CHAPTER III

LINEAR SYSTEM RESPONSE

3.1 OBJECTIVES

The output produced by an operational amplifier (or any other dynamic system) in response to a particular type or class of inputs normally provides the most important characterization of the system. The purpose of this chapter is to develop the analytic tools necessary to determine the response of a system to a specified input.

While it is always possible to determine the response of a linear system to a given input exactly, we shall frequently find that greater insight into the design process results when a system response is approximated by the known response of a simpler configuration. For example, when designing a low-level preamplifier intended for audio signals, we might be interested in keeping the frequency response of the amplifier within $\pm 5\%$ of its mid-band value over a particular bandwidth. If it is possible to approximate the amplifier as a two- or three-pole system, the necessary constraints on pole location are relatively straightforward. Similarly, if an oscilloscope vertical amplifier is to be designed, a required specification might be that the overshoot of the amplifier output in response to a step input be less than 3% of its final value. Again, simple constraints result if the system transfer function can be approximated by three or fewer poles.

The advantages of approximating the transfer functions of linear systems can only be appreciated with the aid of examples. The LM301A integrated-circuit operational amplifier[1] has 13 transistors included in its signal-transmission path. Since each transistor can be modeled as having two capacitors, the transfer function of the amplifier must include 26 poles. Even this estimate is optimistic, since there is distributed capacitance, comparable to transistor capacitances, associated with all of the other components in the signal path.

Fortunately, experimental measurements of performance can save us from the conclusion that this amplifier is analytically intractable. Figure 3.1*a* shows the LM301A connected as a unity-gain inverter. Figures 3.1*b* and 3.1*c* show the output of this amplifier with the input a −50-mV step

[1] This amplifier is described in Section 10.4.1.

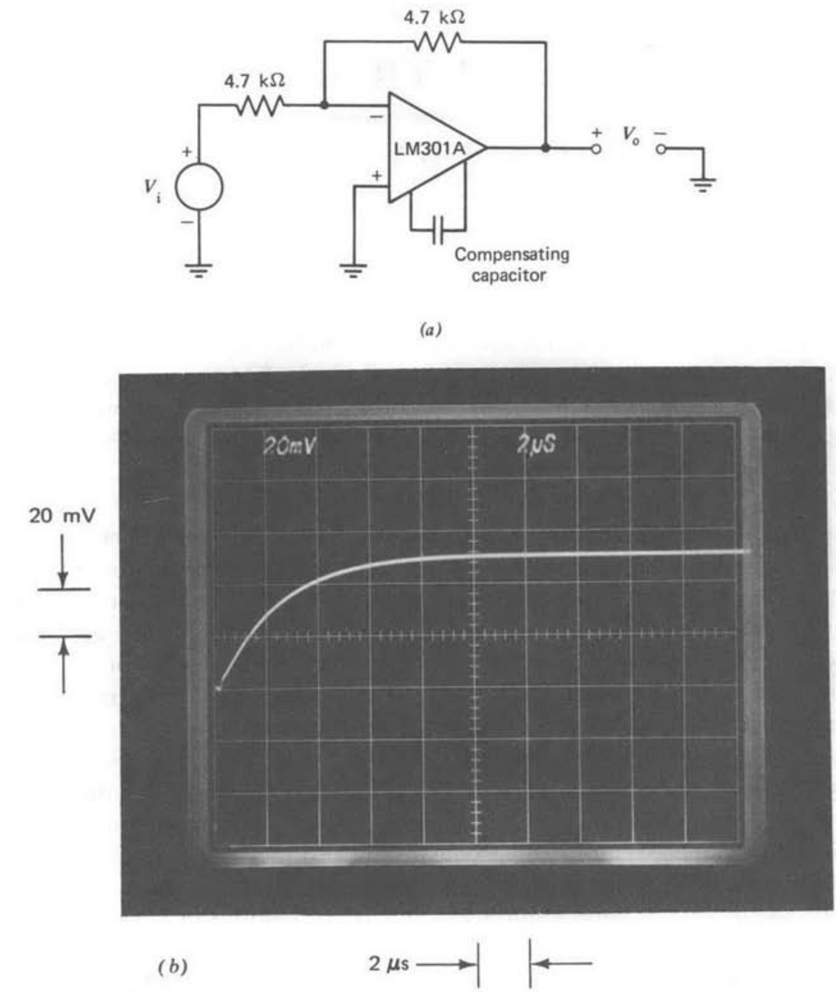
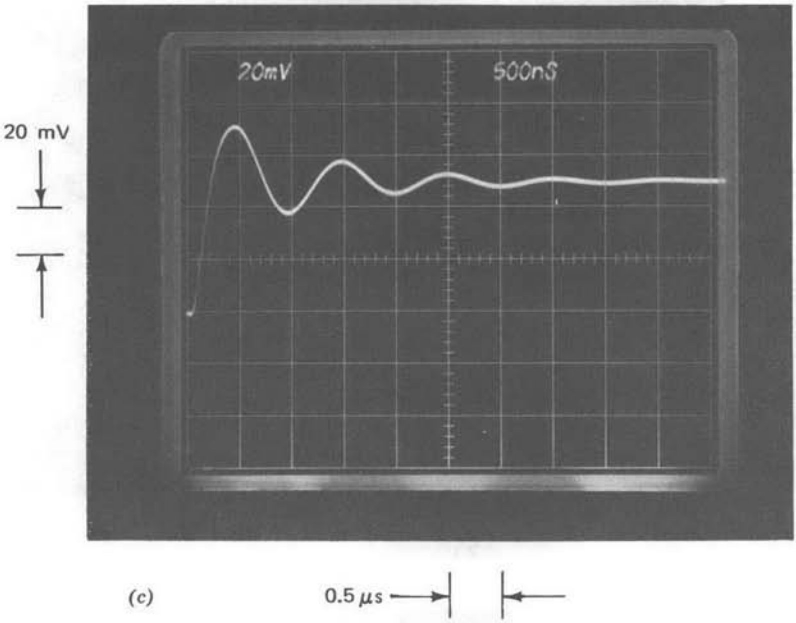


Figure 3.1 Step responses of inverting amplifier. (*a*) Connection. (*b*) Step response with 220-pF compensating capacitor. (*c*) Step response with 12-pF compensating capacitor.

for two different values of compensating capacitor.[2] The responses of an *R-C* network and an *R-L-C* network when excited with +50-mV steps supplied from the same generator used to obtain the previous transients are shown in Figs. 3.2*a* and 3.2*b*, respectively. The network transfer functions are

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{2.5 \times 10^{-6}s + 1} \tag{3.1}$$



(c)

Figure 3.1—Continued

for the response shown in Fig. 3.2*a* and

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{2.5 \times 10^{-14}s^2 + 7 \times 10^{-8}s + 1} \tag{3.2}$$

for that shown in Fig. 3.2*b*. We conclude that there are many applications where the first- and second-order transfer functions of Eqns. 3.1 and 3.2 adequately model the closed-loop transfer function of the LM301A when connected and compensated as shown in Fig. 3.1.

This same type of modeling process can also be used to approximate the open-loop transfer function of the operational amplifier itself. Assume that the input impedance of the LM301A is large compared to 4.7 kΩ and that its output impedance is small compared to this value at frequencies of interest. The closed-loop transfer function for the connection shown in Fig. 3.1 is then

$$\frac{V_o(s)}{V_i(s)} = \frac{-a(s)}{2 + a(s)} \tag{3.3}$$

[2] Compensation is a process by which the response of a system can be modified advantageously, and is described in detail in subsequent sections.

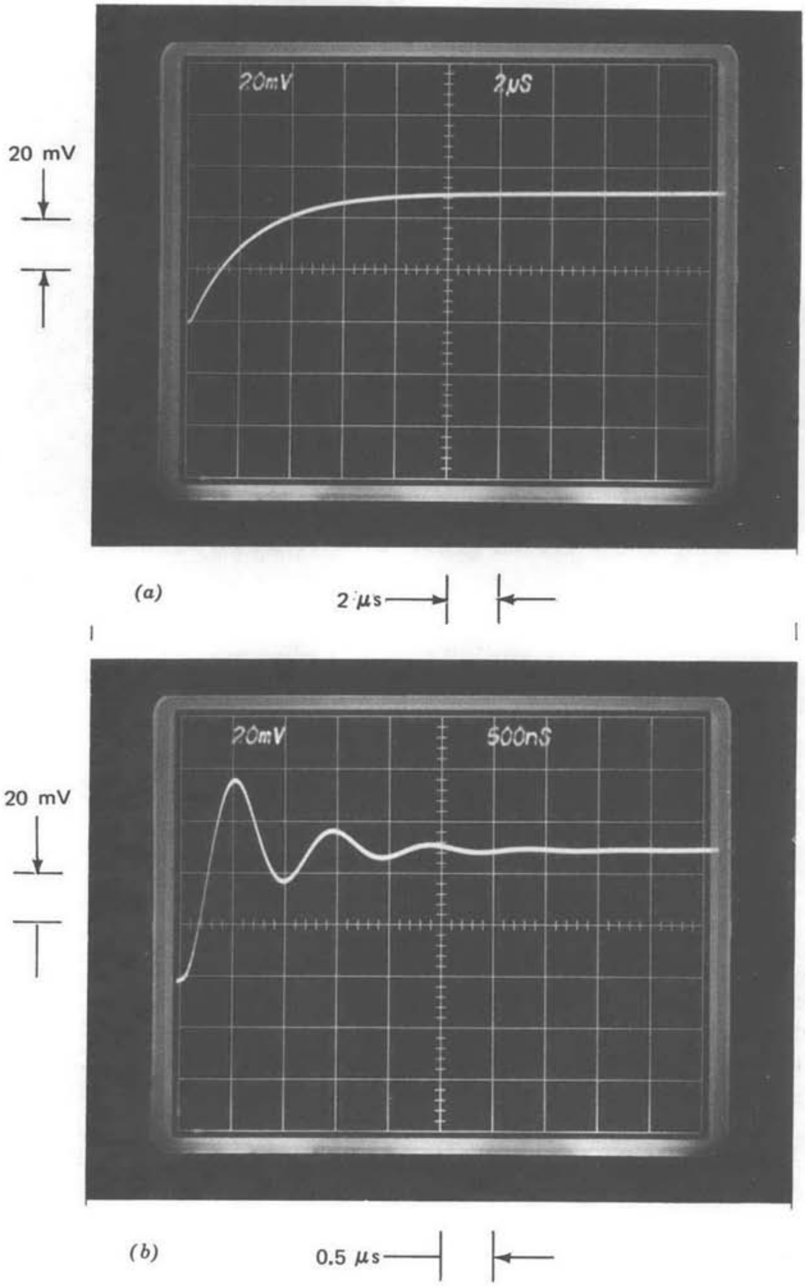


Figure 3.2 Step responses for first- and second-order networks. (*a*) Step response for $V_o(s)/V_i(s) = 1/(2.5 \times 10^{-6}s + 1)$. (*b*) Step response for $V_o(s)/V_i(s) = 1/(2.5 \times 10^{-14}s^2 + 7 \times 10^{-8}s + 1)$.

where $a(s)$ is the unloaded open-loop transfer function of the amplifier. Substituting approximate values for closed-loop gain (the negatives of Eqns. 3.1 and 3.2) into Eqn. 3.3 and solving for $a(s)$ yields

$$a(s) \simeq \frac{8 \times 10^5}{s} \tag{3.4}$$

and

$$a(s) \simeq \frac{2.8 \times 10^7}{s(3.5 \times 10^{-7}s + 1)} \tag{3.5}$$

as approximate open-loop gains for the amplifier when compensated with 220-pF and 12-pF capacitors, respectively. We shall see that these approximate values are quite accurate at frequencies where the magnitude of the loop transmission is near unity.

3.2 LAPLACE TRANSFORMS[3]

Laplace Transforms offer a method for solving any linear, time-invariant differential equation, and thus can be used to evaluate the response of a linear system to an arbitrary input. Since it is assumed that most readers have had some contact with this subject, and since we do not intend to use this method as our primary analytic tool, the exposure presented here is brief and directed mainly toward introducing notation and definitions that will be used later.

3.2.1 Definitions and Properties

The Laplace transform of a time function $f(t)$ is defined as

$$\mathcal{L}[f(t)] \triangleq F(s) \triangleq \int_0^{\infty} f(t)e^{-st}\,dt \tag{3.6}$$

where s is a complex variable $\sigma + j\omega$. The inverse Laplace transform of the complex function $F(s)$ is

$$\mathcal{L}^{-1}[F(s)] \triangleq f(t) \triangleq \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds \tag{3.7}$$

[3] A complete discussion is presented in M. F. Gardner and J. L. Barnes, *Transients in Linear Systems*, Wiley, New York, 1942.

In this section we temporarily suspend the variable and subscript notation used elsewhere and conform to tradition by using a lower-case variable to signify a time function and the corresponding capital for its transform.

The direct-inverse transform pair is unique[4] so that

$$\mathcal{L}^{-1}\mathcal{L}[f(t)] = f(t) \tag{3.8}$$

if $f(t) = 0$, $t < 0$, and if $\int_0^\infty |f(t)|\, e^{-\sigma_1 t}\, dt$ is finite for some real value of σ_1.

