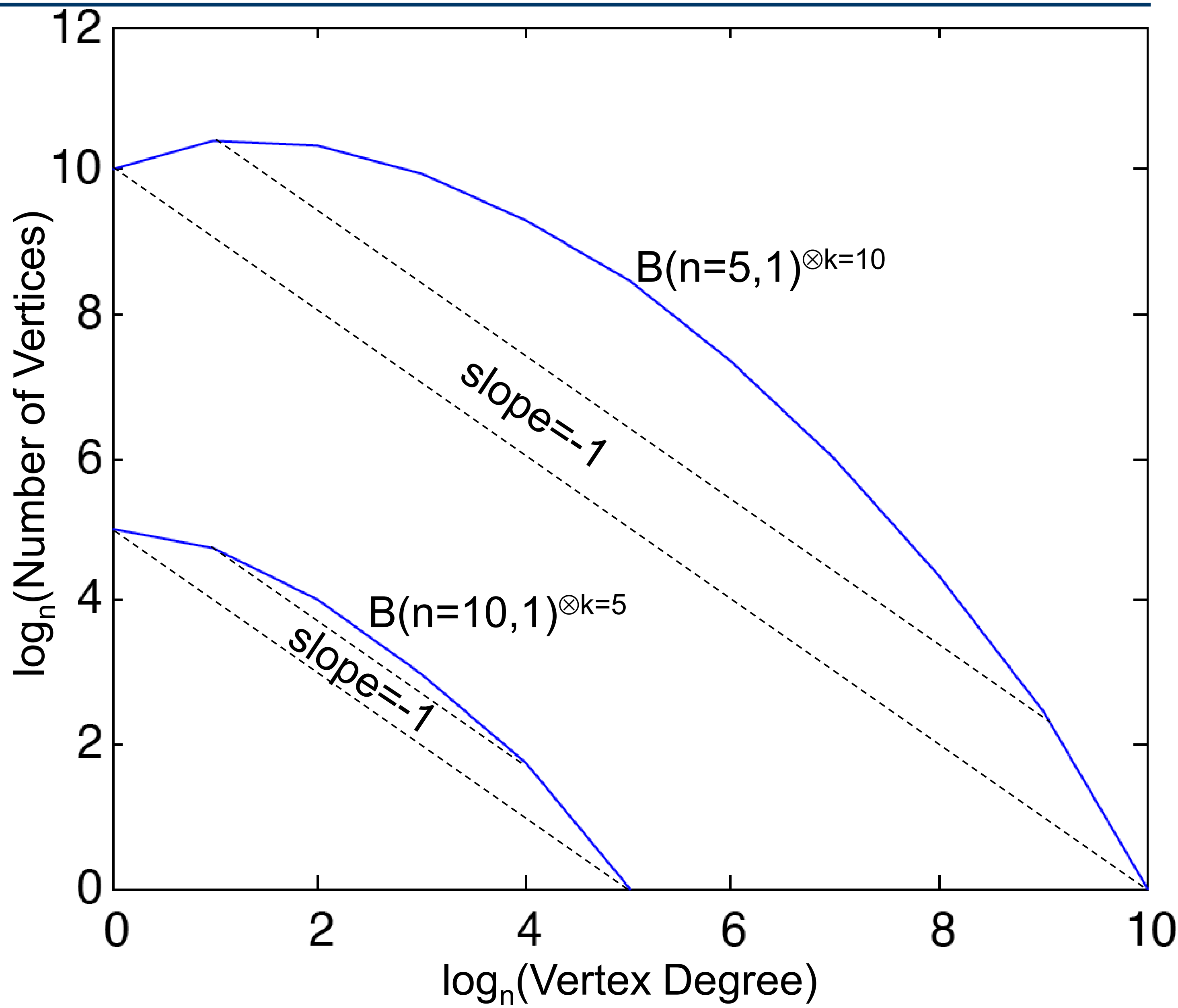
$$B(n,m)^{\otimes k} \stackrel{P}{=} \bigcup_{r=0}^{k-1} \bigcup^{\binom{k-1}{r}} B(n^{k-r}m^r, n^r m^{k-r})$$

- **Only k+1 different kinds of nodes in this graph, with degree distribution**

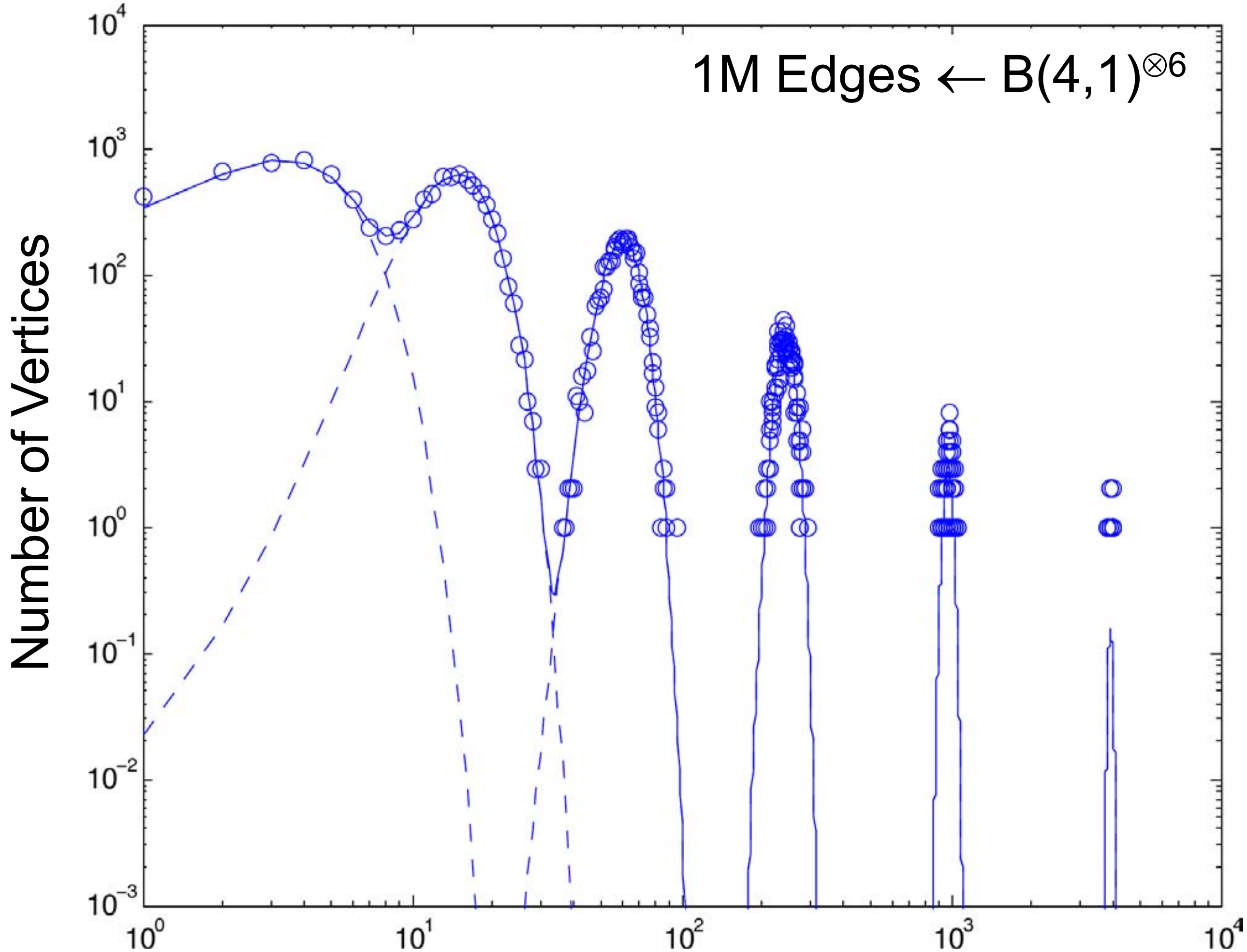
$$Count[Deg = n^r m^{k-r}] = \binom{k}{r} n^{k-r} m^r$$

