## Mathematical background

- Sets and De Morgan's laws
- Sequences and ther limits
- Infinite series
  - The geometric series
- Sums with multiple indices
- Countable and uncountable sets

## Sets

- A collection of distinct elements

$\{a,b,c,d\}$ finite

$\mathbb{R}$: real numbers infinite

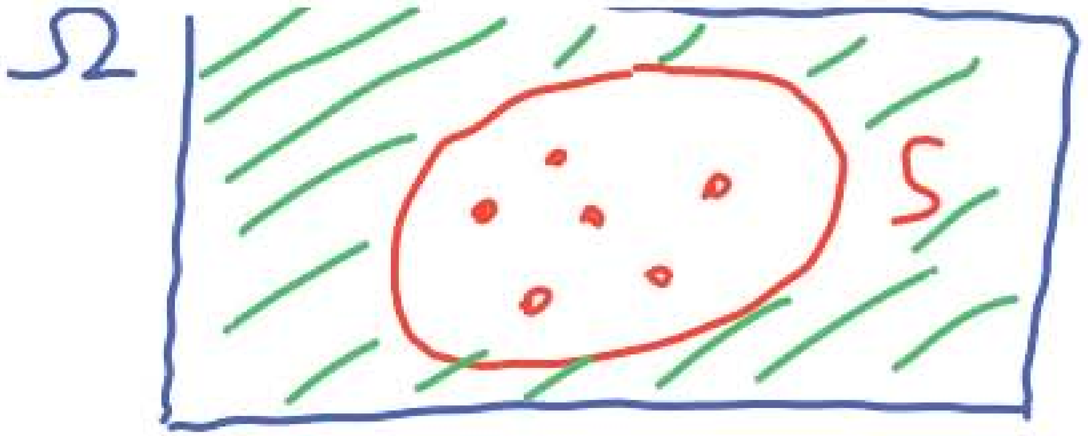
$\{x \in \mathbb{R} : \cos(x) > 1/2\}$

$\Omega$ : universal set

$\emptyset$ : empty set $\Omega^c = \emptyset$


$S \subset T : x \in S \Rightarrow x \in T$

$\subseteq$



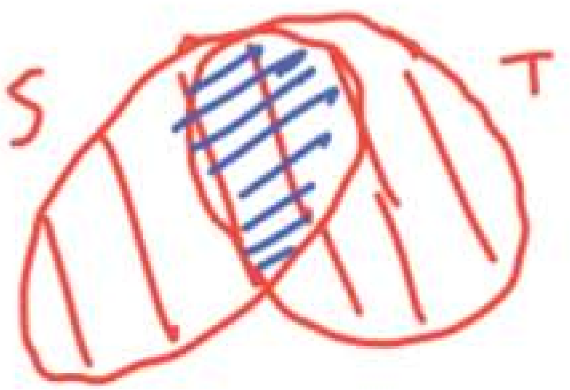

$x \in S$ $x \notin S$

$S^c$

$x \in S^c$ if $x \in \Omega$,
$x \notin S$

$(S^c)^c = S$

## Unions and intersections




$S \cup T$ $\quad$ $x \in S \cup T \iff x \in S$ or $x \in T$

$S \cap T$ $\quad$ $x \in S \cap T \iff x \in S$ and $x \in T$

$S_n$ $\quad$ $n = 1, 2, \ldots$

$x \in \bigcup_n S_n$ iff $x \in S_n$, for some $n$

$x \in \bigcap_n S_n$ iff $x \in S_n$, for all $n$

## Set properties

$S \cup T \cup U$

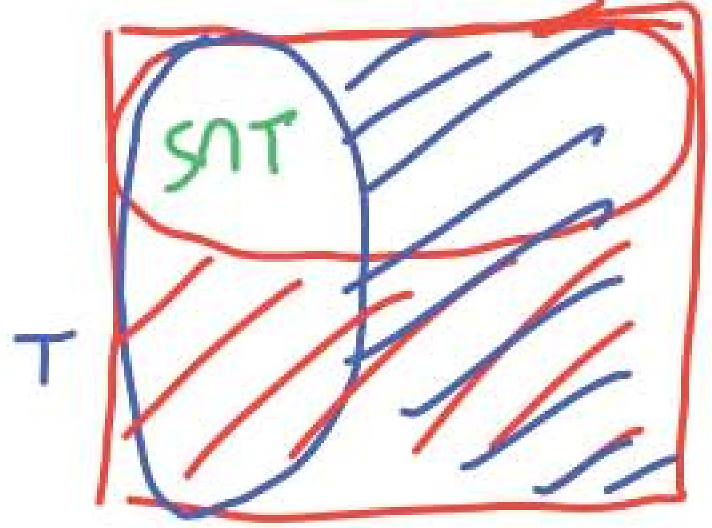
$$
\begin{aligned}
S \cup T &= T \cup S, & S \cup (T \cup U) &= (S \cup T) \cup U, \\
S \cap (T \cup U) &= (S \cap T) \cup (S \cap U), & S \cup (T \cap U) &= (S \cup T) \cap (S \cup U), \\
(S^c)^c &= S, & S \cap S^c &= \emptyset, \\
S \cup \Omega &= \Omega, & S \cap \Omega &= S.
\end{aligned}
$$