A number of theorems useful for the analysis of dynamic systems can be developed from the definitions of the direct and inverse transforms for functions that satisfy the conditions of Eqn. 3.8. The more important of these theorems include the following.

1. *Linearity*

$$\mathcal{L}[af(t) + bg(t)] = [aF(s) + bG(s)]$$

where a and b are constants.

2. *Differentiation*

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t\to 0^+} f(t)$$

(The limit is taken by approaching $t = 0$ from positive t.)

3. *Integration*

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

4. *Convolution*

$$\mathcal{L}\left[\int_0^t f(\tau)g(t - \tau)\, d\tau\right] = \mathcal{L}\left[\int_0^t f(t - \tau)g(\tau)\, d\tau\right] = F(s)G(s)$$

5. *Time shift*

$$\mathcal{L}[f(t - \tau)] = F(s)e^{-s\tau}$$

if $f(t - \tau) = 0$ for $(t - \tau) < 0$, where τ is a positive constant.

6. *Time scale*

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]$$

where a is a positive constant.

7. *Initial value*

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

[4] There are three additional constraints called the Direchlet conditions that are satisfied for all signals of physical origin. The interested reader is referred to Gardner and Barnes.

8. *Final value*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Theorem 4 is particularly valuable for the analysis of linear systems, since it shows that the Laplace transform of a system output is the product of the transform of the input signal and the transform of the impulse response of the system.

3.2.2 Transforms of Common Functions

The defining integrals can always be used to convert from a time function to its transform or vice versa. In practice, tabulated values are frequently used for convenience, and many mathematical or engineering references[5] contain extensive lists of time functions and corresponding Laplace transforms. A short list of Laplace transforms is presented in Table 3.1.

The time functions corresponding to ratios of polynomials in s that are not listed in the table can be evaluated by means of a *partial fraction* expansion. The function of interest is written in the form

$$F(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + s_1)(s + s_2) \cdots (s + s_n)} \tag{3.9}$$

It is assumed that the order of the numerator polynomial is less than that of the denominator. If all of the roots of the denominator polynomial are *first order* (i.e., $s_i \neq s_j$, $i \neq j$),

$$F(s) = \sum_{k=1}^{n} \frac{A_k}{s + s_k} \tag{3.10}$$

where

$$A_k = \lim_{s\to -s_k} [(s + s_k)F(s)] \tag{3.11}$$

If one or more roots of the denominator polynomial are *multiple roots*, they contribute terms of the form

$$\sum_{k=1}^{m} \frac{B_k}{(s + s_i)^k} \tag{3.12}$$

[5] See, for example, A. Erdeyli (Editor) *Tables of Integral Transforms*, Vol. 1, Bateman Manuscript Project, McGraw-Hill, New York, 1954 and R. E. Boly and G. L. Tuve, (Editors), *Handbook of Tables for Applied Engineering Science*, The Chemical Rubber Company, Cleveland, 1970.

Table 3.1 Laplace Transform Pairs

$F(s)$	$f(t),\ t \geq 0$ $[f(t) = 0,\ t < 0]$
1	Unit impulse $u_0(t)$
$\dfrac{1}{s}$	Unit step $u_{-1}(t)$ $[f(t) = 1,\ t \geq 0]$
$\dfrac{1}{s^2}$	Unit ramp $u_{-2}(t)$ $[f(t) = t,\ t \geq 0]$
$\dfrac{1}{s^{n+1}}$	$\dfrac{t^n}{n!}$
$\dfrac{1}{s+a}$	e^{-at}
$\dfrac{1}{(s+a)^{n+1}}$	$\dfrac{t^n}{(n)!}e^{-at}$
$\dfrac{1}{s(\tau s+1)}$	$1 - e^{-t/\tau}$
$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$
$\dfrac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$
$\dfrac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}(\sin \omega_n\sqrt{1-\zeta^2}\,t),\ \zeta < 1$
$\dfrac{1}{s(s^2/\omega_n^2 + 2\zeta s/\omega_n + 1)}$	$1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right)\right],$ $\zeta < 1$

where m is the order of the multiple root located at $s = -s_i$. The B's are determined from the relationship

$$B_k = \lim_{s \to -s_i}\left\{\frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}[(s+s_i)^m F(s)]\right\} \tag{3.13}$$

Because of the linearity property of Laplace transforms, it is possible to find the time function $f(t)$ by summing the contributions of all components of $F(s)$.

The properties of Laplace transforms listed earlier can often be used to determine the transform of time functions not listed in the table. The rectangular pulse shown in Fig. 3.3 provides one example of this technique. The pulse (Fig. 3.3*a*) can be decomposed into two steps, one with an amplitude of $+A$ starting at $t = t_1$, summed with a second step of ampli-

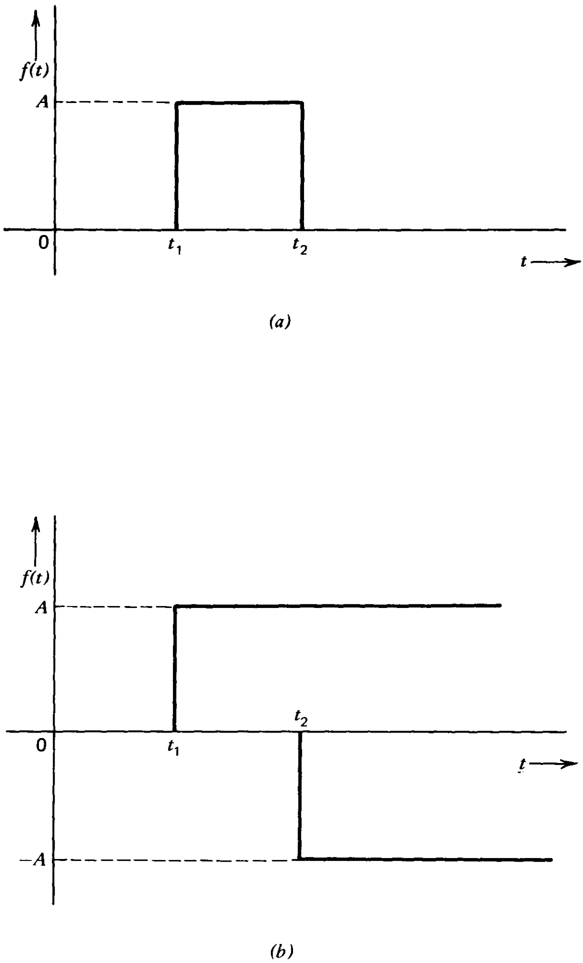
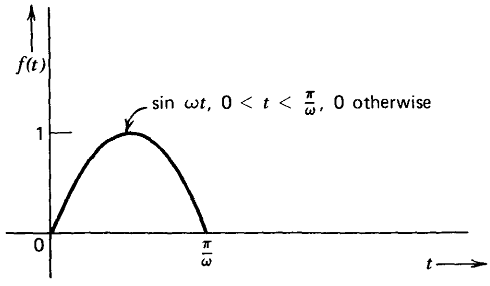


Figure 3.3 Rectangular pulse. (*a*) Signal. (*b*) Signal decomposed in two steps.



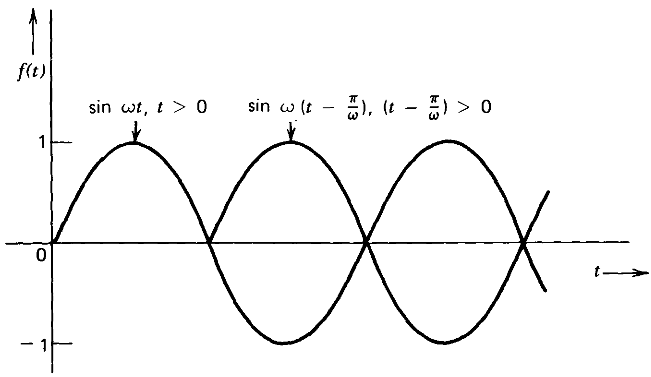


Figure 3.4 Sinusoidal pulse. (*a*) Signal. (*b*) Signal decomposed into two sinusoids. (*c*) First derivative of signal. (*d*) Second derivative of signal.

tude $-A$ starting at $t = t_2$. Theorems 1 and 5 combined with the transform of a unit step from Table 3.1 show that the transform of a step with amplitude A that starts at $t = t_1$ is $(A/s)e^{-st_1}$. Similarly, the transform of the second component is $-(A/s)e^{-st_2}$. Superposition insures that the transform of $f(t)$ is the sum of these two functions, or

$$F(s) = \frac{A}{s}\left(e^{-st_1} - e^{-st_2}\right) \tag{3.14}$$

The sinusoidal pulse shown in Fig. 3.4 is used as a second example. One approach is to represent the single pulse as the sum of two sinusoids

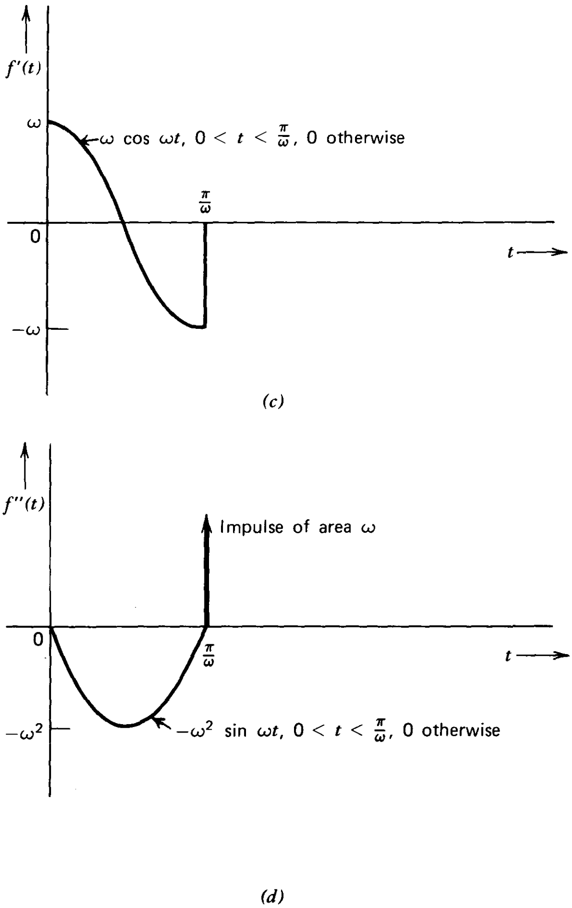


Figure 3.4—Continued

exactly as was done for the rectangular pulse. Table 3.1 shows that the transform of a unit-amplitude sinusoid starting at time $t = 0$ is $\omega/(s^2 + \omega^2)$. Summing transforms of the components shown in Fig. 3.4*b* yields

$$F(s) = \frac{\omega}{s^2 + \omega^2}\,[1 + e^{-s(\pi/\omega)}] \tag{3.15}$$

An alternative approach involves differentiating $f(t)$ twice. The derivative of $f(t)$, $f'(t)$, is shown in Fig. 3.4*c*. Since $f(0) = 0$, theorem 2 shows that

$$\mathcal{L}[f'(t)] = sF(s) \tag{3.16}$$

The second derivative of $f(t)$ is shown in Fig. 3.4*d*. Application of theorem 2 to this function[6] leads to

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - \lim_{t \to 0+} f'(t) = s^2F(s) - \omega \tag{3.17}$$

However, Fig. 3.4*d* indicates that

$$f''(t) = -\omega^2 f(t) + \omega u_0\left(t - \frac{\pi}{\omega}\right) \tag{3.18}$$

Thus

$$\mathcal{L}[f''(t)] = -\omega^2 F(s) + \omega e^{-s(\pi/\omega)} \tag{3.19}$$

Combining Eqns. 3.17 and 3.19 yields

$$s^2F(s) - \omega = -\omega^2 F(s) + \omega e^{-s(\pi/\omega)} \tag{3.20}$$

Equation 3.20 is solved for $F(s)$ with the result that

$$F(s) = \frac{\omega}{s^2 + \omega^2}[1 + e^{-s(\pi/\omega)}] \tag{3.21}$$

Note that this development, in contrast to the one involving superposition, does not rely on knowledge of the transform of a sinusoid, and can even be used to determine this transform.

3.2.3 Examples of the Use of Transforms

Laplace transforms offer a convenient method for the solution of linear, time-invariant differential equations, since they replace the integration and differentiation required to solve these equations in the time domain by algebraic manipulation. As an example, consider the differential equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t} \qquad t > 0 \tag{3.22}$$

subject to the initial conditions

$$x(0^+) = 2 \qquad \frac{dx}{dt}(0^+) = 0$$

The transform of both sides of Eqn. 3.22 is taken using theorem 2 (applied twice in the case of the second derivative) and Table 3.1 to determine the Laplace transform of e^{-t}.

$$s^2X(s) - sx(0^+) - \frac{dx}{dt}(0^+) + 3sX(s) - 3x(0^+) + 2X(s) = \frac{1}{s+1} \tag{3.23}$$

[6] The portion of this expression involving lim $t \to 0^+$ could be eliminated if a second impulse $\omega u_0(t)$ were included in $f''(t)$.