Explicit Degree Distribution

- **Kronecker exponent of bipartite graph naturally produces exponential distribution**



Instance Degree Distribution



- **An instance graph drawn from a stochastic bipartite graph is just the sum of Poisson distributions taken from the explicit bipartite graph**

Outline

- Introduction
- $B^{\otimes K}$ Graphs
- $(B+I)^{\otimes K}$ Graphs
  - Bipartite + Identity Graphs
  - Permutations and substructure
  - Degree Distribution
  - Iso Parametric Ratio
- Performance
- Summary

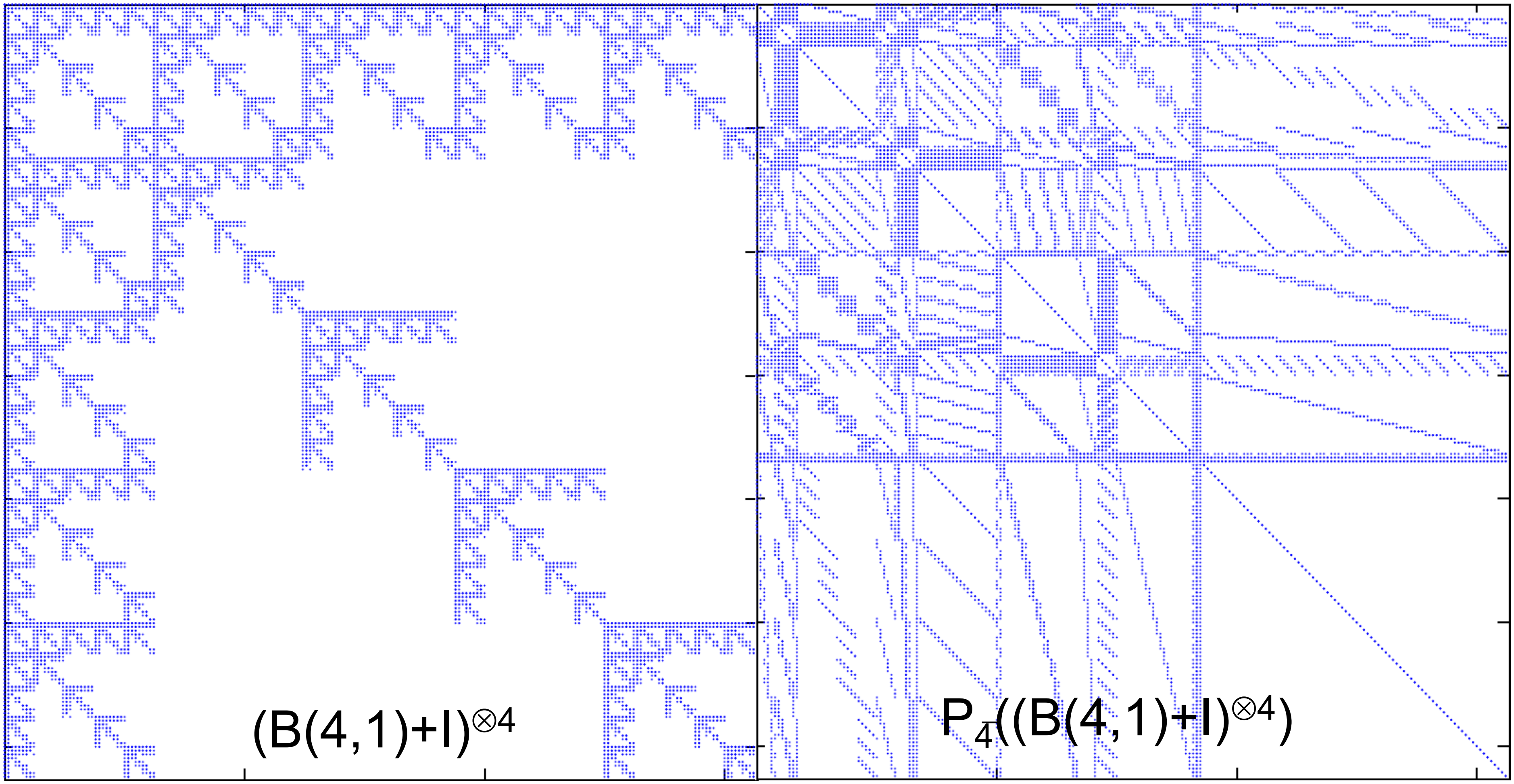
Theory

- **Bipartite Kronecker graphs highlight the fundamental structures in a Kronecker graph, but**
  - **Are not connected (i.e. many independent bipartite graphs)**
- **Adding identity matrix creates connections on all scales**
  - **Resulting explicit graph has diameter = 2**
  - **Sub-structures in the graph are given by**

$$(B+I)^{\otimes k} \stackrel{P}{=} \sum_{r=1}^{k} \text{``}\binom{k}{r}\text{''} \bigcup^{N^{k-1}} B^{\otimes k}$$

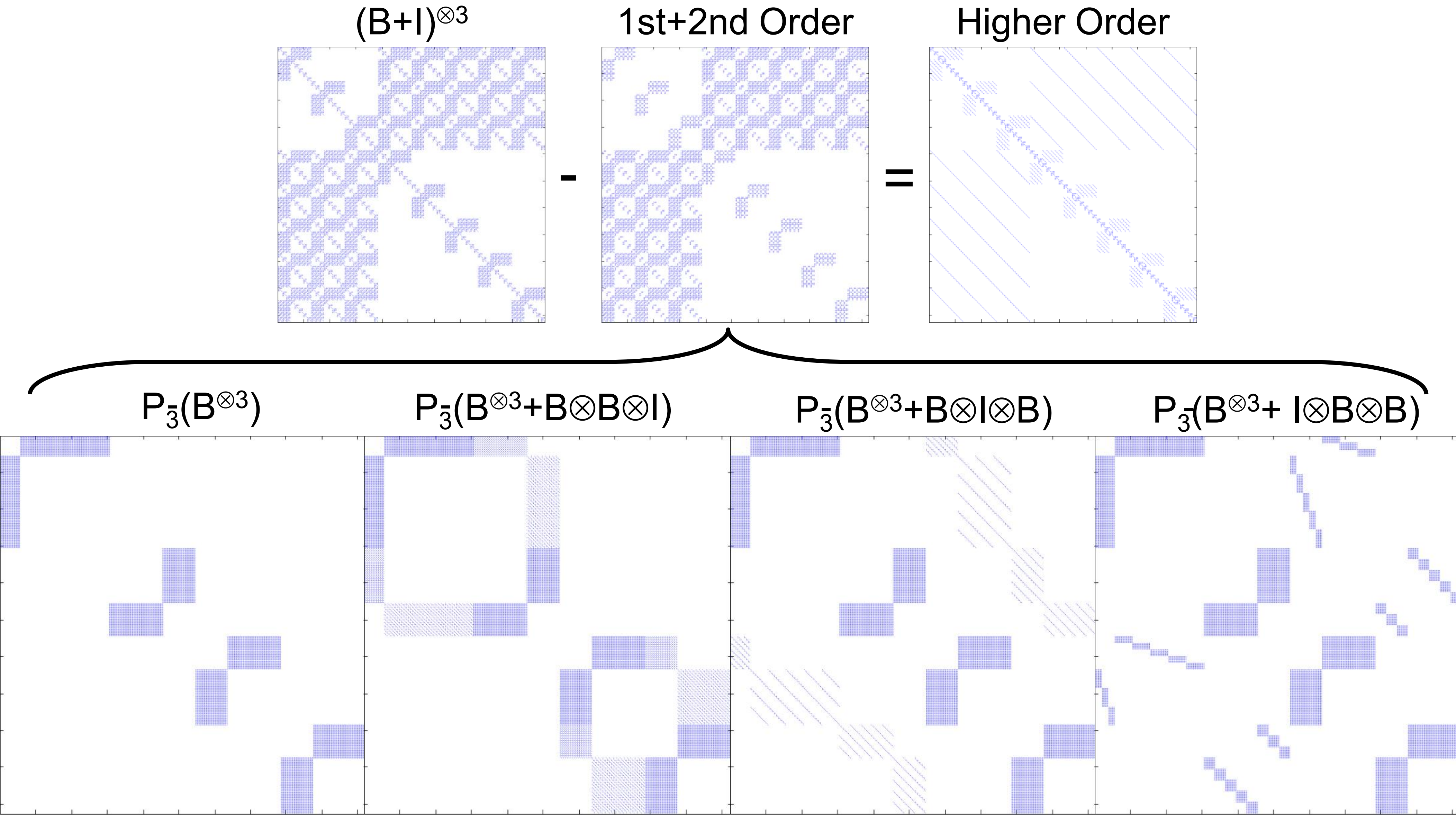
  - **Where “” indicates permutations are required to add the matrices**
- **Sub-structure can be revealed by applying permutation that “groups” vertices by their bipartite sub-graph**