$S \cap (T \cap U) = (S \cap T) \cap U$


$$\left.\begin{array}{l} S \subset T \\ T \subset S \end{array}\right\} \Rightarrow S = T$$

# De Morgan's laws

$$\boxed{(S\cap T)^c = S^c \cup T^c}$$




$$S \to S^c \qquad T \to T^c$$

$$S^c \to S \qquad T^c \to T$$

$$(S^c \cap T^c)^c = S \cup T$$

$$\boxed{S^c \cap T^c = (S \cup T)^c}$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$$x \in (S\cap T)^c \iff x \notin S\cap T \iff \left\{\begin{array}{c} x\notin S \\ \text{or} \\ x \notin T \end{array}\right\} \iff \left\{\begin{array}{c} x\in S^c \\ \text{or} \\ x \in T^c \end{array}\right\} \iff x \in S^c \cup T^c$$

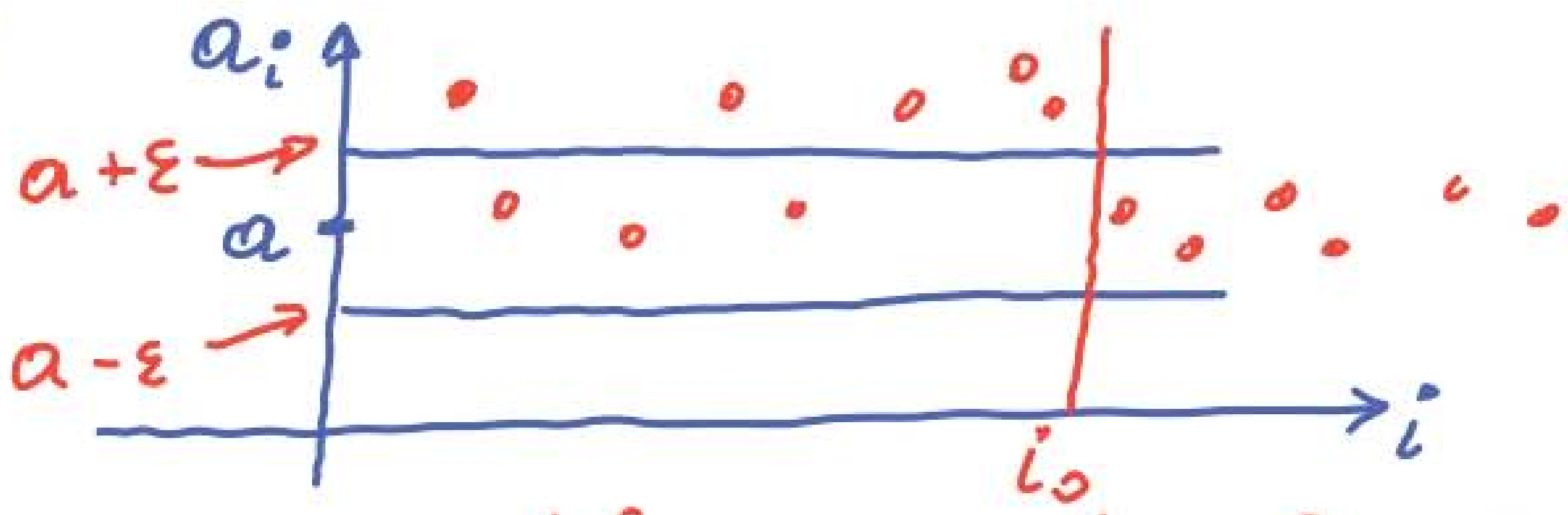
## Mathematical background: Sequences and their limits