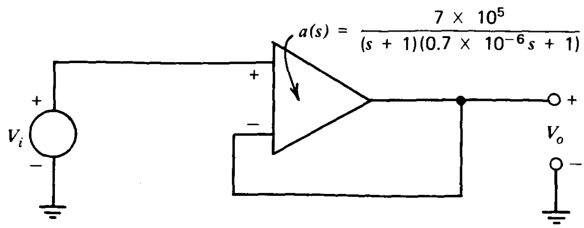


Figure 3.5 Unity-gain follower.

Collecting terms and solving for $X(s)$ yields

$$X(s) = \frac{2s^2 + 8s + 7}{(s+1)^2(s+2)} \tag{3.24}$$

Equations 3.10 and 3.12 show that since there is one first-order root and one second-order root,

$$X(s) = \frac{A_1}{(s+2)} + \frac{B_1}{(s+1)} + \frac{B_2}{(s+1)^2} \tag{3.25}$$

The coefficients are evaluated with the aid of Eqns. 3.11 and 3.13, with the result that

$$X(s) = \frac{-1}{s+2} + \frac{3}{s+1} + \frac{1}{(s+1)^2} \tag{3.26}$$

The inverse transform of $X(s)$, evaluated with the aid of Table 3.1, is

$$x(t) = -e^{-2t} + 3e^{-t} + te^{-t} \tag{3.27}$$

The operational amplifier connected as a unity-gain noninverting amplifier (Fig. 3.5) is used as a second example illustrating Laplace techniques. If we assume loading is negligible,

$$\frac{V_o(s)}{V_i(s)} = \frac{a(s)}{1 + a(s)} = \frac{7 \times 10^5}{(s+1)(0.7 \times 10^{-6}s + 1) + 7 \times 10^5}$$
$$\simeq \frac{1}{10^{-12}s^2 + 1.4 \times 10^{-6}s + 1} \tag{3.28}$$

If the input signal is a unit step so that $V_i(s)$ is $1/s$,

$$V_o(s) = \frac{1}{s(10^{-12}s^2 + 1.4 \times 10^{-6}s + 1)}$$
$$= \frac{1}{s[s^2/(10^6)^2 + 2(0.7)s/10^6 + 1]} \tag{3.29}$$

The final term in Eqn. 3.29 shows that the quadratic portion of the expression has a natural frequency $\omega_n = 10^6$ and a damping ratio $\zeta = 0.7$. The corresponding time function is determined from Table 3.1, with the result

$$f(t) = 1 - \frac{e^{-0.7 \times 10^6 t}}{0.7} \sin (0.7 \times 10^6 t + 45°) \tag{3.30}$$

3.3 TRANSIENT RESPONSE

The *transient response* of an element or system is its output as a function of time following the application of a specified input. The test signal chosen to excite the transient response of the system may be either an input that is anticipated in normal operation, or it may be a mathematical abstraction selected because of the insight it lends to system behavior. Commonly used test signals include the impulse and time integrals of this function.

3.3.1 Selection of Test Inputs

The mathematics of linear systems insures that the same system information is obtainable independent of the test input used, since the transfer function of a system is clearly independent of inputs applied to the system. In practice, however, we frequently find that certain aspects of system performance are most easily evaluated by selecting the test input to accentuate features of interest.

For example, we might attempt to evaluate the d-c gain of an operational amplifier with feedback by exciting it with an impulse and measuring the net area under the impulse response of the amplifier. This approach is mathematically sound, as shown by the following development. Assume that the closed-loop transfer function of the amplifier is $G(s)$ and that the corresponding impulse response [the inverse transform of $G(s)$] is $g(t)$. The properties of Laplace transforms show that

$$\int_0^t g(t)\, dt = \frac{1}{s} G(s) \tag{3.31}$$

The final value theorem applied to this function indicates that the net area under impulse response is

$$\lim_{t \to \infty} \int_0^t g(t)\, dt = \lim_{s \to 0} s \frac{1}{s} G(s) = G(0) \tag{3.32}$$

Unfortunately, this technique involves experimental pitfalls. The first of these is the choice of the time function used to approximate an impulse.

In order for a finite-duration pulse to approximate an impulse satisfactorily, it is necessary to have[7]

$$t_p \ll \frac{1}{|s_m|} \tag{3.33}$$

where t_p is the width of the pulse and s_m is the frequency of the pole of $G(s)$ that is located furthest from the origin.

It may be difficult to find a pulse generator that produces pulses narrow enough to test high-frequency amplifiers. Furthermore, the narrow pulse frequently leads to a small-amplitude output with attendant measurement problems. Even if a satisfactory impulse response is obtained, the tedious task of integrating this response (possibly by counting boxes under the output display on an oscilloscope) remains. It should be evident that a far more accurate and direct measurement of d-c gain is possible if a constant input is applied to the amplifier.

Alternatively, high-frequency components of the system response are not excited significantly if slowly time-varying inputs are applied as test inputs. In fact, systems may have high-frequency poles close to the imaginary axis in the s-plane, and thus border on instability; yet they exhibit well-behaved outputs when tested with slowly-varying inputs.

For systems that have neither a zero-frequency pole nor a zero in their transfer function, the step response often provides the most meaningful evaluation of performance. The d-c gain can be obtained directly by measuring the final value of the response to a unit step, while the initial discontinuity characteristic of a step excites high-frequency poles in the system transfer function. Adequate approximations to an ideal step are provided by rectangular pulses with risetimes

$$t_r \ll \frac{1}{|s_m|} \tag{3.34}$$

(s_m as defined earlier) and widths

$$t_w \gg \frac{1}{|s_n|} \tag{3.35}$$

where s_n is the frequency of the pole in the transfer function located closest to the origin. Pulse generators with risetimes under 1 ns are available, and these generators can provide useful information about amplifiers with bandwidths on the order of 100 MHz.

[7] While this statement is true in general, if only the d-c gain of the system is required, any pulse can be used. An extension of the above development shows that the area under the response to any unit-area input is identical to the area under the impulse response.

3.3.2 Approximating Transient Responses

Examples in Section 3.1 indicated that in some cases it is possible to approximate the transient response of a complex system by using that of a much simpler system. This type of approximation is possible whenever the transfer function of interest is dominated by one or two poles.

Consider an amplifier with a transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{a_0 \prod_{i=1}^{m} (\tau_{zi}s + 1)}{\prod_{j=1}^{n} (\tau_{pj}s + 1)} \qquad n > m, \quad \text{all } \tau > 0 \tag{3.36}$$

The response of this system to a unit-step input is

$$v_o(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\frac{V_o(s)}{V_i(s)}\right] = a_o + \sum_{k=1}^{n} A_k e^{-t/\tau_{pk}} \tag{3.37}$$

The A's obtained from Eqn. 3.11 after slight rearrangement are

$$A_k = -a_0 \frac{\prod_{i=1}^{m}\left(-\frac{\tau_{zi}}{\tau_{pk}} + 1\right)}{\prod_{\substack{j=1\\ j\neq k}}^{n}\left(-\frac{\tau_{pj}}{\tau_{pk}} + 1\right)} \tag{3.38}$$

Assume that $\tau_{p1} \gg$ all other τ's. In this case, which corresponds to one pole in the system transfer function being much closer to the origin than all other singularities, Eqn. 3.38 can be used to show that $A_1 \simeq a_0$ and all other A's $\simeq 0$ so that

$$v_o(t) \simeq a_0(1 - e^{-t/\tau_{p1}}) \tag{3.39}$$

This single-exponential transient response is shown in Fig. 3.6. Experience shows that the single-pole response is a good approximation to the actual response if remote singularities are a factor of five further from the origin than the dominant pole.

The approximate result given above holds even if some of the remote singularities occur in complex conjugate pairs, providing that the pairs are located at much greater distances from the origin in the s plane than the dominant pole. However, if the real part of the complex pair is not more negative than the location of the dominant pole, small-amplitude, high-frequency damped sinusoids may persist after the dominant transient is completed.

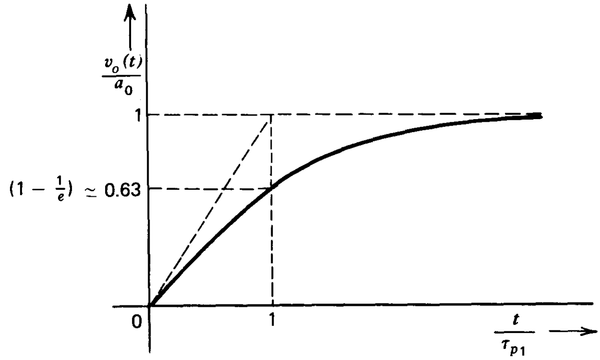


Figure 3.6 Step response of first-order system.

Another common singularity pattern includes a complex pair of poles much closer to the origin in the s plane than all other poles and zeros. An argument similar to that given above shows that the transfer function of an amplifier with this type of singularity pattern can be approximated by the complex pair alone, and can be written in the standard form

$$\frac{V_o(s)}{V_i(s)} = \frac{a_o}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1} \tag{3.40}$$

The equation parameters ω_n and ζ are called the *natural frequency* (expressed in radians per second) and the *damping ratio*, respectively. The physical significance of these parameters is indicated in the s-plane plot shown as Fig. 3.7. The relative pole locations shown in this diagram correspond to the *underdamped* case ($\zeta < 1$). Two other possibilities are the *critically damped* pair ($\zeta = 1$) where the two poles coincide on the real axis and the *overdamped* case ($\zeta > 1$) where the two poles are separated on the real axis. The denominator polynomial can be factored into two roots with real coefficients for the later two cases and, as a result, the form shown in Eqn. 3.40 is normally not used. The output provided by the amplifier described by Eqn. 3.40 in response to a unit step is (from Table 3.1).

$$v_o(t) = a_0\left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1 - \zeta^2}\, \omega_n t + \Phi\right)\right] \tag{3.41}$$

where

$$\Phi = \tan^{-1}\left[\frac{\sqrt{1 - \zeta^2}}{\zeta}\right]$$

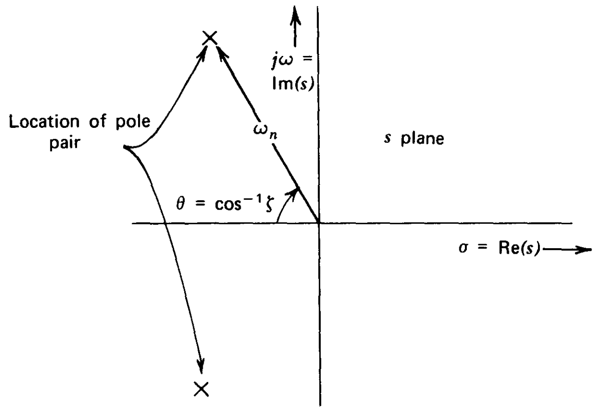


Figure 3.7 s-plane plot of complex pole pair.

Figure 3.8 is a plot of $v_o(t)$ as a function of normalized time $\omega_n t$ for various values of damping ratio. Smaller damping ratios, corresponding to complex pole pairs with the poles nearer the imaginary axis, are associated with step responses having a greater degree of overshoot.

The transient responses of third- and higher-order systems are not as easily categorized as those of first- and second-order systems since more parameters are required to differentiate among the various possibilities. The situation is simplified if the relative pole positions fall into certain patterns. One class of transfer functions of interest are the Butterworth filters. These transfer functions are also called *maximally flat* because of properties of their frequency responses (see Section 3.4). The step responses of Butterworth filters also exhibit fairly low overshoot, and because of these properties feedback amplifiers are at times compensated so that their closed-loop poles form a Butterworth configuration.

The poles of an nth-order Butterworth filter are located on a circle centered at the origin of the s-plane. For n even, the poles make angles $\pm (2k + 1)\, 90°/n$ with the negative real axis, where k takes all possible integral values from 0 to $(n/2) - 1$. For n odd, one pole is located on the negative real axis, while others make angles of $\pm k\,(180°/n)$ with the negative real axis where k takes integral values from 1 to $(n/2) - (1/2)$. Thus, for example, a first-order Butterworth filter has a single pole located at $s = -\omega_n$. The second-order Butterworth filter has its poles located $\pm 45°$ from the negative real axis, corresponding to a damping ratio of 0.707.