Bipartite Permutation



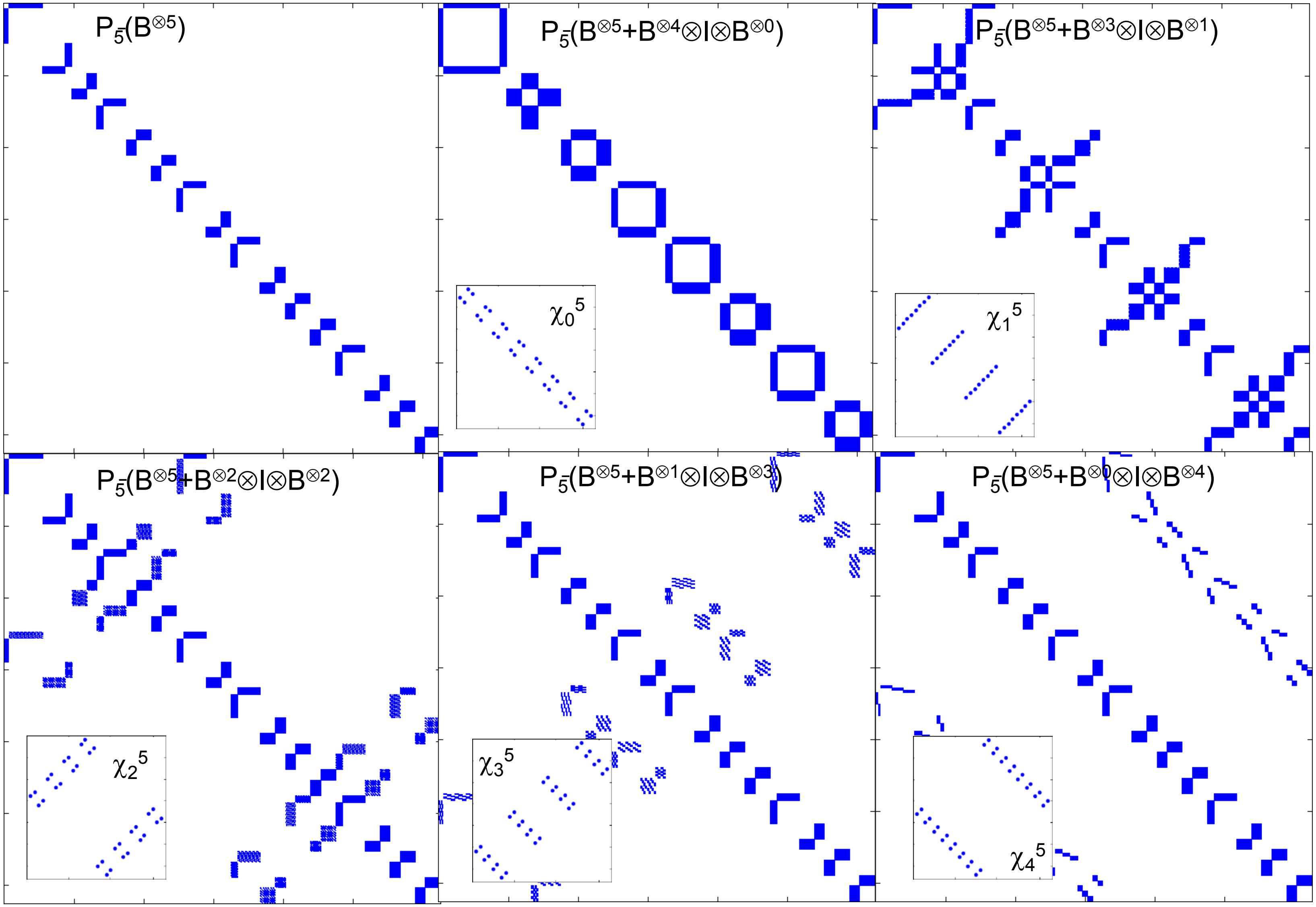
- **Left: unpermuted $(B+I)^{\otimes 4}$ kronecker graph**
- **Right: permuted $(B+I)^{\otimes 4}$ kronecker graph**

Identifying Substructure



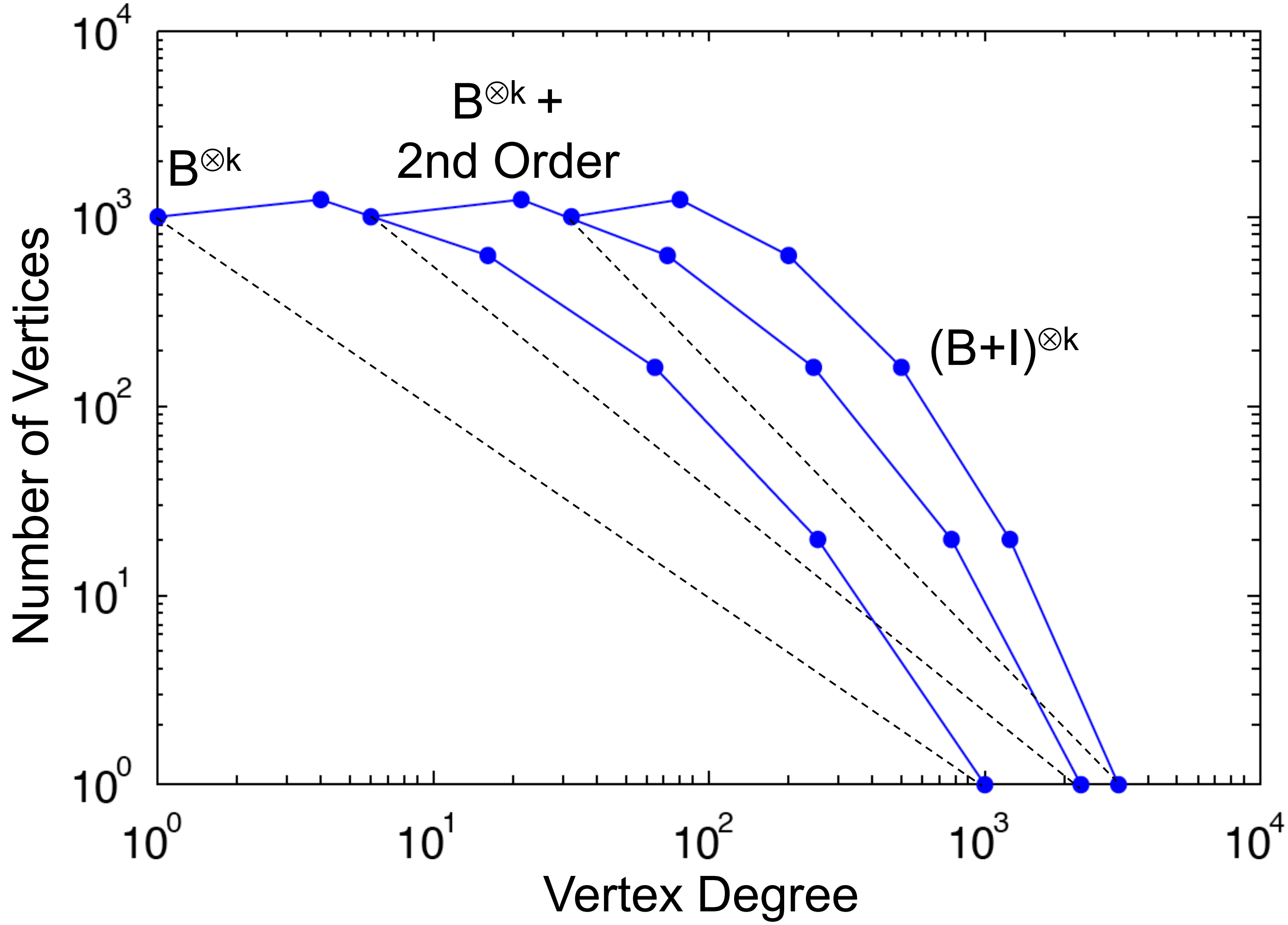
- **Permuting specific terms shows their contributions to the graph**