$a_1, a_2, a_3, \ldots$ $\qquad i \in \mathbb{N} = \{1, 2, 3, \ldots\}$

sequence $a_i$ , $\{a_i\}$ $\qquad a_i \in S \qquad S = \mathbb{R} \qquad \mathbb{R}^n$

function $f: \mathbb{N} \to S$

$f(i) = a_i$




$a_i \xrightarrow[i\to\infty]{} a$

$\lim\limits_{i\to\infty} a_i = a$

For any $\varepsilon > 0$, there exists $i_0$, such that if $i \geq i_0$, then $|a_i - a| < \varepsilon$

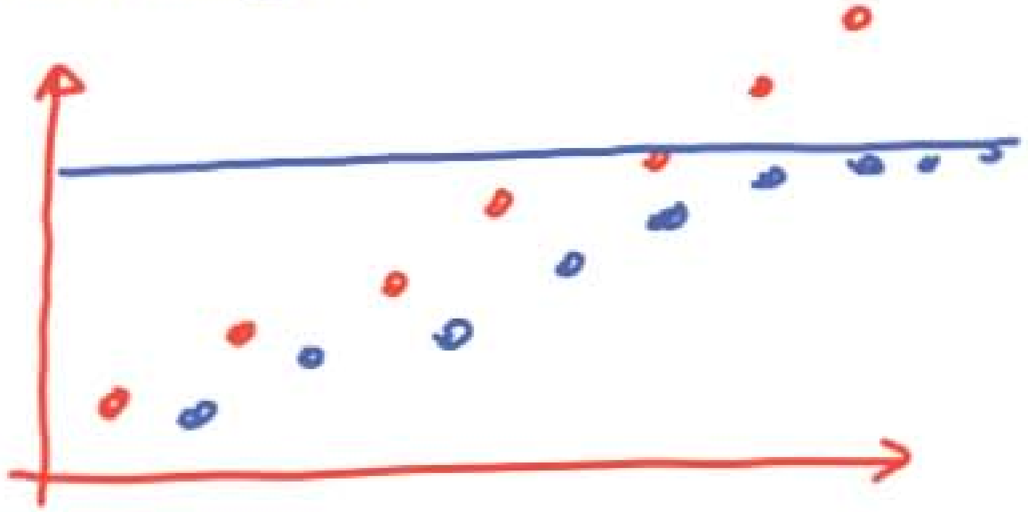
$a_i^2 \to a^2$

$\left.\begin{array}{l} a_i \to a \\ b_i \to b \end{array}\right\} \Rightarrow \begin{array}{l} a_i + b_i \to a + b \\ a_i b_i \to ab \end{array}$

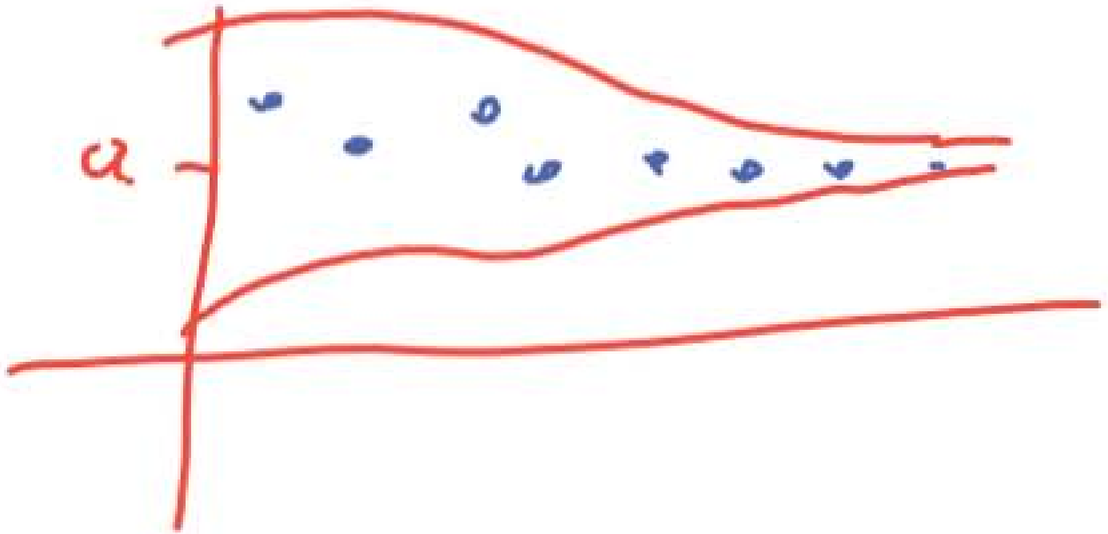
$g$: continuous $\Rightarrow g(a_i) \to g(a)$