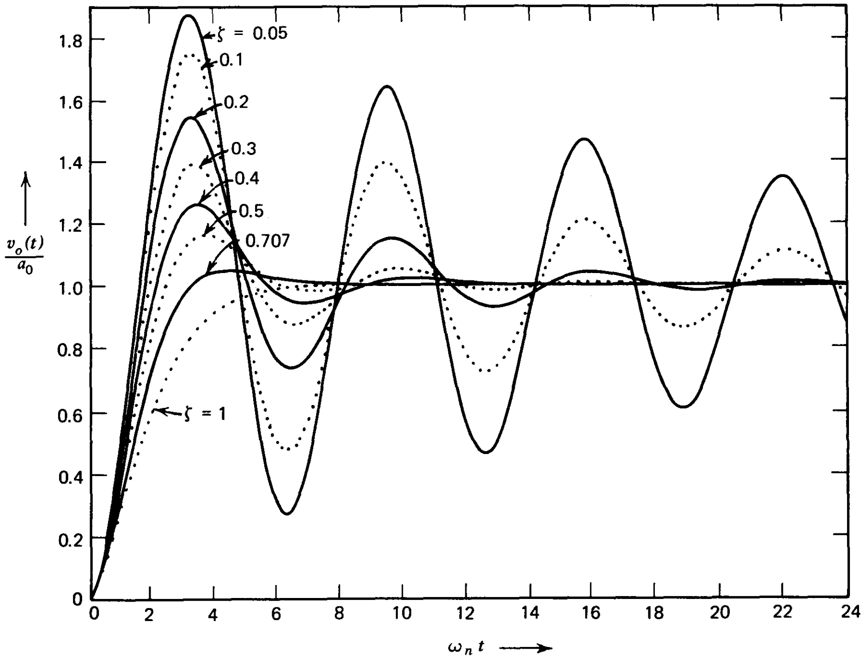


Figure 3.8 Step responses of second-order system.

The transfer functions for third- and fourth-order Butterworth filters are

$$B_3(s) = \frac{1}{s^3/\omega_n^3 + 2s^2/\omega_n^2 + 2s/\omega_n + 1} \tag{3.42}$$

and

$$B_4(s) = \frac{1}{s^4/\omega_n^4 + 2.61s^3/\omega_n^3 + 3.42s^2/\omega_n^2 + 2.61s/\omega_n + 1} \tag{3.43}$$

respectively. Plots of the pole locations of these functions are shown in Fig. 3.9. The transient outputs of these filters in response to unit steps are shown in Fig. 3.10.

3.4 FREQUENCY RESPONSE

The frequency response of an element or system is a measure of its steady-state performance under conditions of sinusoidal excitation. In

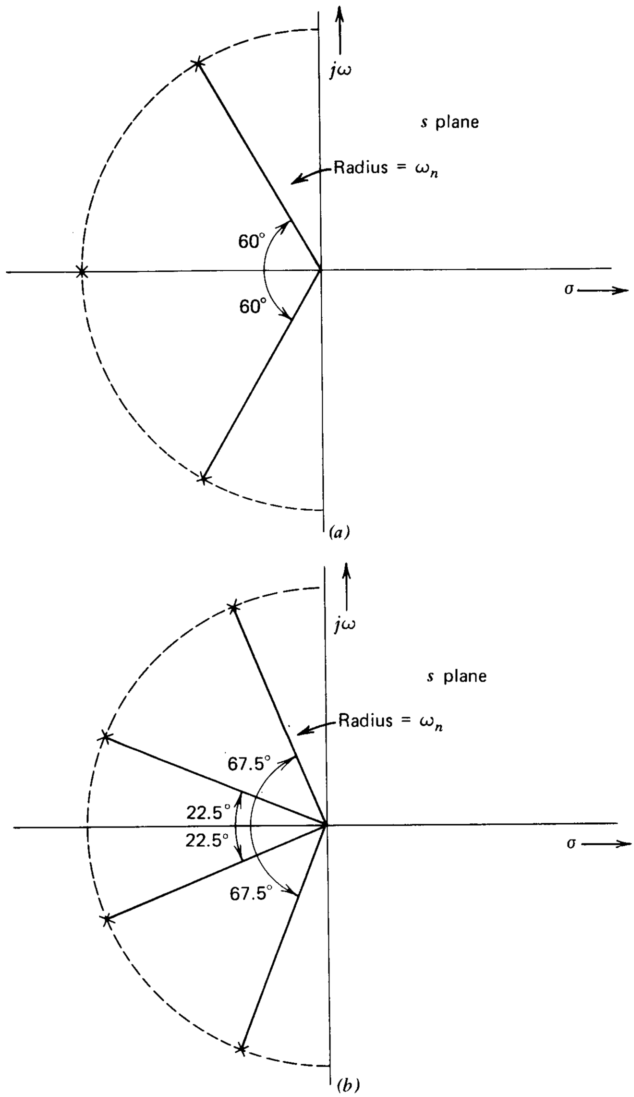


Figure 3.9 Pole locations for third- and fourth-order Butterworth filters. (*a*) Third-Order. (*b*) Fourth-order.

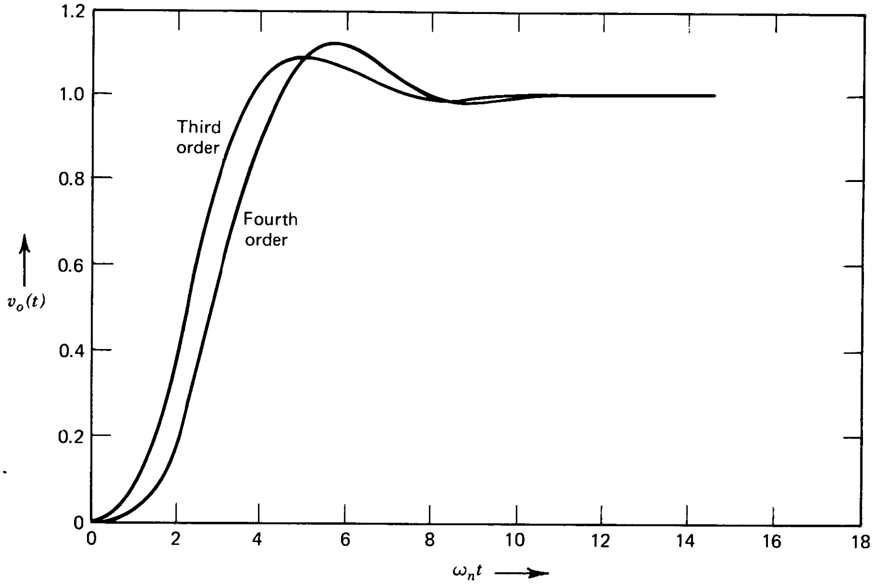


Figure 3.10 Step responses for third- and fourth-order Butterworth filters.

steady state, the output of a linear element excited with a sinusoid at a frequency ω (expressed in radians per second) is purely sinusoidal at frequency ω. The frequency response is expressed as a gain or magnitude $M(\omega)$ that is the ratio of the amplitude of the output to the input sinusoid and a phase angle $\phi(\omega)$ that is the relative angle between the output and input sinusoids. The phase angle is positive if the output leads the input. The two components that comprise the frequency response of a system with a transfer function $G(s)$ are given by

$$M(\omega) = |G(j\omega)| \tag{3.44a}$$

$$\phi(\omega) = \measuredangle G(j\omega) = \tan^{-1} \frac{\mathrm{Im}[G(j\omega)]}{\mathrm{Re}[G(j\omega)]} \tag{3.44b}$$

It is frequently necessary to determine the frequency response of a system with a transfer function that is a ratio of polynomials in s. One possible method is to evaluate the frequency response by substituting $j\omega$ for s at all frequencies of interest, but this method is cumbersome, particularly for high-order polynomials. An alternative approach is to present the information concerning the frequency response graphically, as described below.

The transfer function is first factored so that both the numerator and denominator consist of products of first- and second-order terms with real coefficients. The function can then be written in the general form

$$G(s) = \frac{a_0}{s^n}\left[\prod_{\substack{\text{first-}\\\text{order}\\\text{zeros}}} (\tau_h s + 1)\right]\left[\prod_{\substack{\text{complex}\\\text{zero}\\\text{pairs}}} \left(\frac{s^2}{\omega_{ni}^2} + \frac{2\zeta_i s}{\omega_{ni}} + 1\right)\right]$$
$$\times\left[\prod_{\substack{\text{first-}\\\text{order}\\\text{poles}}} \frac{1}{(\tau_j s + 1)}\right]\left[\prod_{\substack{\text{complex}\\\text{pole}\\\text{pairs}}} \frac{1}{(s^2/\omega_{nk}^2 + 2\zeta_k s/\omega_{nk} + 1)}\right] \tag{3.45}$$

While several methods such as Lin's method[8] are available for factoring polynomials, this operation can be tedious unless machine computation is employed, particularly when the order of the polynomial is large. Fortunately, in many cases of interest the polynomials are either of low order or are available from the system equations in factored form.

Since $G(j\omega)$ is a function of a complex variable, its angle $\phi(\omega)$ is the sum of the angles of the constituent terms. Similarly, its magnitude $M(\omega)$ is the product of the magnitudes of the components. Furthermore, if the magnitudes of the components are plotted on a logarithmic scale, the log of M is given by the sum of the logs corresponding to the individual components.[9]

Plotting is simplified by recognizing that only four types of terms are possible in the representation of Eqn. 3.45:

1. Constants, a_0.
2. Single- or multiple-order differentiations or integrations, s^n, where n can be positive (differentiations) or negative (integrations).
3. First-order terms $(\tau s + 1)$, or its reciprocal.
4. Complex conjugate pairs $s^2/\omega_n^2 + 2\zeta s/\omega_n + 1$, or its reciprocal.

[8] S. N. Lin, "A Method of Successive Approximations of Evaluating the Real and Complex Roots of Cubic and Higher-Order Equations," *J. Math. Phys.*, Vol. 20, No. 3, August, 1941, pp. 231–242.

[9] The decibel, equal to 20 $\log_{10}$ [magnitude] is often used for these manipulations. This usage is technically correct only if voltage gains or current gains between portions of a circuit with identical impedance levels are considered. The issue is further confused when the decibel is used indiscriminately to express dimensioned quantities such as transconductances. We shall normally reserve this type of presentation for loop-transmission manipulations (the loop transmission of any feedback system must be dimensionless), and simply plot signal ratios on logarithmic coordinates.

It is particularly convenient to represent each of these possible terms as a plot of M (on a logarithmic magnitude scale) and ϕ (expressed in degrees) as a function of ω (expressed in radians per second) plotted on a logarithmic frequency axis. A logarithmic frequency axis is used because it provides adequate resolution in cases where the frequency range of interest is wide and because the relative shape of a particular response curve on the log axis does not change as it is frequency scaled. The magnitude and angle of any rational function can then be determined by adding the magnitudes and angles of its components. This representation of the frequency response of a system or element is called a *Bode plot.*

The magnitude of a term a_0 is simply a frequency-independent constant, with an angle equal to 0° or 180° depending on whether the sign of a_0 is positive or negative, respectively.

Both differentiations and integrations are possible in feedback systems. For example, a first-order high-pass filter has a single zero at the origin and, thus, its voltage transfer ratio includes a factor s. A motor (frequently used in mechanical feedback systems) includes a factor $1/s$ in the transfer function that relates mechanical shaft angle to applied motor voltage, since a constant input voltage causes unlimited shaft rotation. Similarly, various types of phase detectors are examples of purely electronic elements that have a pole at the origin in their transfer functions. This pole results because the voltage out of such a circuit is proportional to the phase-angle difference between two input signals, and this angle is equal to the integral of the frequency difference between the two signals. We shall also see that it is often convenient to approximate the transfer function of an amplifier with high d-c gain and a single low-frequency pole as an integration.

The magnitude of a term s^n is equal to ω^n, a function that passes through 1 at $\omega = 1$ and has a slope of n on logarithmic coordinates. The angle of this function is $n \times 90°$ at all frequencies.

The magnitude of a first order pole $1/(\tau s + 1)$ is

$$M = \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \tag{3.46}$$

while the angle of this function is

$$\phi = -\tan^{-1}\tau\omega \tag{3.47}$$

The magnitude and angle for the first-order pole are plotted as a function of normalized frequency in Fig. 3.11. An essential feature of the magnitude function is that it can be approximated by two straight lines, one lying along the $M = 1$ line and the other with a slope of -1, which intersect at $\omega = 1/\tau$. (This frequency is called the *corner frequency.*) The maximum

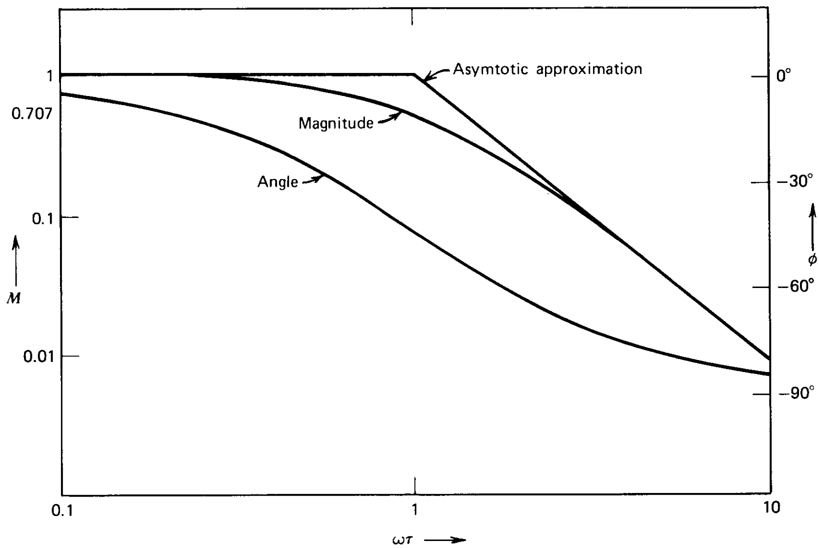


Figure 3.11 Frequency response of first-order system.

departure of the actual curves from the asymptotic representation is a factor of 0.707 and occurs at the corner frequency. The magnitude and angle for a first-order zero are obtained by inverting the curves shown for the pole, so that the magnitude approaches an asymptotic slope of +1 beyond the corner frequency, while the angle changes from 0 to +90°.