Quantifying Substructure



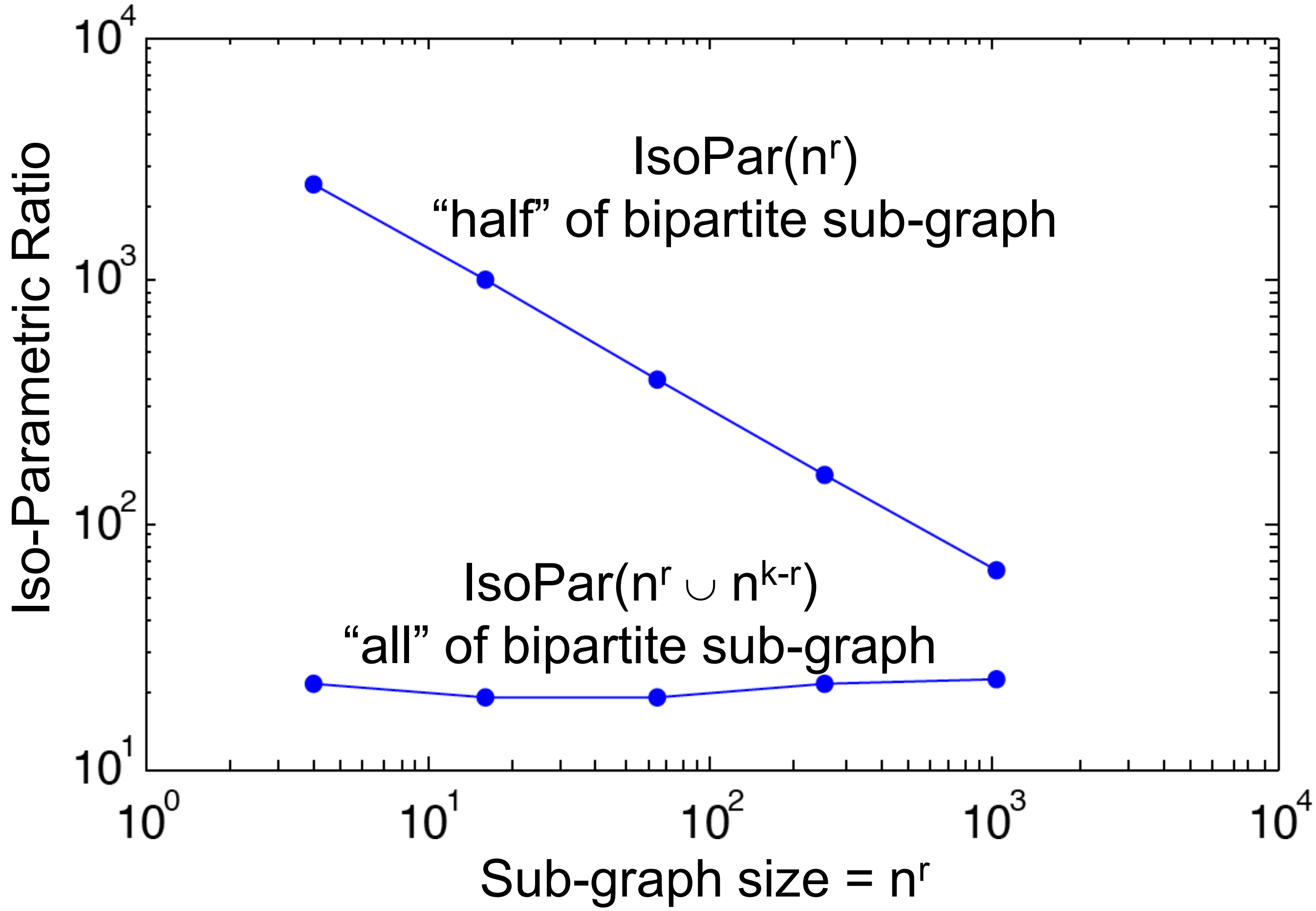
- **Connections between bipartite subgraphs are the Kronecker product of corresponding 2x2 matrices, e.g. $B(1,1)^{\otimes 4}\otimes I(2)$**

Substructure Degree Distribution



- **Only k+1 different kinds of nodes in this graph, with same degree distribution, only differing values of vertex degree**
- **$(B+I)^{\otimes k}$ is steeper than $B^{\otimes k}$**

Example Result: Iso-Parametric Ratio



- Iso-parametric ratios measure the "surface" to "volume" of a sub-graph
- Can analytically compute for a Kronecker graph: $(B+I)^{\otimes k}$
- Shows large effect of including "half" or "all" of bipartite sub-graph

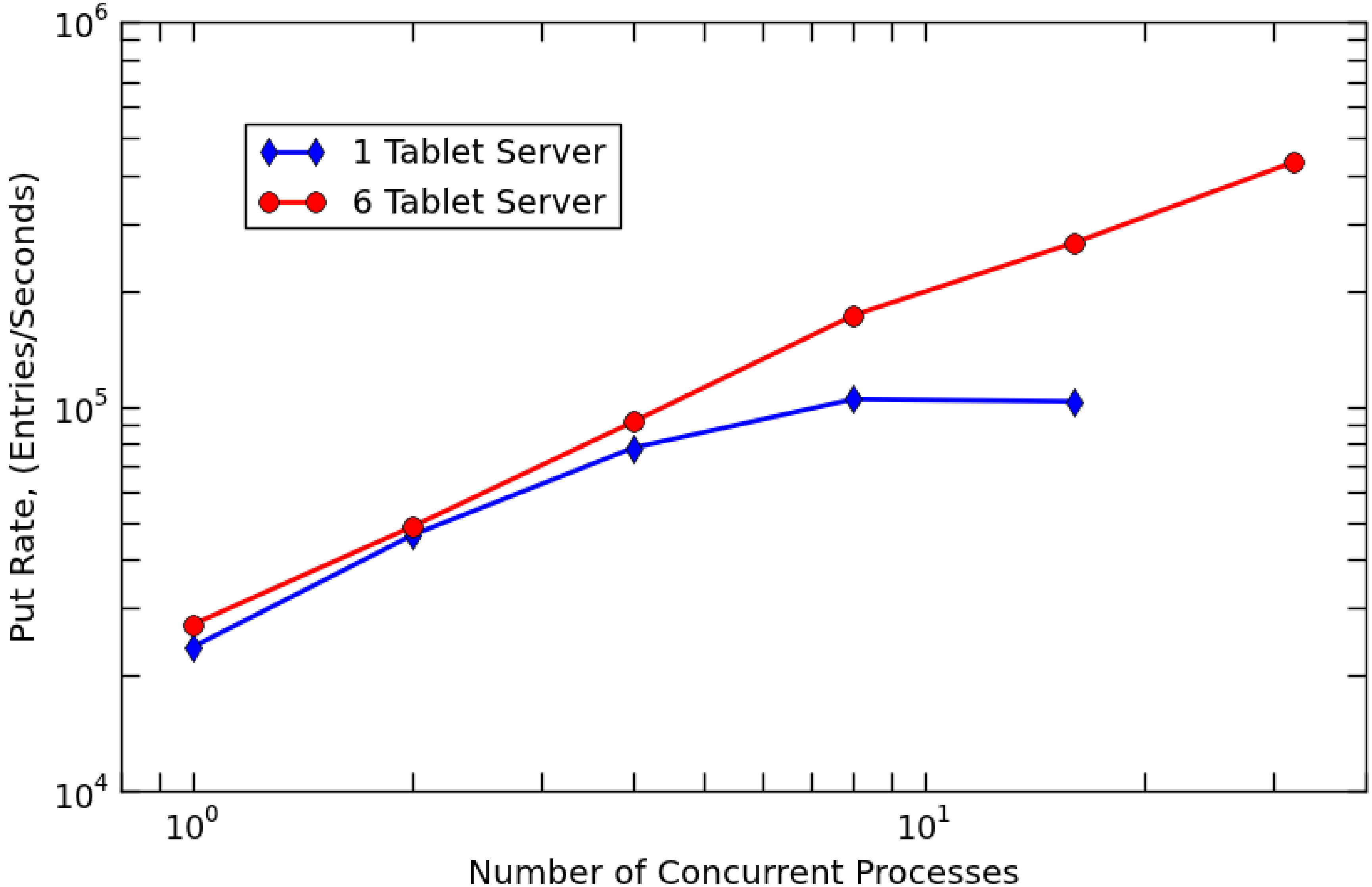
Kronecker Graph Theory
-Summary of Current Results-