## Mathematical background: When does a sequence converge?

- If $a_i \leq a_{i+1}$, for all $i$, then either:
  - the sequence "converges to $\infty$"
  - the sequence converges to some real number $a$



- If $|a_i - a| \leq b_i$, for all $i$, and $b_i \to 0$, then $a_i \to a$



## Mathematical background: Infinite series

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \, .$$ provided limit exists

- If $a_i \geq 0$: limit exists ←

- if terms $a_i$ do not all have the same sign:
  - limit need not exist
  - limit may exist but be different if we sum in a different order
  - **Fact:** limit exists and independent of order of summation if $\underline{\sum_{i=1}^{\infty} |a_i| < \infty}$

# Mathematical background: Geometric series

$$S = \sum_{i=0}^{\infty} \alpha^i = 1 + \alpha + \alpha^2 + \cdots \quad = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

$$(1-\alpha)(1+\alpha+\cdots+\alpha^n) = 1-\alpha^{n+1}$$

$n \to \infty$

$$(1-\alpha)S = 1$$

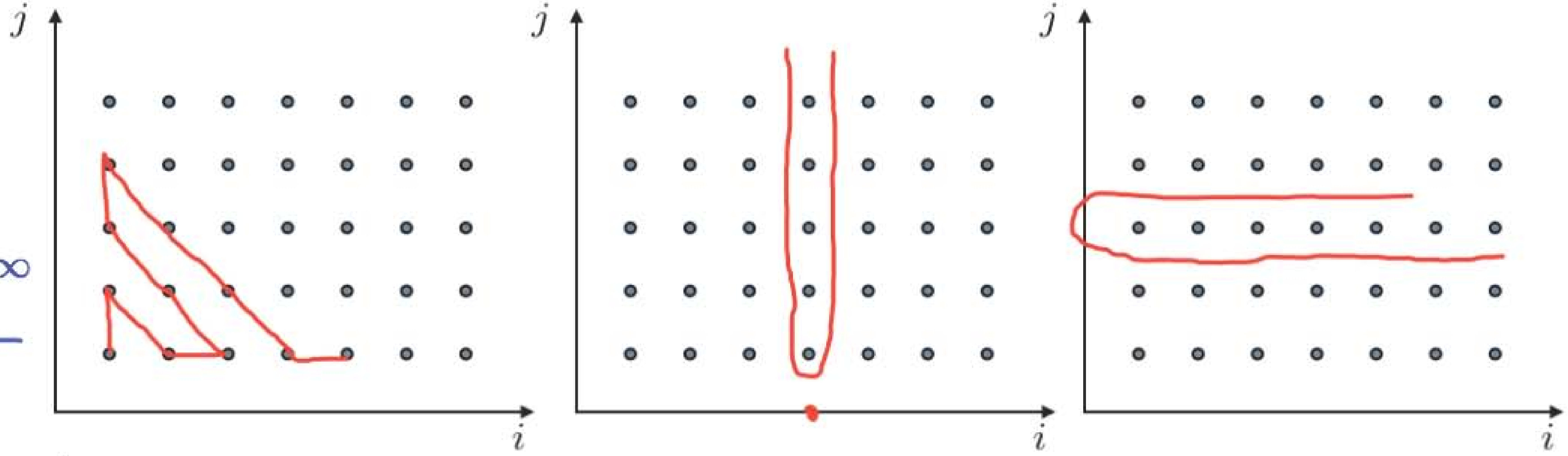
$$S = 1 + \sum_{i=1}^{\infty} \alpha^i = 1 + \alpha \sum_{i=0}^{\infty} \alpha^i = 1 + \alpha S \Rightarrow S(1-\alpha) = 1$$