The magnitude for a complex-conjugate pole pair

$$\frac{1}{s^2/\omega_n^2 + 2\zeta s/\omega_n + 1}$$

is

$$M = \frac{1}{\sqrt{\dfrac{4\zeta^2\omega^2}{\omega_n^2} + \left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2}} \tag{3.48}$$

with the corresponding angle

$$\phi = -\tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \tag{3.49}$$

These functions are shown in Bode-plot form as a parametric family of curves plotted against normalized frequency ω/ω_n in Fig. 3.12. Note that

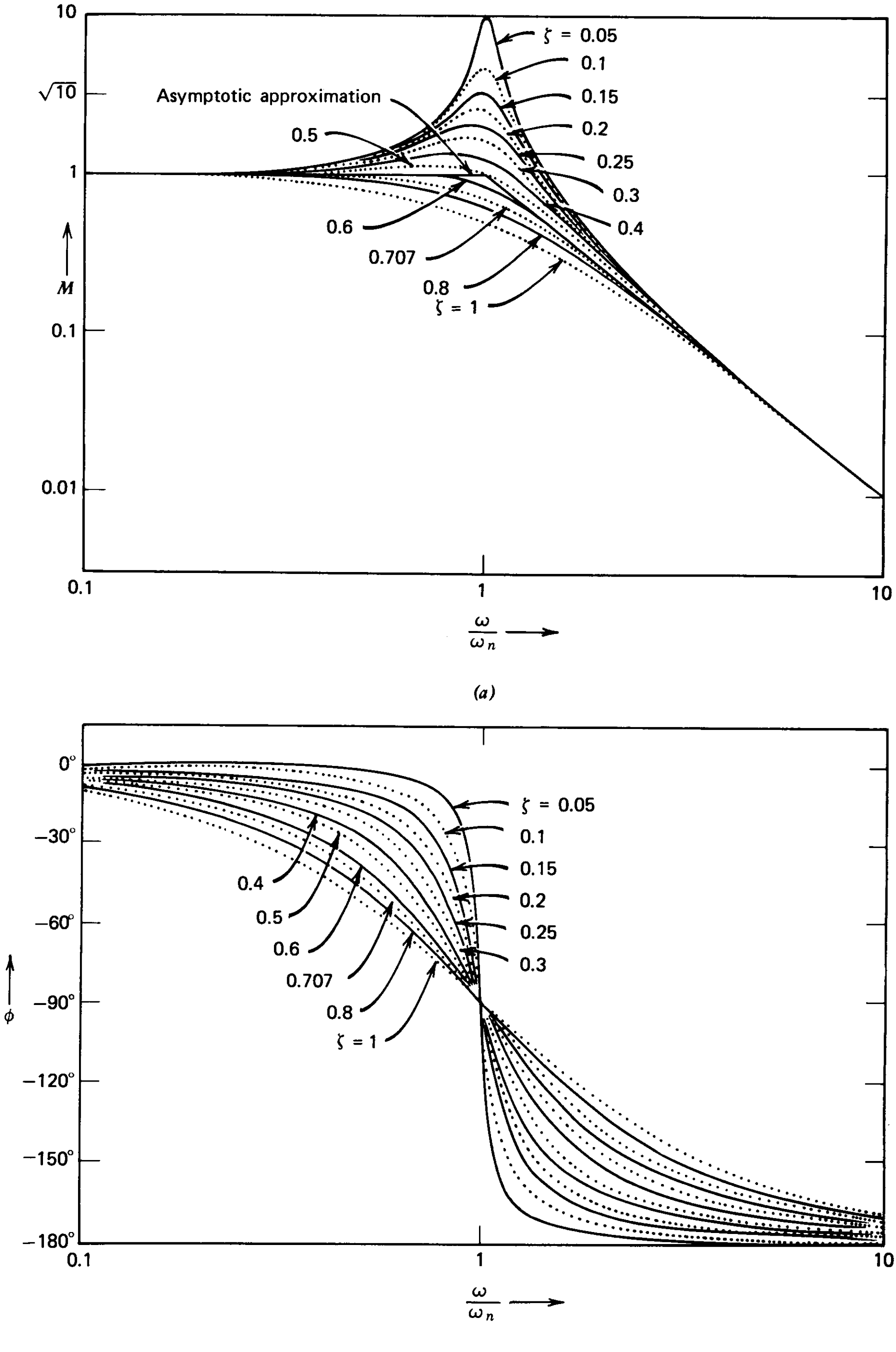


Figure 3.12 Frequency response of second-order system. (*a*) Magnitude. (*b*) Angle.

the asymptotic approximation to the magnitude is reasonably accurate providing that the damping ratio exceeds 0.25. The corresponding curves for a complex-conjugate zero are obtained by inverting the curves shown in Fig. 3.12.

It was stated in Section 3.3.2 that feedback amplifiers are occasionally adjusted to have Butterworth responses. The frequency responses for third- and fourth-order Butterworth filters are shown in Bode-plot form in Fig. 3.13. Note that there is no peaking in the frequency response of these maximally-flat transfer functions. We also see from Fig. 3.12 that the damping ratio of 0.707, corresponding to the two-pole Butterworth configuration, divides the second-order responses that peak from those which do not. The reader should recall that the flatness of the Butterworth response refers to its *frequency response*, and that the step responses of all Butterworth filters exhibit overshoot.

The value associated with Bode plots stems in large part from the ease with which the plot for a complex system can be obtained. The overall system transfer function can be obtained by the following procedure. First, the magnitude and phase curves corresponding to all the terms included in the transfer function of interest are plotted. When the first- and second-order curves (Figs. 3.11 and 3.12) are used, they are located along the frequency axis so that their corner frequencies correspond to those of the represented factors. Once these curves have been plotted, the magnitude of the complete transfer function at any frequency is obtained by adding the linear distances from unity magnitude of all components at the frequency of interest. The same type of graphical addition can be used to obtain the complete phase curve. Dividers, or similar aids, can be used to perform the graphical addition.

In practice, the asymptotic magnitude curve is usually sketched by drawing a series of intersecting straight lines with appropriate slope changes at intersections. Corrections to the asymptotic curve can be added in the vicinity of singularities if necessary.

The information contained in a Bode plot can also be presented as a *gain-phase* plot, which is a more convenient representation for some operations. Rectangular coordinates are used, with the ordinate representing the magnitude (on a logarithmic scale) and the abscissa representing the phase angle in degrees. Frequency expressed in radians per second is a parameter along the gain-phase curve. Gain-phase plots are frequently drawn by transferring data from a Bode plot.

The transfer function

$$G(s) = \frac{10^7(10^{-4}s + 1)}{s(0.01s + 1)(s^2/10^{12} + 2(0.2)s/10^6 + 1)} \tag{3.50}$$

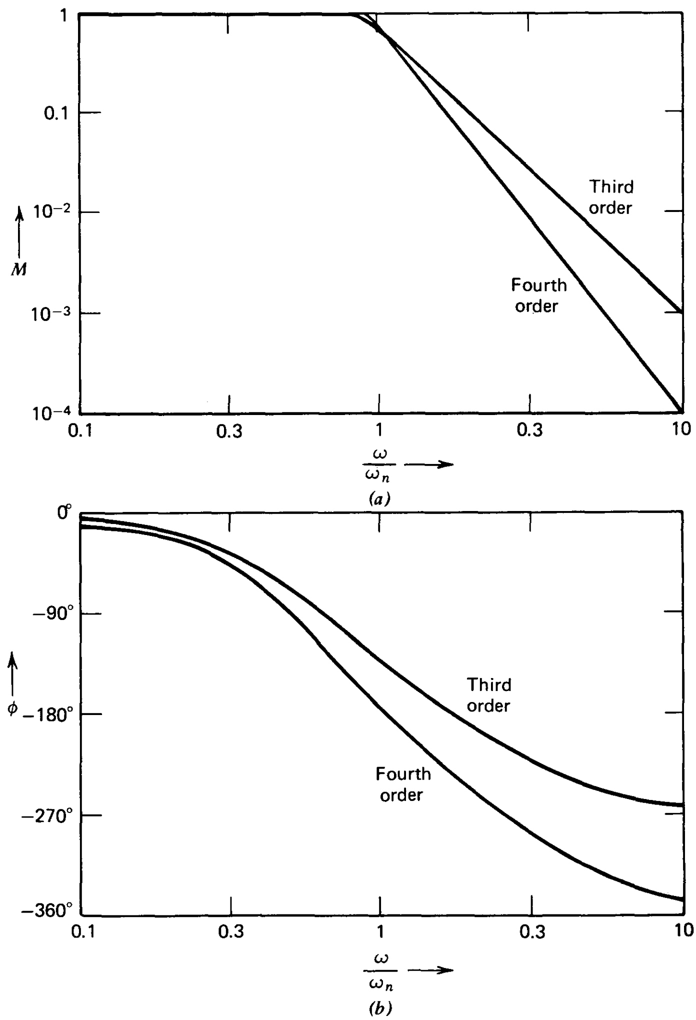


Figure 3.13 Frequency response of third- and fourth-order Butterworth filters. (*a*) Magnitude. (*b*) Angle.

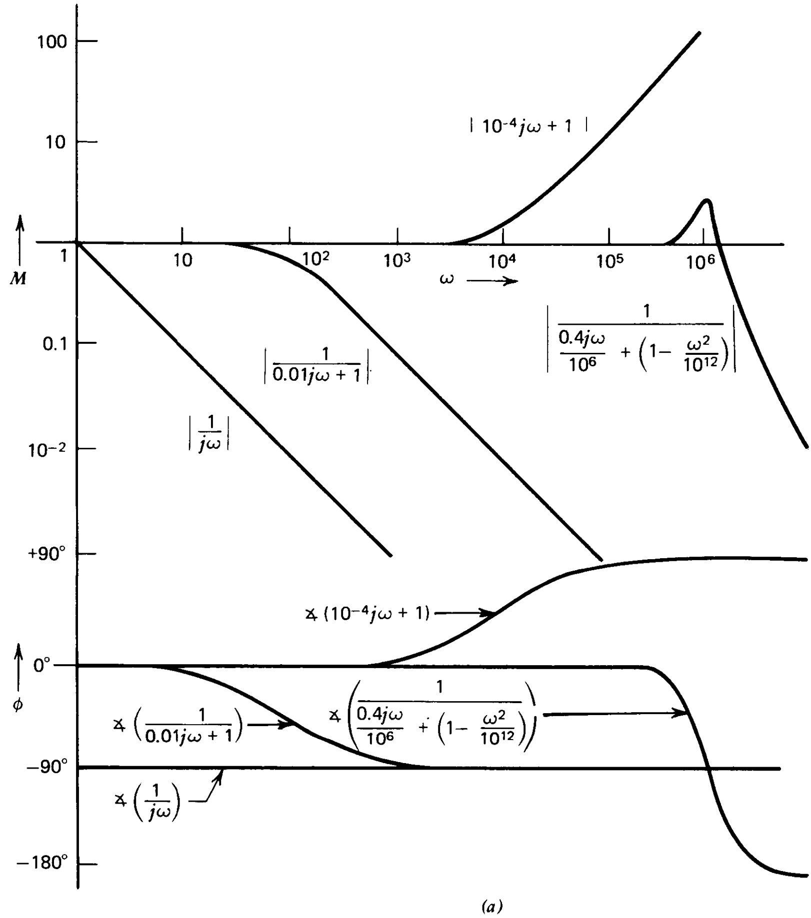
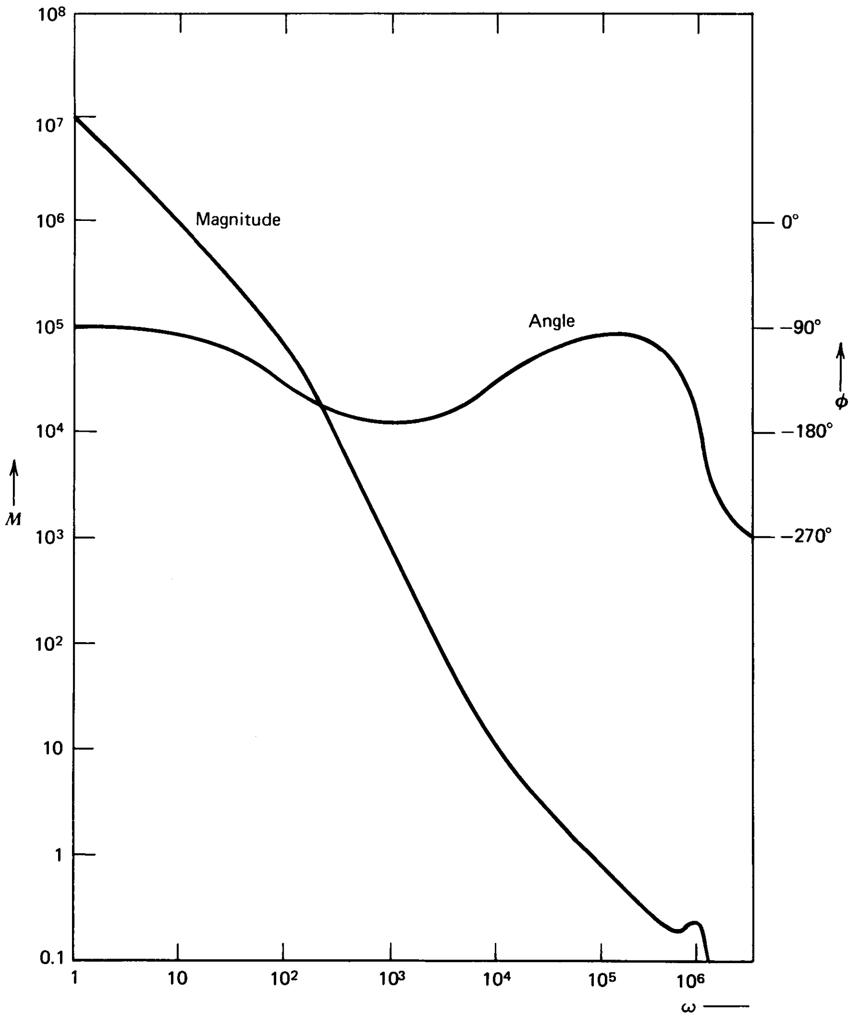


Figure 3.14 Bode plot of $\dfrac{10^7(10^{-4}s + 1)}{s(0.01s + 1)(s^2/10^{12} + 2(0.2)s/10^6 + 1)}$. (*a*) Individual factors. (*b*) Bode plot.

is used to illustrate construction of Bode and gain-phase plots. This function includes these five factors:

1. A constant 10^7.
2. A single integration.
3. A first-order pole with a time constant of 0.01 second, corresponding to a corner frequency of 100 radians per second.