Quantity	Graph: $B(n,m)^{\otimes k}$	Graph: $(B+I)^{\otimes k}$
Degree Distribution	$Count[Deg = n^r m^{k-r}] = \binom{k}{r} n^{k-r} m^r$	$Count[Deg = (n+1)^r (m+1)^{k-r}] = \binom{k}{r} n^{k-r} m^r$
Betweenness Centrality	$Count[C_b = (n/m)^{2r-k}(n^{k-r}m^r - 1)] = \binom{k}{r} n^{k-r} m^r$	
Diameter	$Diam(B^{\otimes k}) = \infty$	$Diam((B+I)^{\otimes k}) = 2$
Eigenvalues	$eig(B(n,m)^{\otimes k}) = \{\overbrace{(nm)^{k/2}, ..., (nm)^{k/2}}^{2^{k-1}}, \overbrace{-(nm)^{k/2}, ..., -(nm)^{k/2}}^{2^{k-1}}\}$	$eig((B+I)^{\otimes k}) = \{((nm)^{1/2}+1)^k, ((nm)^{1/2}+1)^{k-1}, ((nm)^{1/2}-1)^2((nm)^{1/2}+1)^{k-2}, ...\}$
Iso-parametric Ratio "half"	$IsoPar(n_k(i)) = \infty$	$IsoPar(n_k(i)) = 2(n+1)^{k-r}(m+1)^r - 2$
Iso-parametric Ratio "all"	$IsoPar(n_k(i) \cup m_k(i)) = 0$	$IsoPar(n_k(i) \cup m_k(i)) = 2\frac{n^r m^{k-r}(n+1)^{k-r}(m+1)^r + n^{k-r}m^r(n+1)^r(m+1)^{k-r}}{2n^k m^k + n^r m^{k-r} + n^{k-r}m^r + [\chi \text{ terms}]} - 2$

Outline

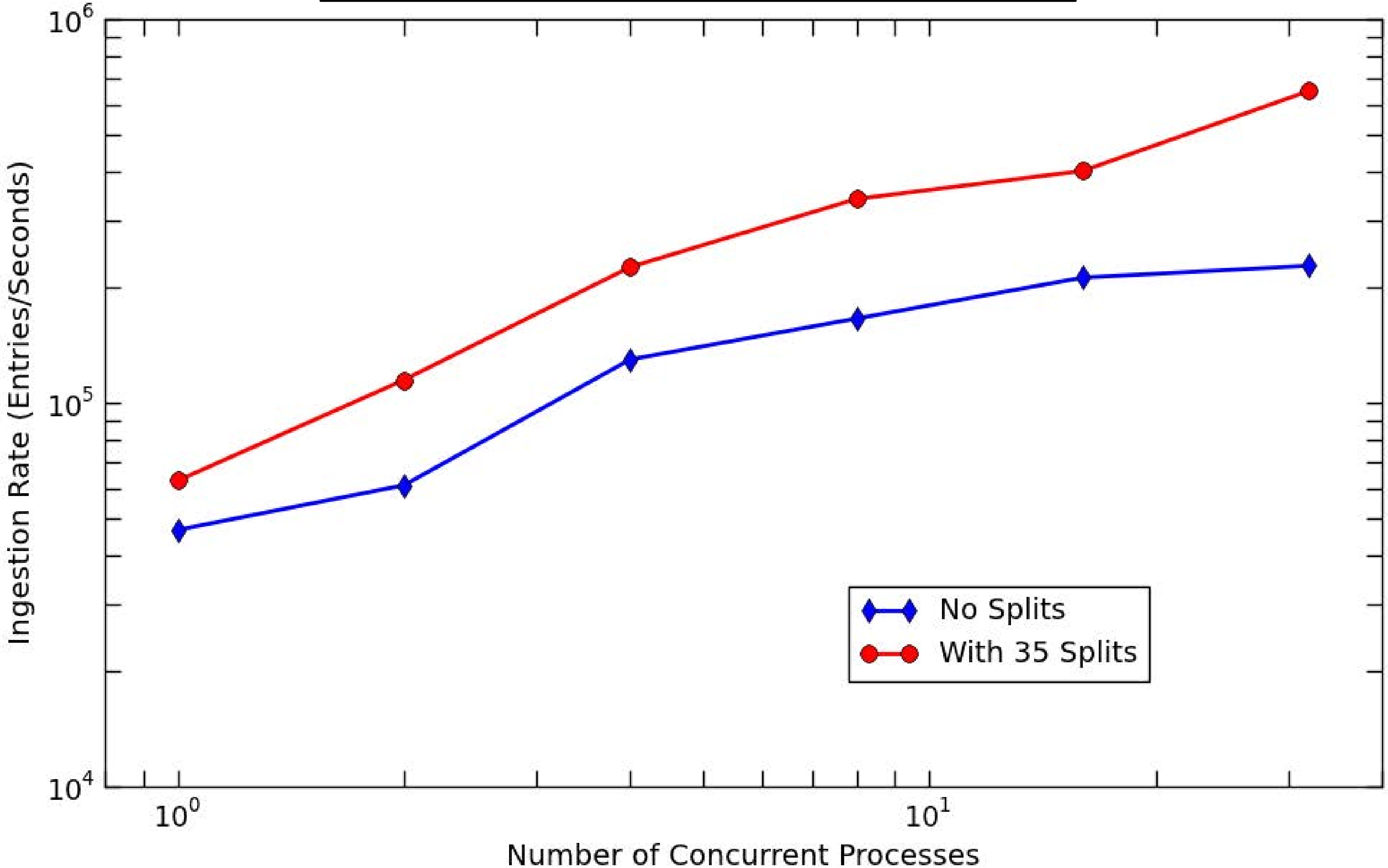
- Introduction
- $B^{\otimes K}$ Graphs
- $(B+I)^{\otimes K}$ Graphs
- Performance
  - Insert
  - Query
  - Matrix multiply
- Summary