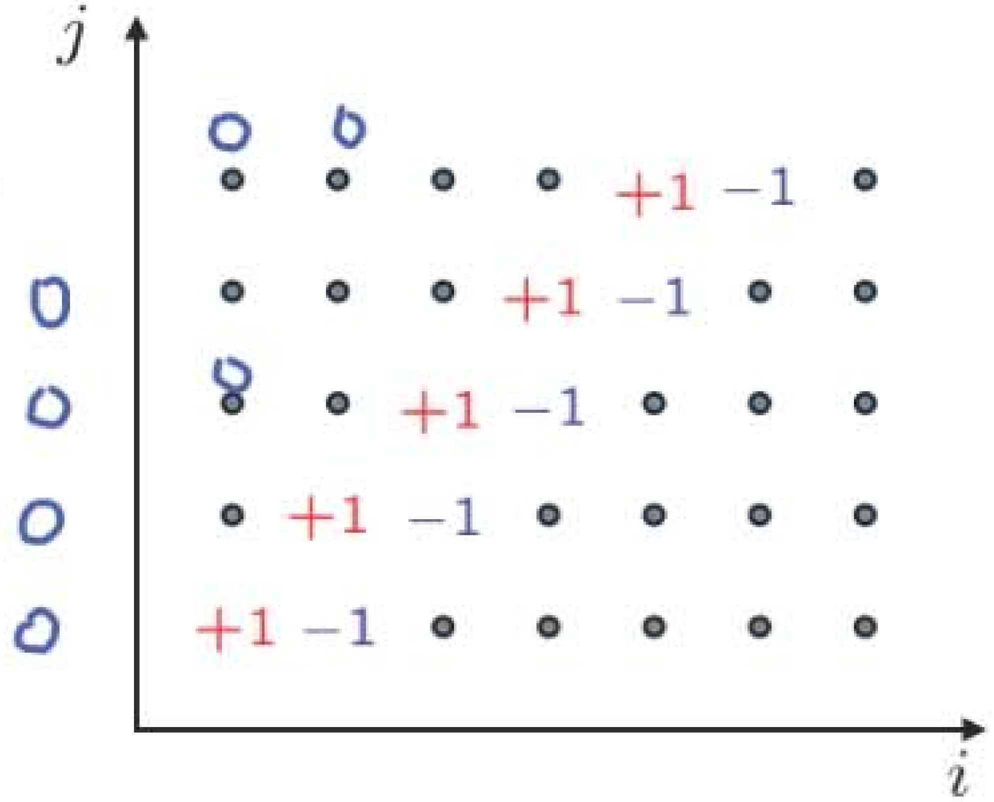
$S < \infty$ taken for granted

## About the order of summation in series with multiple indices

$$\sum_{i\geq 1, j\geq 1} a_{ij}$$

$$\sum |a_{ij}| < \infty$$