(b)

Figure 3.14—Continued

4. A first-order zero with a time constant of 10^{-4} seconds, corresponding to a corner frequency of 10^4 radians per second.
5. A complex-conjugate pole pair with a natural frequency of 10^6 radians per second and a damping ratio of 0.2.

The individual factors are shown in Bode-plot form on a common frequency scale in Fig. 3.14*a*. These factors are combined to yield the Bode plot for the complete transfer function in Fig. 3.14*b*. The same information is presented in gain-phase form in Fig. 3.15.

3.5 RELATIONSHIPS BETWEEN TRANSIENT RESPONSE AND FREQUENCY RESPONSE

It is clear that either the impulse response (or the response to any other transient input) of a linear system or its frequency response completely characterize the system. In many cases experimental measurements on a closed-loop system are most easily made by applying a transient input. We may, however, be interested in certain aspects of the frequency response of the system such as its *bandwidth* defined as the frequency where its gain drops to 0.707 of the midfrequency value.

Since either the transient response or the frequency response completely characterize the system, it should be possible to determine performance in one domain from measurements made in the other. Unfortunately, since the measured transient response does not provide an equation for this response, Laplace techniques cannot be used directly unless the time response is first approximated analytically as a function of time. This section lists several approximate relationships between transient response and frequency response that can be used to estimate one performance measure from the other. The approximations are based on the properties of first- and second-order systems.

It is assumed that the feedback path for the system under study is frequency independent and has a magnitude of unity. A system with a frequency-independent feedback path f_0 can be manipulated as shown in Fig. 3.16 to yield a scaled, unity-feedback system. The approximations given are valid for the transfer function V_a/V_i, and V_o can be determined by scaling values for V_a by $1/f_0$.

It is also assumed that the magnitude of the d-c loop transmission is very large so that the closed-loop gain is nearly one at d-c. It is further assumed that the singularity closest to the origin in the s plane is either a pole or a complex pair of poles, and that the number of poles of the function exceeds the number of zeros. If these assumptions are satisfied, many practical systems have time domain-frequency domain relationships similar to those of first- or second-order systems.

The parameters we shall use to describe the transient response and the frequency response of a system include the following.

(a) Rise time t_r. The time required for the step response to go from 10 to 90% of final value.

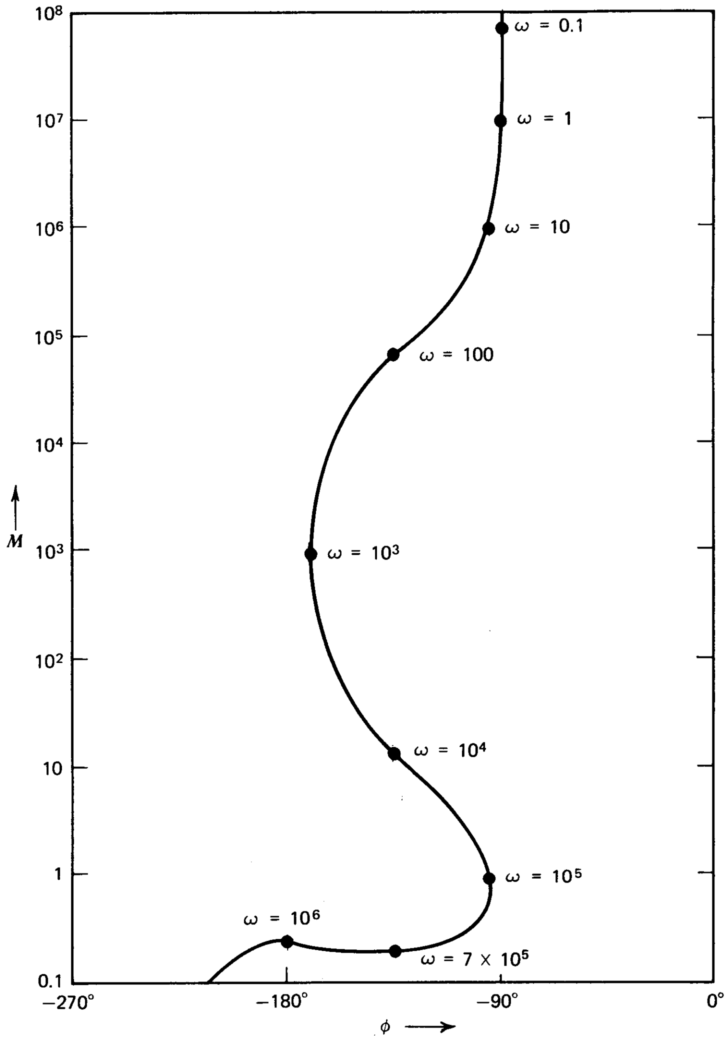


Figure 3.15 Gain phase plot of $\dfrac{10^7(10^{-4}s + 1)}{s(0.1s + 1)(s^2/10^{12} + 2(0.2)s/10^6 + 1)}$.

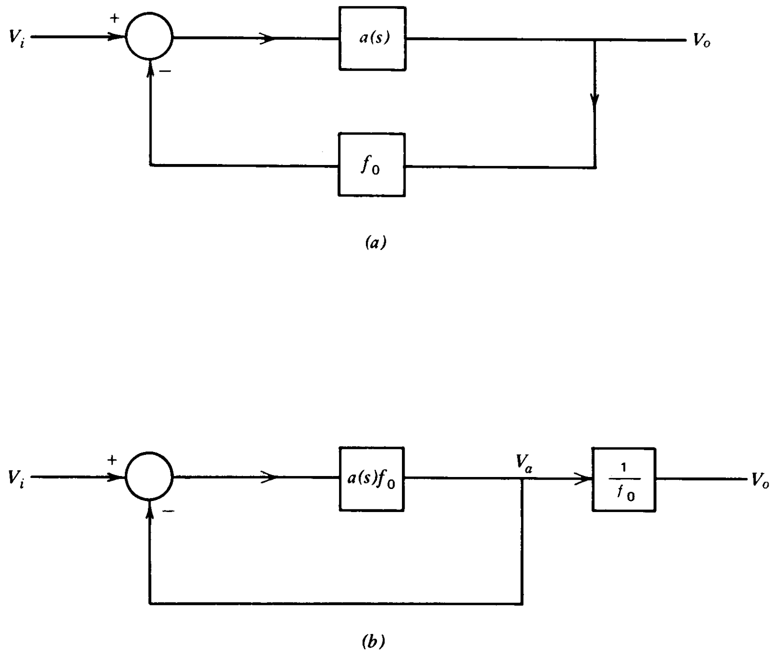


Figure 3.16 System topology for approximate relationships. (*a*) System with frequency-independent feedback path. (*b*) System represented in scaled, unity-feedback form.

(b) The maximum value of the step response P_0.

(c) The time at which P_0 occurs t_p.

(d) Settling time t_s. The time after which the system step response remains within 2% of final value.

(e) The error coefficient e_1. (See Section 3.6.) This coefficient is equal to the time delay between the output and the input when the system has reached steady-state conditions with a ramp as its input.

(f) The bandwidth in radians per second ω_h or hertz f_h ($f_h = \omega_h/2\pi$). The frequency at which the response of the system is 0.707 of its low-frequency value.

(g) The maximum magnitude of the frequency response M_p.

(h) The frequency at which M_p occurs ω_p.

These definitions are illustrated in Fig. 3.17.

For a first-order system with $V_o(s)/V_i(s) = 1/(\tau s + 1)$, the relationships are

$$t_r = 2.2\tau = \frac{2.2}{\omega_h} = \frac{0.35}{f_h} \tag{3.51}$$

$$P_0 = M_p = 1 \tag{3.52}$$

$$t_p = \infty \tag{3.53}$$

$$t_s = 4\tau \tag{3.54}$$

$$e_1 = \tau \tag{3.55}$$

$$\omega_p = 0 \tag{3.56}$$

For a second-order system with $V_o(s)/V_i(s) = 1/(s^2/\omega_n^2 + 2\zeta s/\omega_n + 1)$ and $\theta \triangleq \cos^{-1}\zeta$ (see Fig. 3.7) the relationships are

$$t_r \simeq \frac{2.2}{\omega_h} = \frac{0.35}{f_h} \tag{3.57}$$

$$P_0 = 1 + \exp\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = 1 + e^{-\pi/\tan\theta} \tag{3.58}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n \sin\theta} \tag{3.59}$$

$$t_s \simeq \frac{4}{\zeta\omega_n} = \frac{4}{\omega_n\cos\theta} \tag{3.60}$$

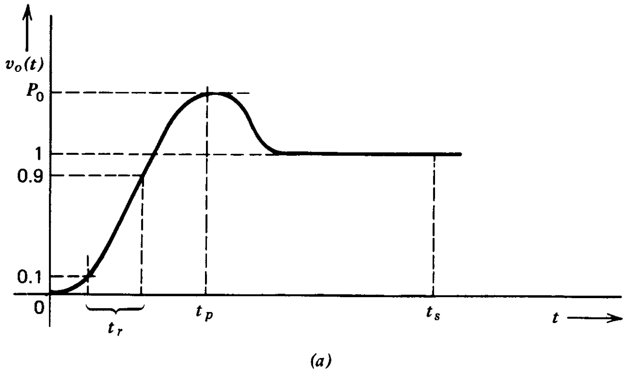
$$e_1 = \frac{2\zeta}{\omega_n} = \frac{2\cos\theta}{\omega_n} \tag{3.61}$$

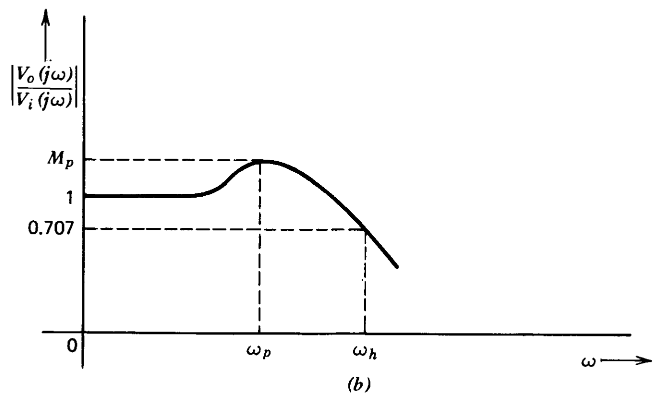
$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{\sin 2\theta} \qquad \zeta < 0.707,\ \theta > 45° \tag{3.62}$$

$$\omega_p = \omega_n\sqrt{1-2\zeta^2} = \omega_n\sqrt{-\cos 2\theta} \qquad \zeta < 0.707,\ \theta > 45° \tag{3.63}$$

$$\omega_h = \omega_n\left(1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right)^{1/2} \tag{3.64}$$

If a system step response or frequency response is similar to that of an approximating system (see Figs. 3.6, 3.8, 3.11, and 3.12) measurements of t_r, P_0, and t_p permit estimation of ω_h, ω_p, and M_p or vice versa. The steady-state error in response to a unit ramp can be estimated from either set of measurements.