Accumulo Data Ingestion Scalability pMATLAB Application Using D4M



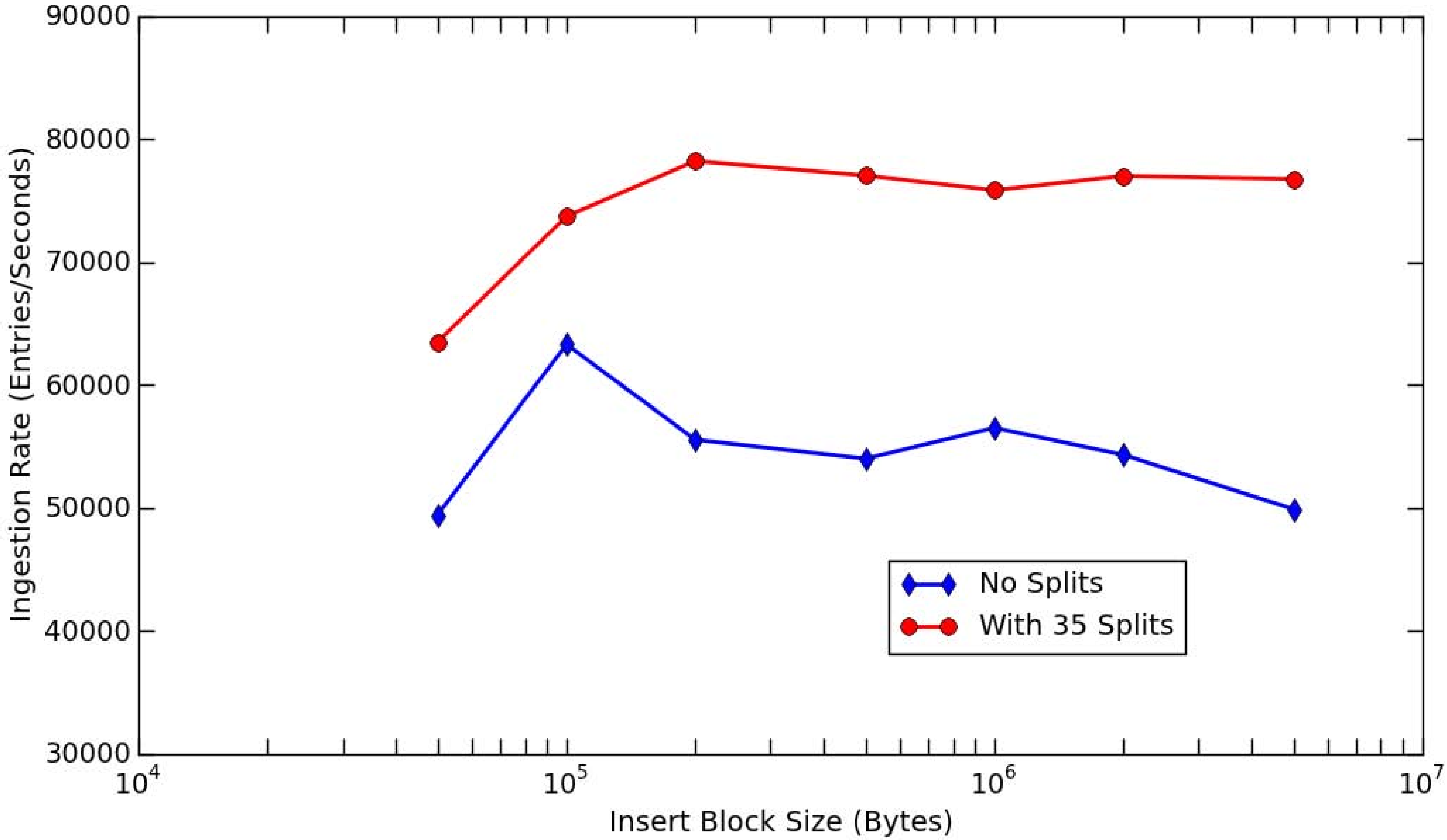
Effect of Pre-Split

Accumulo with 8 tablet servers



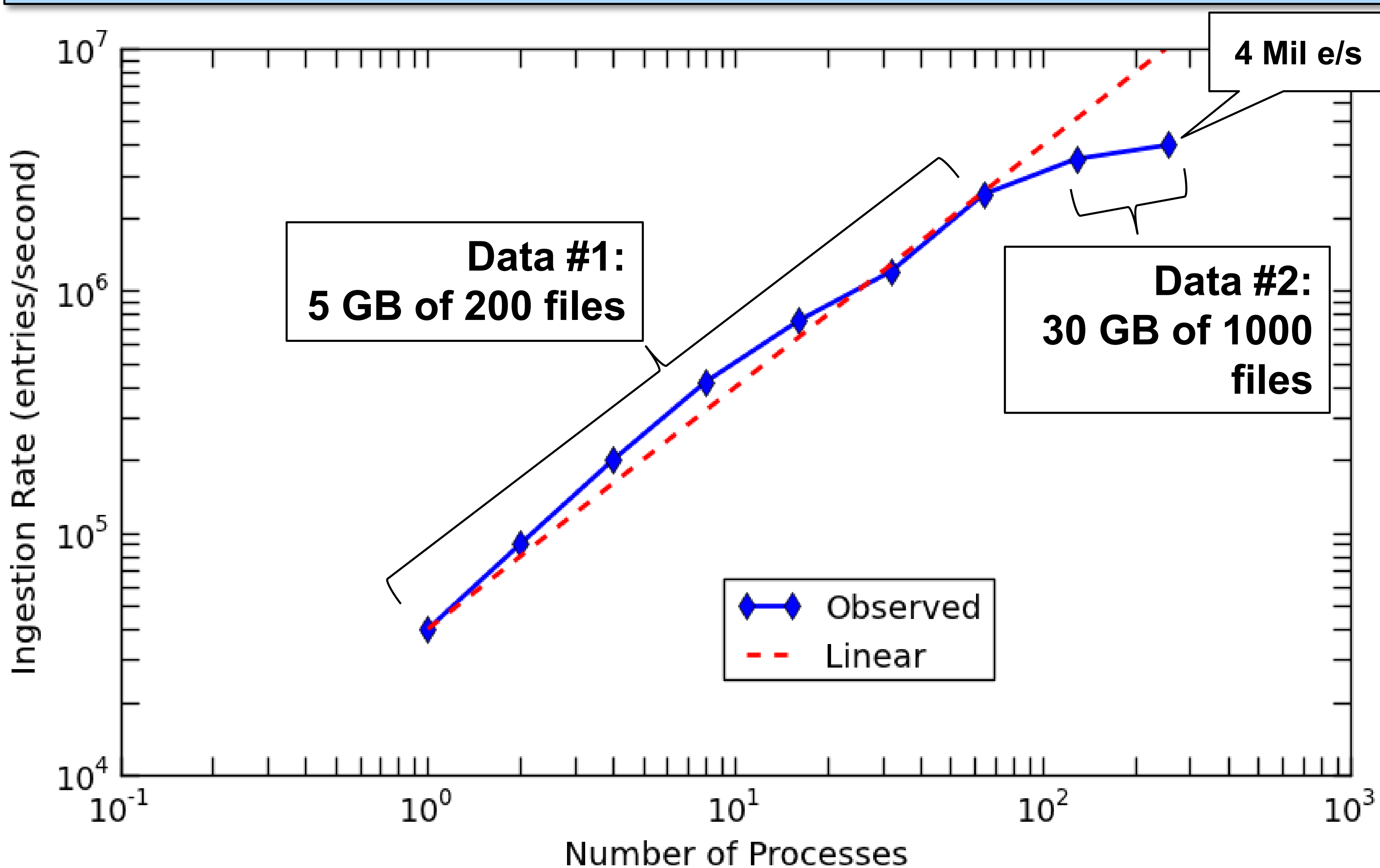
Effect of Ingestion Block Size