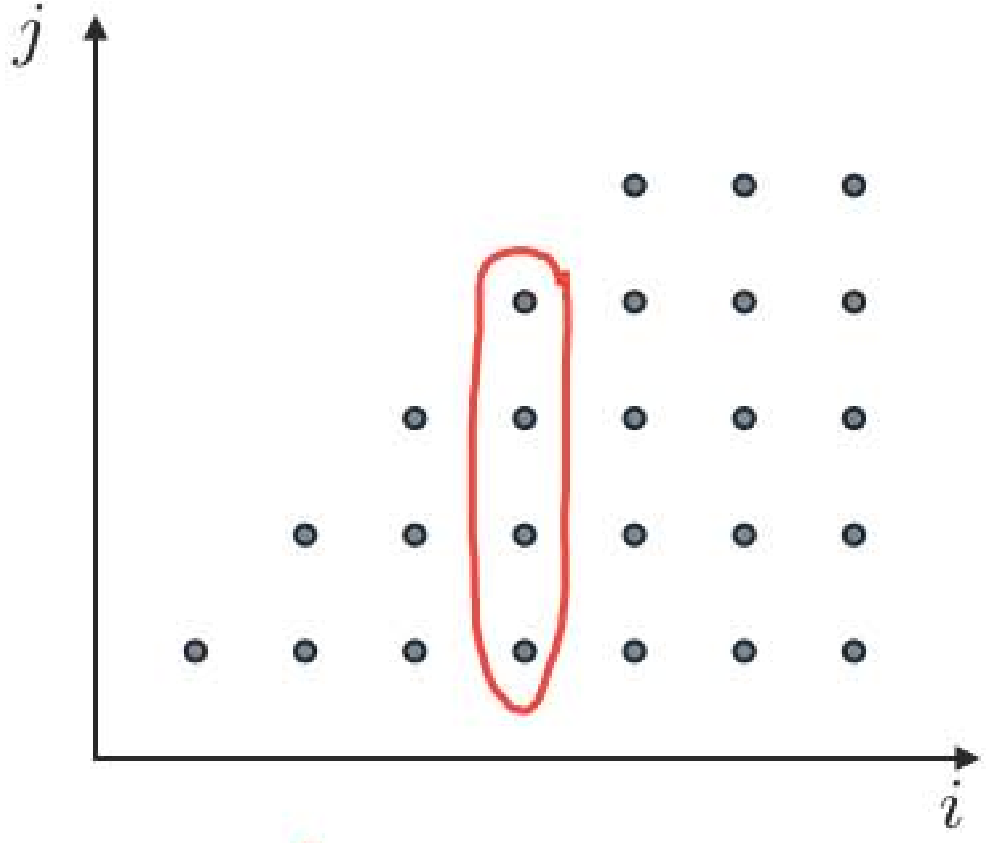

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

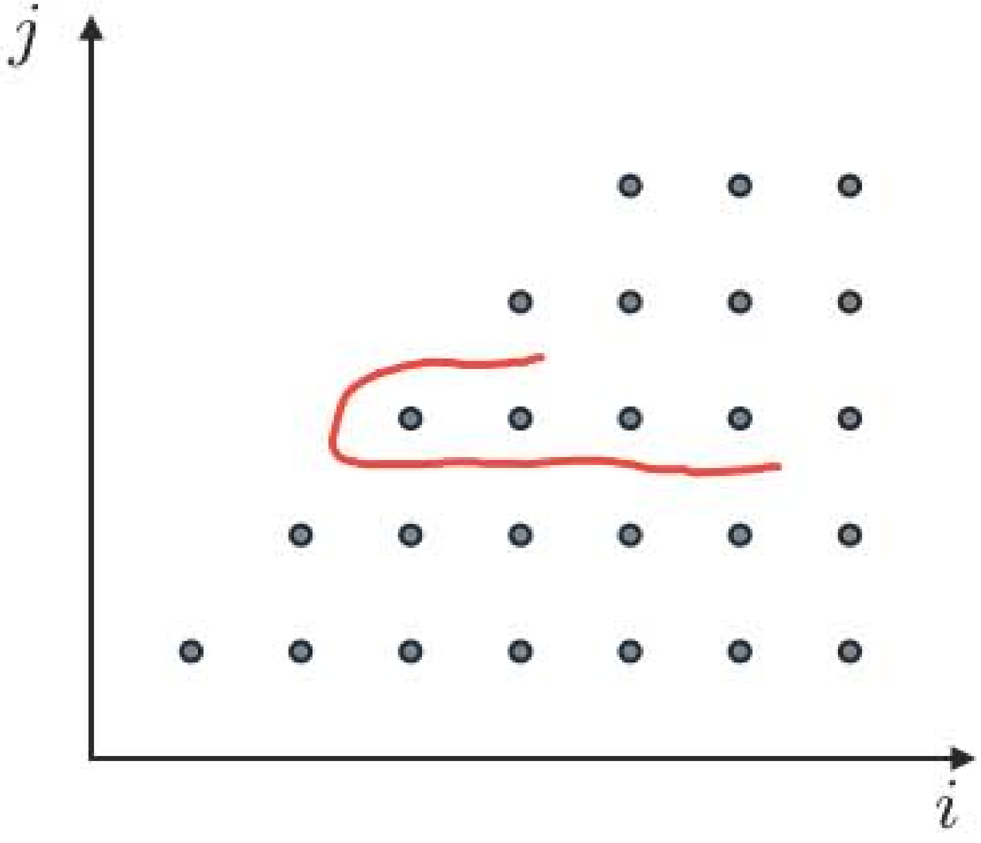
$1 + 0 + 0 \cdots = 1$ $\qquad$ $\sum 0 = 0$