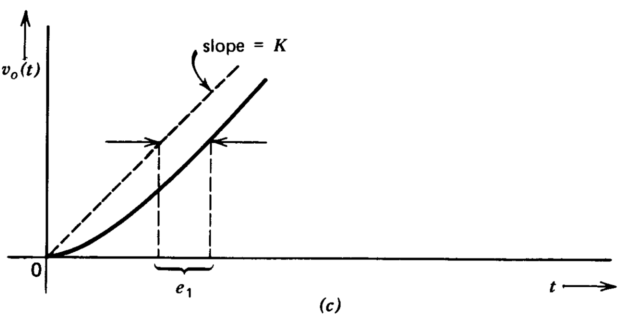


Figure 3.17 Parameters used to describe transient and frequency responses. (*a*) Unit-step response. (*b*) Frequency response. (*c*) Ramp response.

One final comment concerning the quality of the relationship between 0.707 bandwidth and 10 to 90% step risetime (Eqns. 3.51 and 3.57) is in order. For virtually any system that satisfies the original assumptions, independent of the order or relative stability of the system, the product $t_r f_h$ is within a few percent of 0.35. This relationship is so accurate that it really isn't worth measuring f_h if the step response can be more easily determined.

3.6 ERROR COEFFICIENTS

The response of a linear system to certain types of transient inputs may be difficult or impossible to determine by Laplace techniques, either because the transform of the transient is cumbersome to evaluate or because the transient violates the conditions necessary for its transform to exist. For example, consider the angle that a radar antenna makes with a fixed reference while tracking an aircraft, as shown in Fig. 3.18. The pointing angle determined from the geometry is

$$\theta = \tan^{-1}\left[\frac{v}{l}t\right] \tag{3.65}$$

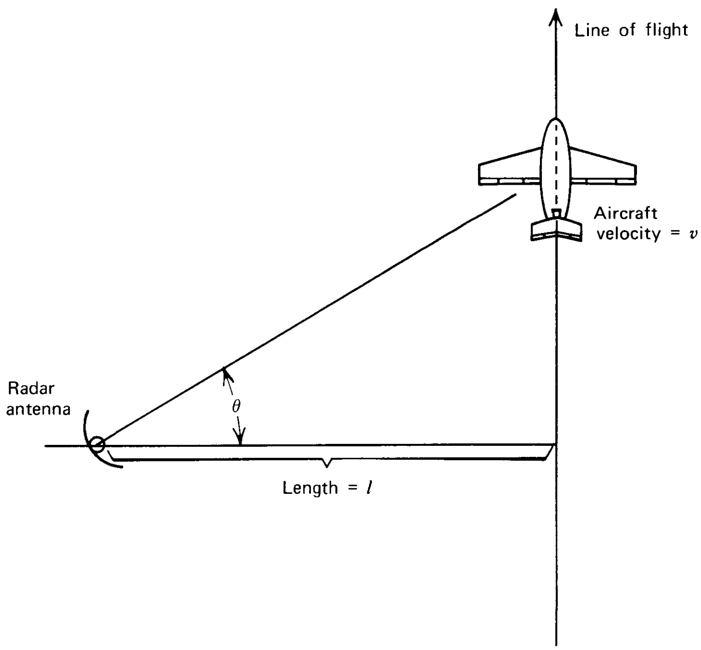


Figure 3.18 Radar antenna tracking an airplane.

assuming that $\theta = 0$ at $t = 0$. This function is not transformable using our form of the Laplace transform, since it is nonzero for negative time and since no amount of time shift makes it zero for negative time. The expansion introduced in this section provides a convenient method for evaluating the performance of systems excited by transient inputs, such as Eqn. 3.65, for which all derivatives exist at all times.

3.6.1 The Error Series

Consider a system, initially at rest and driven by a single input, with a transfer function $G(s)$. Furthermore, assume that $G(s)$ can be expanded in a power series in s, or that

$$G(s) = g_0 + g_1 s + g_2 s^2 + \cdots + \tag{3.66}$$

If the system is excited by an input $v_i(t)$, the output signal as a function of time is

$$\begin{aligned} v_o(t) &= \mathcal{L}^{-1}[G(s)V_i(s)] \\ &= \mathcal{L}^{-1}[g_0 V_i(s) + g_1 s V_i(s) + g_2 s^2 V_i(s) + \cdots +] \end{aligned} \tag{3.67}$$

If Eqn. 3.66 is inverse transformed term by term, and the differentiation property of Laplace transforms is used to simplify the result, we see that[8]

$$v_o(t) = g_0 v_i(t) + g_1 \frac{dv_i(t)}{dt} + g_2 \frac{d^2 v_i(t)}{dt^2} + \cdots + \tag{3.68}$$

The complete series yields the correct value for $v_o(t)$ in cases where the function $v_i(t)$ and all its derivatives exist at all times.

In practice, the method is normally used to evaluate the error (or difference between ideal and actual output) that results for a specified input. If Eqn. 3.68 is rewritten using the error $e(t)$ as the dependent parameter, the resultant series

$$e(t) = e_0 v_i(t) + e_1 \frac{dv_i(t)}{dt} + e_2 \frac{d^2 v_i(t)}{dt^2} + \cdots + \tag{3.69}$$

is called an *error series*, and the e's on the right-hand side of this equation are called *error coefficients*.

The error coefficients can be obtained by two equivalent expansion methods. A formal mathematical approach shows that

$$e_k = \frac{1}{k!} \frac{d^k}{ds^k} \left[\frac{V_e(s)}{V_i(s)} \right]_{s=0} \tag{3.70}$$

[8] A mathematically satisfying development is given in G. C. Newton, Jr., L. A. Gould, and J. F. Kaiser, *Analytical Design of Linear Feedback Controls*, Wiley, New York, 1957, Appendix C. An expression that bounds the error when the series is truncated is also given in this reference.

where $V_e(s)/V_i(s)$ is the input-to-error transfer function for the system. Alternatively, synthetic division can be used to write the input-to-error transfer function as a series in ascending powers of s. The coefficient of the s^k term in this series is e_k.

While the formal mathematics require that the complete series be used to determine the error, the series converges rapidly in cases of practical interest where the error is small compared to the input signal. (Note that if the error is the same order of magnitude as the input signal in a unity-feedback system, comparable results can be obtained by turning off the system.) Thus in reasonable applications, a few terms of the error series normally suffice. Furthermore, the requirement that all derivatives of the input signal exist can be usually relaxed if we are interested in errors at times separated from the times of discontinuities by at least the settling time of the system. (See Section 3.5 for a definition of settling time.)

3.6.2 Examples

Some important properties of feedback amplifiers can be illustrated by applying error-coefficient analysis methods to the inverting-amplifier connection shown in Fig. 3.19*a*. A block diagram obtained by assuming negligible loading at the input and output of the amplifier is shown in Fig. 3.19*b*. An error signal is generated in this diagram by comparing the actual output of the amplifier with the ideal value, $-V_i$. The input-to-error transfer function from this block diagram is

$$\frac{V_e(s)}{V_i(s)} = \frac{-1}{1 + a(s)/2} \tag{3.71}$$

Operational amplifiers are frequently designed to have an approximately single-pole open-loop transfer function, implying

$$a(s) \simeq \frac{a_0}{\tau s + 1} \tag{3.72}$$

The error coefficients assuming this value for $a(s)$ are easily evaluated by means of synthetic division since

$$\begin{aligned}\frac{V_e(s)}{V_i(s)} &= \frac{-1}{1 + a_0/2(\tau s + 1)} = \frac{-2 - 2\tau s}{a_0 + 2 + 2\tau s} \\ &= -\frac{2}{a_0 + 2} - \frac{2\tau}{a_0 + 2}\left(1 - \frac{2}{a_0 + 2}\right)s \\ &\qquad + \frac{4\tau^2}{(a_0 + 2)^2}\left(1 - \frac{2}{a_0 + 2}\right)s^2 + \cdots +\end{aligned} \tag{3.73}$$

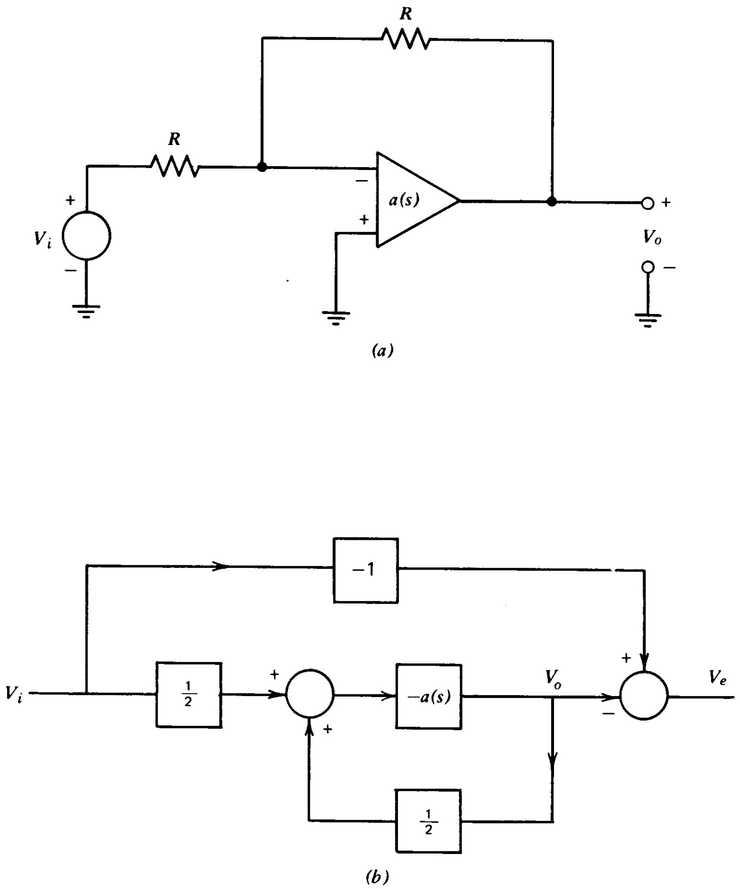


Figure 3.19 Unity-gain inverter. (*a*) Connection. (*b*) Block diagram including error signal.

If a_0, the amplifier d-c gain, is large, the error coefficients are

$$e_0 \simeq -\frac{2}{a_0}$$

$$e_1 \simeq -\frac{2\tau}{a_0}$$

$$e_2 \simeq \frac{4\tau^2}{a_0{}^2}$$

.
.
.

$$e_n = \frac{(-2)^n \tau^n}{a_0{}^n} \qquad n \geq 1 \tag{3.74}$$

The error coefficients are easily interpreted in terms of the loop transmission of the amplifier-feedback network combination in this example. The magnitude of the zero-order error coefficient is equal to the reciprocal of the d-c loop transmission. The first-order error-coefficient magnitude is equal to the reciprocal of the frequency (in radians per second) at which the loop transmission is unity, while the magnitude of each subsequent higher-order error coefficient is attenuated by a factor equal to this frequency. These results reinforce the conclusion that feedback-amplifier errors are reduced by large loop transmissions and unity-gain frequencies.

If this amplifier is excited with a ramp $v_i(t) = Rt$, the error after any start-up transient has died out is

$$v_e(t) = e_0 v_i(t) + e_1 \frac{dv_i(t)}{dt} + \cdots + = -\frac{2Rt}{a_0} - \frac{2R\tau}{a_0} \tag{3.75}$$

Because the maximum input-signal level is limited by linearity considerations, (the voltage Rt must be less than the voltage at which the amplifier saturates) the second term in the error series frequently dominates, and in these cases the error is

$$v_e(t) \simeq -\frac{2R\tau}{a_0} \tag{3.76}$$

implying the actual ramp response of the amplifier lags behind the ideal output by an amount equal to the slope of the ramp divided by the unity-loop-transmission frequency. The ramp response of the amplifier, assuming that the error series is dominated by the e_1 term, is compared with the ramp response of a system using an infinite-gain amplifier in Fig. 3.20. The steady-state ramp error, introduced earlier in Eqns. 3.55 and 3.61 and illustrated in Fig. 3.17*c*, is evident in this figure.

One further observation lends insight into the operation of this type of system. If the relative magnitudes of the input signal and its derivatives are constrained so that the first-order (or higher) terms in the error series dominate, the open-loop transfer function of the amplifier can be approximated as an integration.

$$a(s) \simeq \frac{a_0}{\tau s} \tag{3.77}$$

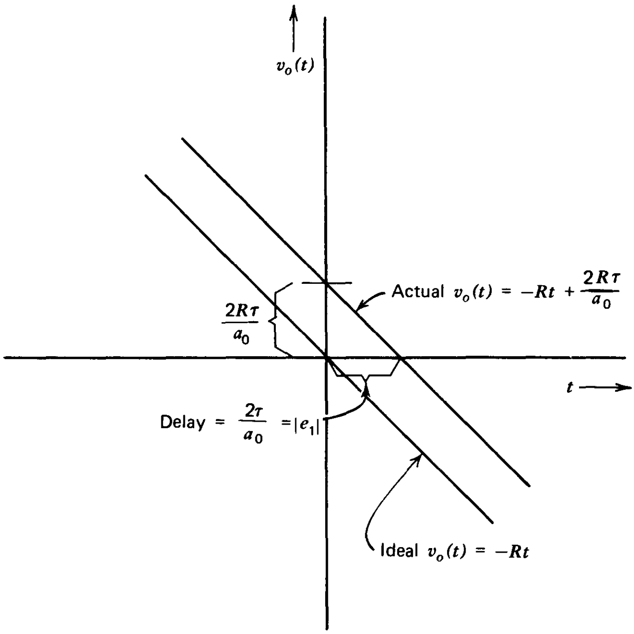


Figure 3.20 Ideal and actual ramp responses.