Accumulo with 8 tablet servers

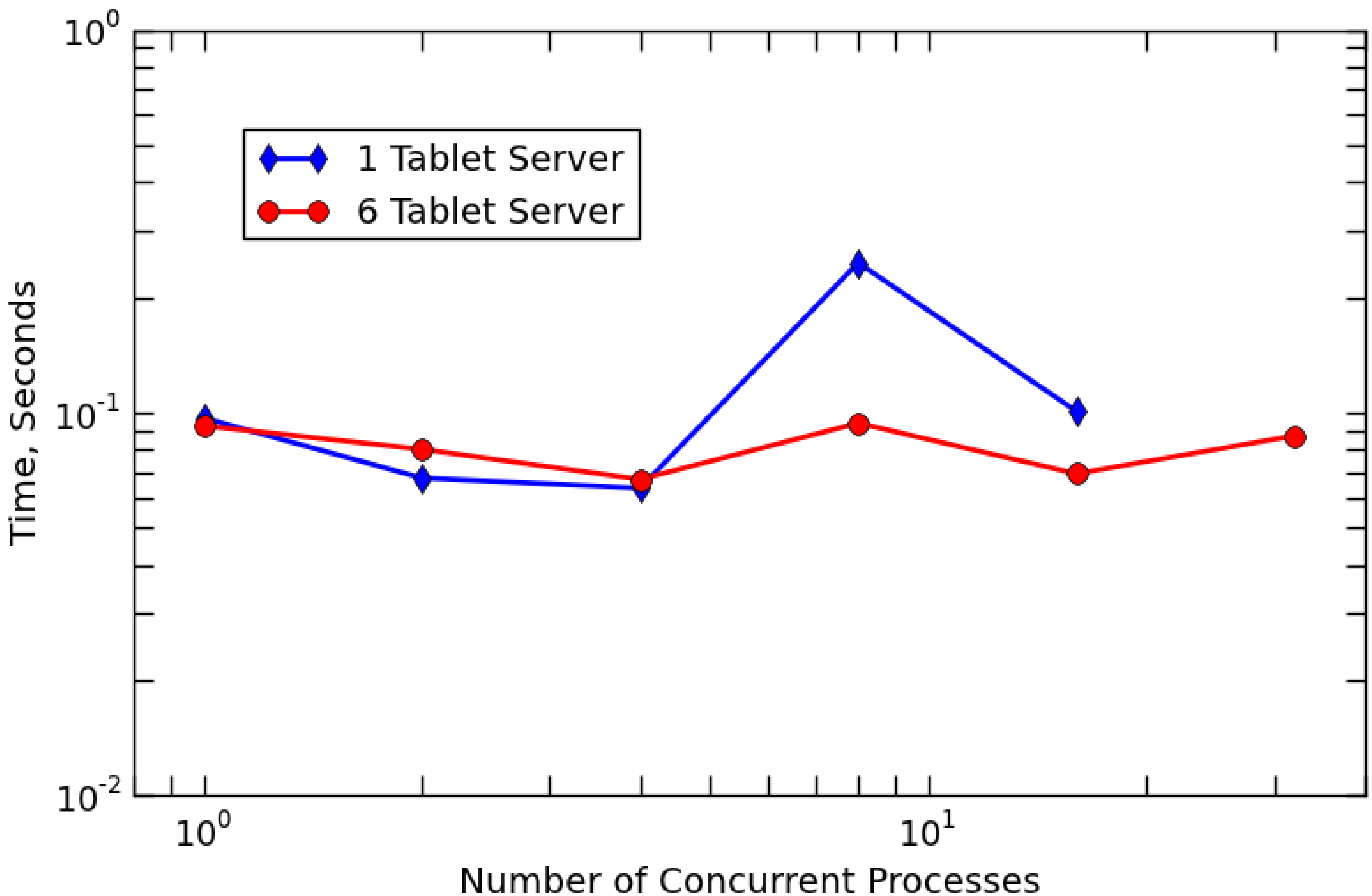


Accumulo Ingestion Scalability Study LLGrid MapReduce With A Python Application

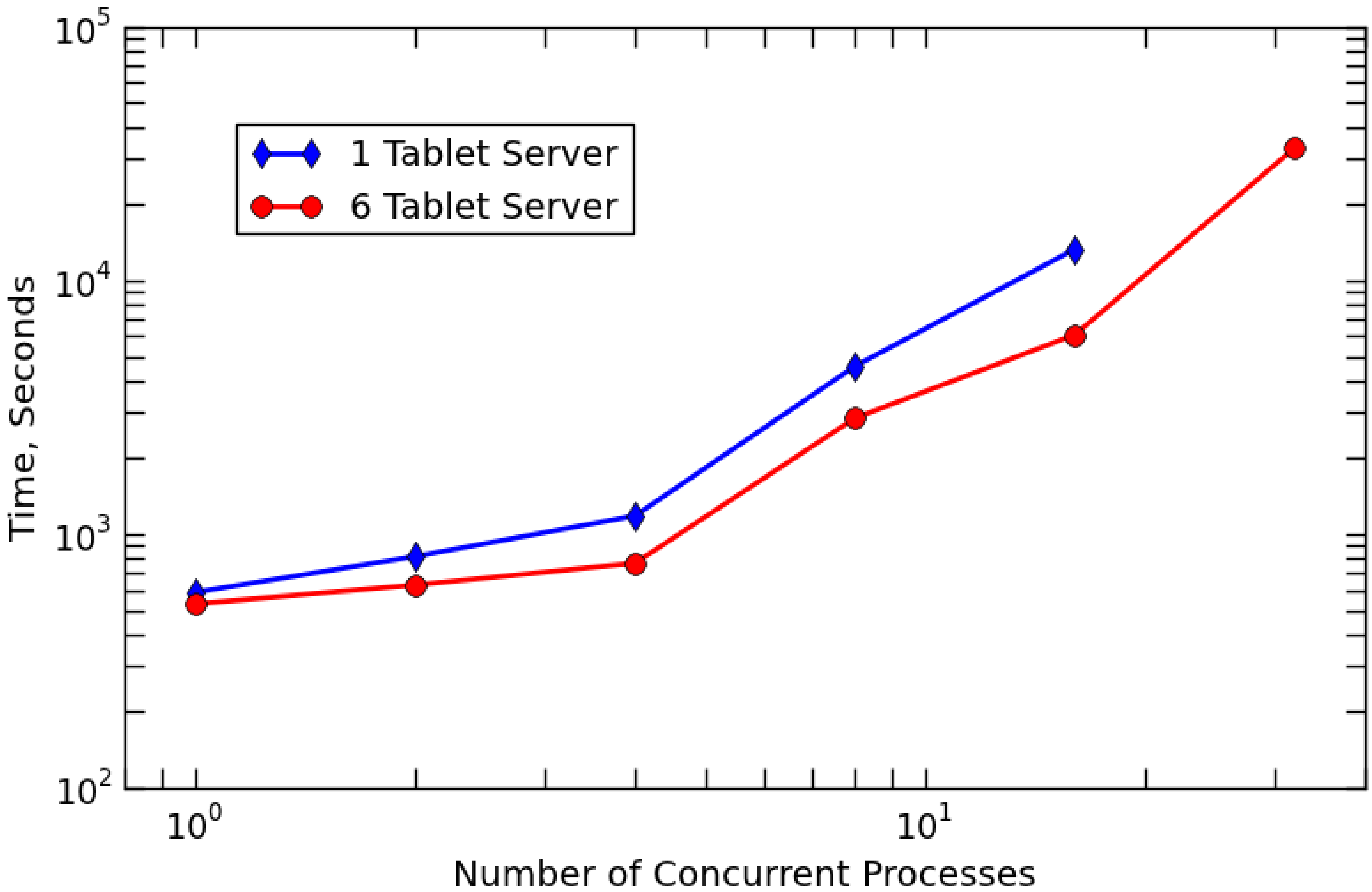
Accumulo Database: 1 Master + 7 Tablet servers (24 cores/each)