## About the order of summation in series with multiple indices

if

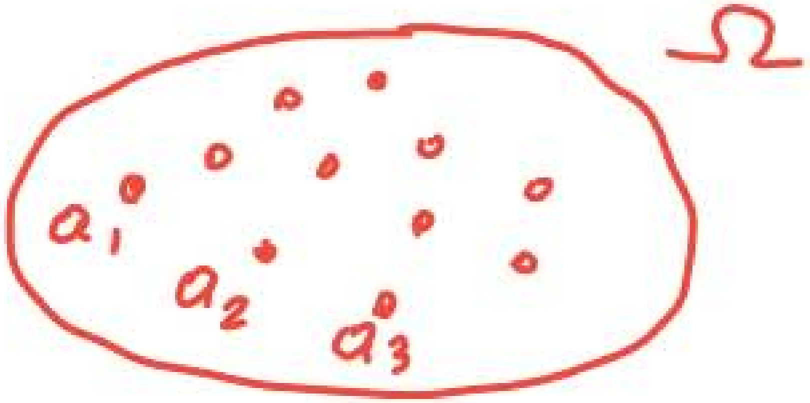
$$\sum_{(i,j):\, j\le i} |a_{ij}| < \infty$$




$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} a_{ij} \quad \dot{=} \quad \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} a_{ij}$$

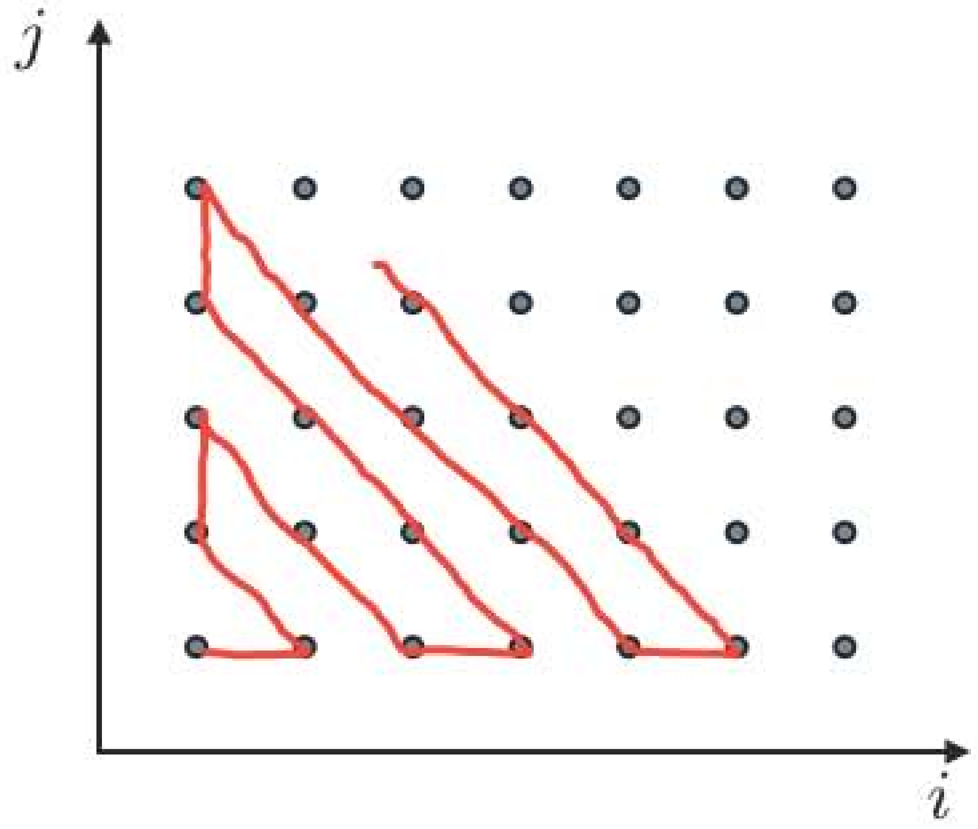
## Countable versus uncountable infinite sets




- Countable: can be put in 1-1 correspondence with positive integers
  - positive integers $1, 2, 3, \ldots$
  - integers $0, 1, -1, 2, -2, 3, -3, \ldots$

$\{a_1, a_2, a_3, \ldots\} = \Omega$

  - pairs of positive integers




  - rational numbers $q$, with $0 < q < 1$

$\underline{\underline{1/2}}, 1/3, 2/3, 1/4, \cancel{2/4}, 3/4, 1/5, 2/5 \ldots$

- Uncountable: not countable
  - the interval $[0, 1]$
  - the reals, the plane,...

## The reals are uncountable

- Cantor's diagonalization argument

$\rightarrow \{x \in (0,1) :$ decimal expansion only has $3,4\}$

If countable " $\underline{\{x_1, x_2, x_3, \ldots\}}$

$x_1$: $.343443\ldots$

$x_2$: $.4443443$

$x_3$: $.3343444$

$\underline{\underline{.433\ldots}} = x$

$\neq x_i$

for all $i$