In order for the output of an amplifier with this type of open-loop gain to be a ramp, it is necessary to have a constant error signal applied to the amplifier input.

Pursuing this line of reasoning further shows how the open-loop transfer function of the amplifier should be chosen to reduce ramp error. Error is clearly reduced if the quantity a_0/τ is increased, but such an increase requires a corresponding increase in the unity-loop-gain frequency. Unfortunately oscillations result for sufficiently high unity-gain frequencies. Alternatively, consider the result if the amplifier open-loop transfer function approximates a double integration

$$a(s) \simeq \frac{a_0(\tau s + 1)}{s^2} \tag{3.78}$$

(The zero is necessary to insure stability. See Chapter 4.) The reader should verify that both e_0 and e_1 are zero for an amplifier with this open-loop transfer function, implying that the steady-state ramp error is zero. Further manipulation shows that if the amplifier open-loop transfer function includes an nth order integration, the error coefficients e_0 through e_{n-1} are zero.

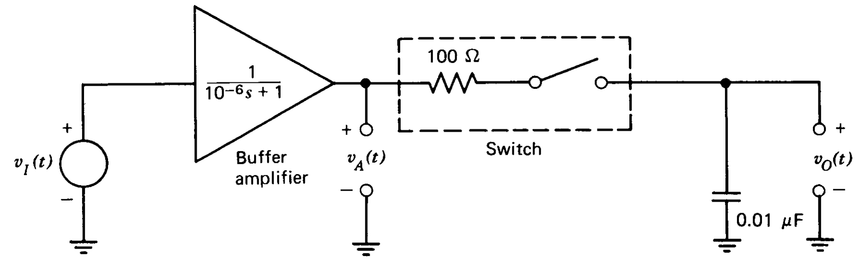


Figure 3.21 Sample-and-hold circuit.

The use of error coefficients to analyze systems excited by pulse signals is illustrated with the aid of the sample-and-hold circuit shown in Fig. 3.21. This circuit consists of a buffer amplifier followed by a switch and capacitor. In practice the switch is frequently realized with a field-effect transistor, and the 100-Ω resistor models the on resistance of the transistor. When the switch is closed, the capacitor is charged toward the voltage v_I through the switch resistance. If the switch is opened at a time t_A, the voltage $v_O(t)$ should ideally maintain the value $v_I(t_A)$ for all time greater than t_A. The buffer amplifier is included so that the capacitor charging current is supplied by the amplifier rather than the signal source. A second buffer amplifier is often included following the capacitor to isolate it from loads, but this second amplifier is not required for the present example.

There are a variety of effects that degrade the performance of a sample-and-hold circuit. One important source of error stems from the fact that $v_O(t)$ is generally not equal to $v_I(t)$ unless $v_I(t)$ is time invariant because of the dynamics of the buffer amplifier and the switch-capacitor combination. Thus an incorrect value is held when the switch is opened.

Error coefficients can be used to predict the magnitude of this tracking error as a function of the input signal and the system dynamics. For purposes of illustration, it is assumed that the buffer amplifier has a single-pole transfer function such that

$$\frac{V_a(s)}{V_i(s)} = \frac{1}{10^{-6}s + 1} \tag{3.79}$$

Since the time constant associated with the switch-capacitor combination is also 1 μs, the input-to-output transfer function with the switch closed (in which case the system is linear, time-invariant) is

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{(10^{-6}s + 1)^2} \tag{3.80}$$

With the switch closed the output is ideally equal to the input, and thus the input-to-error transfer function is

$$\frac{V_e(s)}{V_i(s)} = 1 - \frac{V_o(s)}{V_i(s)} = \frac{10^{-12}s^2 + 2 \times 10^{-6}s}{(10^{-6}s + 1)^2} \tag{3.81}$$

The first three error coefficients associated with Eqn. 3.81, obtained by means of synthetic division, are

$$\begin{aligned} e_0 &= 0 \\ e_1 &= 2 \times 10^{-6} \text{ sec} \\ e_2 &= -3 \times 10^{-12} \text{ sec}^2 \end{aligned} \tag{3.82}$$

Sample-and-hold circuits are frequently used to process pulses such as radar echos after these signals have passed through several amplifier stages. In many cases the pulse following amplification can be well approximated by a Gaussian signal, and for this reason a signal

$$v_i(t) = e^{-(10^{10}t^2/2)} \tag{3.83}$$

is used as a test input.

The first two derivatives of $v_i(t)$ are

$$\frac{dv_i(t)}{dt} = -10^{10}te^{-(10^{10}t^2/2)} \tag{3.84}$$

and

$$\frac{d^2v_i(t)}{dt^2} = -10^{10}e^{-(10^{10}t^2/2)} + 10^{20}t^2e^{-(10^{10}t^2/2)} \tag{3.85}$$

The maximum magnitude of dv_i/dt is 6.07×10^4 volts per second occurring at $t = \pm 10^{-5}$ seconds, and the maximum magnitude of d^2v_i/dt^2 is 10^{10} volts per second squared at $t = 0$. If the first error coefficient is used to estimate error, we find that a tracking error of approximately 0.12 volt (12% of the peak-signal amplitude) is predicted if the switch is opened at $t = \pm 10^{-5}$ seconds. The error series converges rapidly in this case, with its second term contributing a maximum error of 0.03 volt at $t = 0$.

PROBLEMS

P3.1

An operational amplifier is connected to provide a noninverting gain of 10. The small-signal step response of the connection is approximately first order with a 0 to 63% risetime of 1 μs. Estimate the quantity $a(s)$ for the amplifier, assuming that loading at the amplifier input and output is insignificant.

P3.2

The transfer function of a linear system is

$$A(s) = \frac{1}{(s^2 + 0.5s + 1)(0.1s + 1)}$$

Determine the step response of this system. Estimate (do not calculate exactly) the percentage overshoot of this system in response to step excitation.

P3.3

Use the properties of Laplace transforms to evaluate the transform of the triangular pulse signal shown in Fig. 3.22.

P3.4

Use the properties of Laplace transforms to evaluate the transform of the pulse signal shown in Fig. 3.23.

P3.5

The response of a certain linear system is approximately second order, with a d-c gain of one. Measured performance shows that the peak value of the response to a unit step is 1.38 and that the time for the step response to first pass through one is 0.5 μs. Determine second-order parameters that can be used to model the system. Also estimate the peak value of the output that results when a unit impulse is applied to the input of the system and the time required for the system impulse response to first return to zero. Estimate the quantities M_p and f_h for this system.

P3.6

A high-fidelity audio amplifier has a transfer function

$$A(s) = \frac{100s}{(0.05s + 1)(s^2/4 \times 10^{10} + s/2 \times 10^5 + 1)(0.5 \times 10^{-6}s + 1)}$$

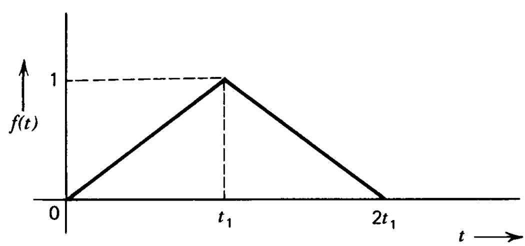


Figure 3.22 Triangular pulse.

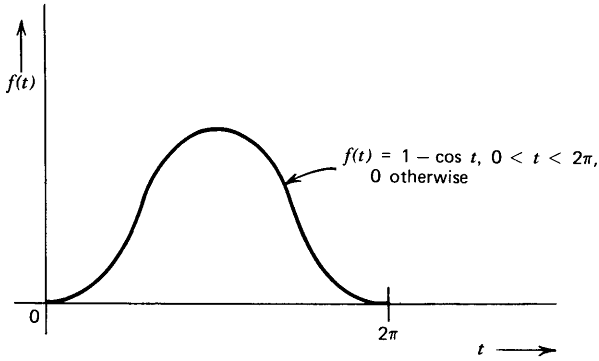


Figure 3.23 Raised cosine pulse.

Plot this transfer function in both Bode and gain-phase form. Recognize that the high- and low-frequency singularities of this amplifier are widely spaced and use this fact to estimate the following quantities when the amplifier is excited with a 10-mV step.

(a) The peak value of the output signal.
(b) The time at which the peak value occurs.
(c) The time required for the output to go from 2 to 18 V.
(d) The time until the output droops to 7.4 V.

P3.7

An oscilloscope vertical amplifier can be modeled as having a transfer function equal to $A_0/(10^{-9}s + 1)^5$. Estimate the 10 to 90% rise time of the output voltage when the amplifier is excited with a step-input signal.

P3.8

An asymptotic plot of the measured open-loop frequency response of an operational amplifier is shown in Fig. 3.24*a*. The amplifier is connected as shown in Fig. 3.24*b*. (You may neglect loading.) Show that lower values of α result in more heavily damped responses. Determine the value of α that results in the closed-loop step response of the amplifier having an overshoot of 20% of final value. What is the 10 to 90% rise time in response to a step for this value of α?

P3.9

A feedback system has a forward gain $a(s) = K/s(\tau s + 1)$ and a feedback gain $f = 1$. Determine conditions on K and τ so that e_0 and e_2 are

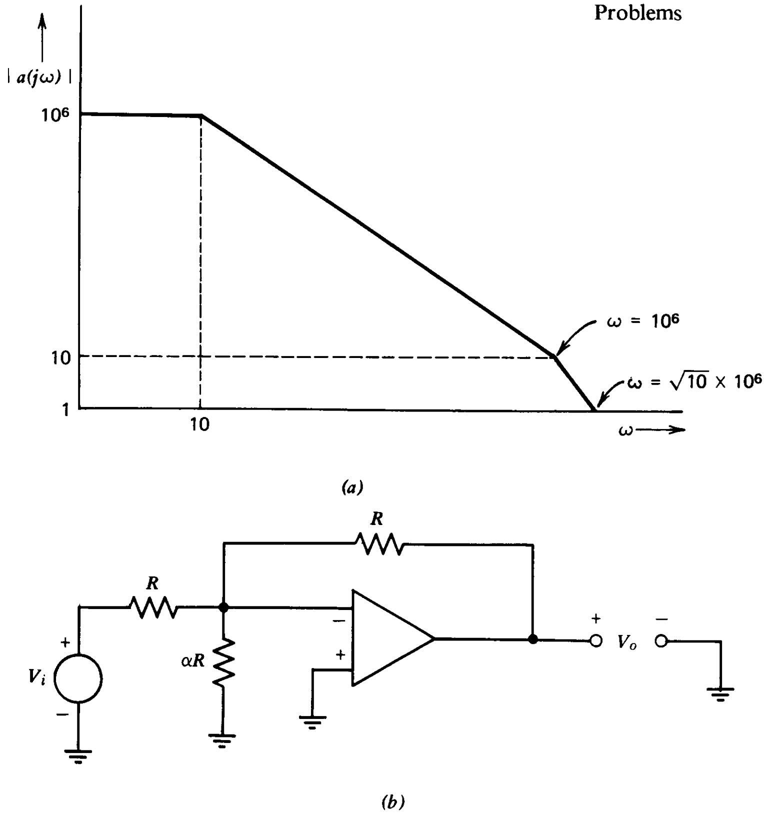


Figure 3.24 Inverting amplifier. (*a*) Amplifier open-loop response. (*b*) Connection.

both zero. What is the steady-state error in response to a unit ramp for this system?

P3.10

An operational amplifier connected as a unity-gain noninverting amplifier is excited with an input signal

$$v_i(t) = 5 \tan^{-1} 10^5 t$$

Estimate the error between the actual and ideal outputs assuming that the open-loop transfer function can be approximated as indicated below. (Note that these transfer functions all have identical values for unity-gain frequency.)

(a) $a(s) = 10^7/s$
(b) $a(s) = 10^{13}(10^{-6}s + 1)/s^2$
(c) $a(s) = 10^{19}(10^{-6}s + 1)^2/s^3$

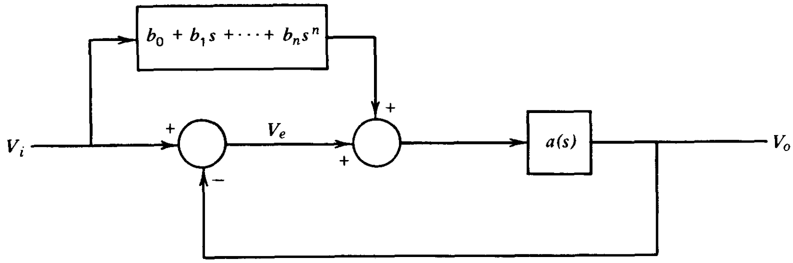


Figure 3.25 System with feedforward path.

P3.11

The system shown in Fig. 3.25 uses a feedforward path to reduce errors. How should the b's be chosen to reduce error coefficients e_0 through e_n to zero? Can you think of any practical disadvantages to this scheme?