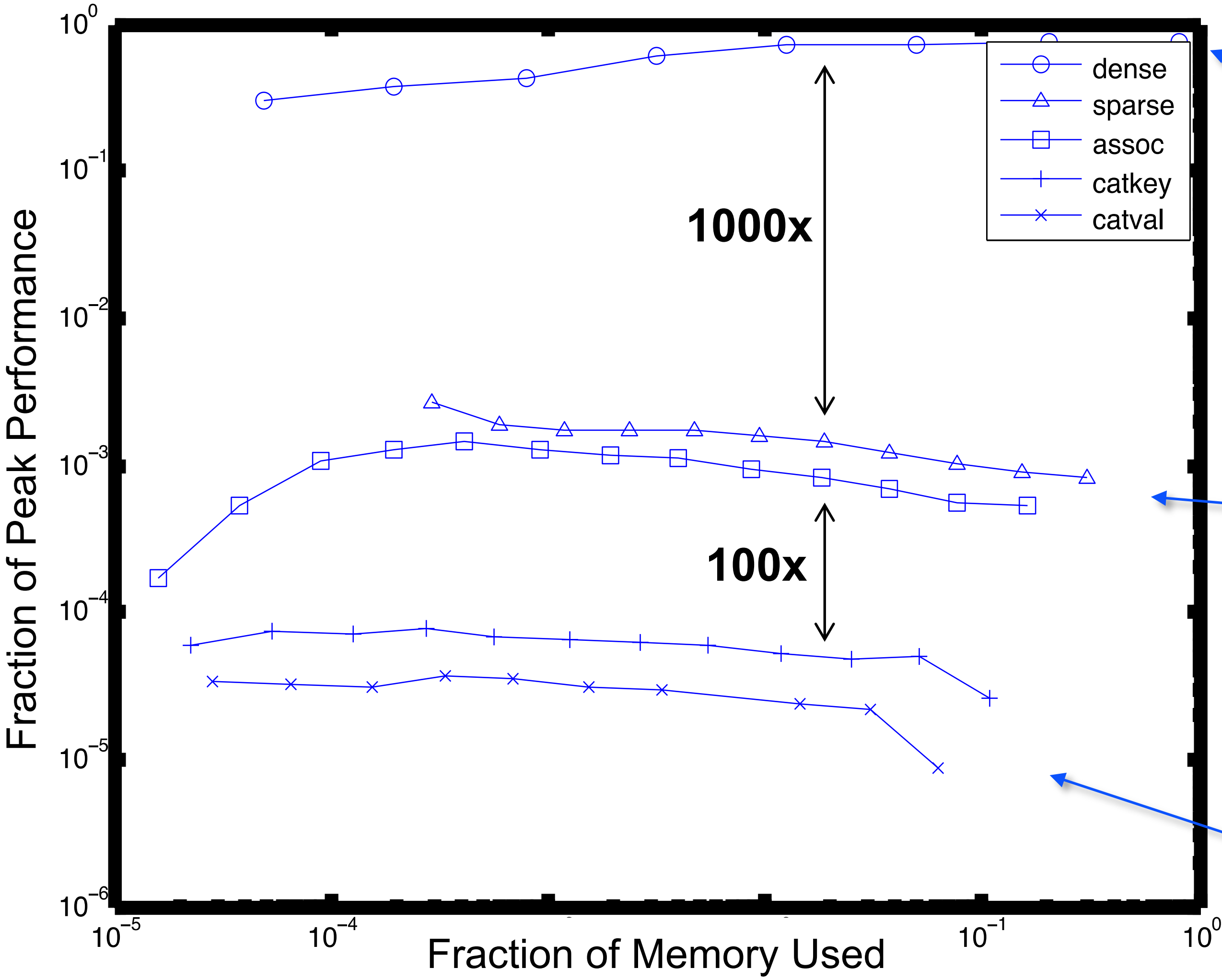
Accumulo Row Query Time
pMATLAB Application Using D4M



Accumulo Column Query Time
pMATLAB Application Using D4M



Matrix Multiply Performance



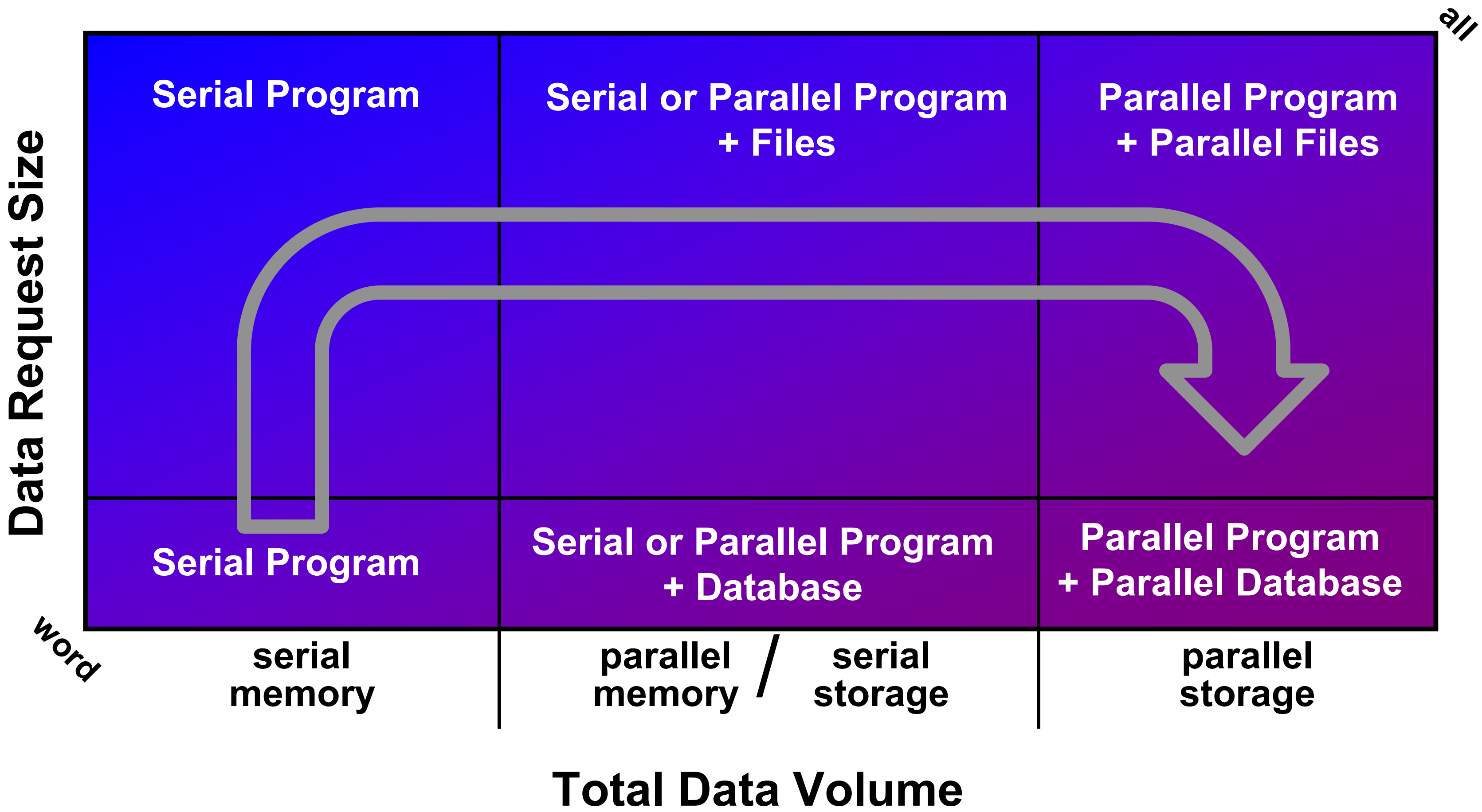
Dense Linear Algebra
~100% Efficient.
What COTS is designed to do.

Sparse Linear Algebra
~0.1% Efficient.
What network analysis requires.

Sparse String Correlation
~0.001% Efficient.
What semantic analysis requires.

- Sparse correlation (matrix multiply) is at the heart of graph algorithms
- Huge efficiency gap between what COTS processors are designed to do and what we need them to do ☹

Data Use Cases



- Data volume and data request size determine best approach
- Always want to start with the simplest and move to the most complex

Summary

- Power law graphs are the dominant type of data
  - Graph500 relies on Kronecker graphs
- Kronecker graphs have a rich theoretical structure that can be exploited for theory
- Parallel computations are implemented in D4M via pMatlab
- Complex graph algorithms are ultimately limited by hardware sparse matrix multiply performance

Example Code & Assignment

- Example Code
  - D4Muser_share/Examples/3Scaling/1KroneckerGraph
  - D4Muser_share/Examples/3Scaling/2ParallelDatabase
  - D4Muser_share/Examples/3Scaling/3MatrixPerformance

- Assignment
  - None