Problem Set 1

Thermodynamics and Climate Change

MOSTEC 2021

1. **Properties of a thermodynamic system:** If we take the top layer of the Pacific ocean as our thermodynamic system, which of the following are properties of that system? Which are not? Briefly state why.

   (a) Temperature of the water

   (b) Density of the water

   (c) The amount of water exchanged with lower layers each month

   (d) The concentration of dissolved oxygen in the water

   (e) The atmospheric pressure at the water's surface

   (f) Concentration of salt in the water

   (g) The amount of water that evaporates each day

   (a), (b), (d), and (f) are properties, as they are directly measurable quantities that do not require knowledge of the *history* of the system. (e) is a property but may not be part of the thermodynamic system, depending on how we define it. If we do not include any atmosphere, this would not be a property *of the system*, but it would influence the pressure of the water just below the surface, which would be a property of the system. (c) and (g) are not properties as there is no way to measure these quantities without knowing something about the history of system. They depend on a transfer of quantities over time.

2. **Concept questions:** Answer each with a brief but specific explanation.

(a) For the following, state the type(s) and method of energy conversion (e.g. mechanical to thermal via frictional dissipation): (i) A block sliding down an incline and coming to a stop. (ii) Rain falling from a cloud and hitting the ground. (iii) The ground heating up in the sun. (iv) A forest fire. (v) Ice melting. (vi) Air expanding as it is heated.

(i) Gravitational potential energy first to kinetic energy and ultimately to thermal energy (ii) Gravitational potential energy to kinetic energy and ultimately to thermal energy (plus some acoustic energy as it hits). Also important to note that heat is transferred from water vapor to air to allow droplets to condense in clouds. Some of this energy also goes into surface tension potential energy holding the drop together, which gets dissipated when the droplet breaks apart upon impact. (iii) Nuclear fusion in the sun to thermal radiation to sensible thermal energy in the ground. (iv) Chemical potential energy to sensible thermal energy and thermal radiation (light). Also forest fires may also cause liquid water to evaporate, which is thermal energy to energy stored in phase change. (v) Latent heat released via thermal energy transfer. Note, ice is less dense than liquid water, so as it melts it shrinks, and thus some work is done on the ice. This observation is captured by a quantity called enthalpy, which we will soon discuss. (vi) Thermal energy to work and also gravitational potential energy as the expanding air will rise.

(b) For an ideal gas at constant pressure: (i) What happens to the volume when you increase its temperature? (ii) What happens to the volume if we remove half of the molecules of gas? (iii) Does heating the gas in this case require more or less energy to achieve the same temperature than if the gas was kept at constant volume instead?

Parts (i) and (ii) require the ideal gas law:

$$PV = nRT \tag{1}$$

So from this equation, we can see directly that (i) the volume must increase if temperature increases to hold the pressure constant. This is captured in *Charles's Law*:

$$V_2 = \frac{T_2}{T_1} V_1 \tag{2}$$

and (ii) that if we reduce half the moles, the volume must also decrease by a factor of 2 to counter the resultant drop in pressure. (iii) If we heat the gas holding the pressure constant, its volume must increase as the temperature rises, and thus the gas does *work* on the environment. From the First Law of Thermodynamics:

$$\Delta U = Q_{cp} - W \tag{3}$$

where $\Delta U = C_v \Delta T$. So the amount of heat necessary, to cause a temperature rise of ΔT, is

$$Q_{cp} = C_v \Delta T + W = C_p \Delta T \tag{4}$$

whereas in the constant volume case, $W = 0$, and thus the heat is simply

$$Q_{cv} = C_v \Delta T \tag{5}$$

Clearly the constant pressure case requires more heat as some of that energy has to also supply the energy needed to do work. Also mathematically, we know that $c_p = c_v + R$ and so $c_p > c_v$ and $Q_{cp} > Q_{cv}$.

(c) In which of the following cases is the First Law of Thermodynamics violated? Why or why not? (i) A solar sail in outer space that accelerates by sunlight shining on. (ii) A balloon that rises when you heat the gas inside. (iii) A block sliding on a frictional surface without slowing down. (iv) A device that extracts mechanical work from a heated block without cooling the block down. (v) The Moon causing tides on Earth.

(i) Not violated as light has momentum which can transfer to the sail, doing work on it, causing it to accelerate. (ii) Not violated if we allow the balloon to expand, then it becomes less dense than the surrounding air and can float upwards, increasing its gravitational potential energy. (iii) Depends. Violated if not on an incline since the friction would heat up the block and/or the surface and that thermal energy would need to come from somewhere i.e. the block's kinetic energy. If on an incline, this is possible as gravitational potential energy is repeatedly cashed in to overcome the energy dissipated by friction. Eventually you would hit the center of the Earth and bad things would happen. (iv) Always violated. Internal energy must decrease to power mechanical work. (v) Not violated of course, but in fact this is causing the moon to drift farther from Earth, slowing the Earth's rotation in the process. So that energy is coming at the cost of the Earth's rotational kinetic energy. Eventually the tides will be locked in place and no more energy can be harvested from the changing tides. (See tidal friction).

(d) (i) Does ice absorb or release thermal energy as it melts? (ii) Does water absorb or release thermal energy as it evaporates?

(i) Ice absorbs energy as it melts, which is why ice packs work. (ii) Water absorbs energy as it evaporates, which is why sweating cools the skin.

3. **An Earth without its atmosphere:** The Earth's atmosphere is essential for maintaining its surface temperature above the freezing point of water.

(a) Estimate the Earth's surface temperature if it had no atmosphere, modeling the system as a uniform rocky sphere exposed to sunlight in a vacuum. Assume that sphere is of uniform temperature. You can use the following average values for the Earth in your computations: solar irradiance at the upper atmosphere is about 1400 W/m^2, an average albedo (fraction of light that is reflected) of 0.3, and an emissivity of 0.8.

First draw a picture. See Fig. 1. Taking a control volume of just the Earth model, shown by the dashed red line, we write the First Law in its time derivative form:

$$\frac{dE_{CV}}{dt} = \frac{dQ_{net}}{dt} - \cancel{\frac{dW}{dt}} = 0 \tag{6}$$

We can set this to zero because in steady state, we know that the temperature is constant and thus the rate of change of the internal energy - which is proportional to temperature - is equal to zero. The net heat transfer, $\frac{dQ_{net}}{dt}$, is purely radiative and must therefore also equal zero. Based on our picture and the information given, we see that the net heat transfer is the sum of the solar energy in, some

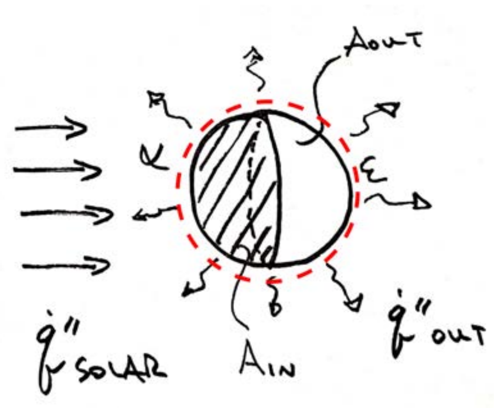

Figure 1: Diagram for Problem 3a.

of which gets reflected, and the thermal radiation of the Earth itself out. Mathematically, this balance is written as:

$$\dot{q}''_{solar} A_{in}(1-\alpha) = A_{out}\sigma\varepsilon T^4 \tag{7}$$

where α is the albedo and the right hand side is given by the Stefan-Boltzmann Law. For this problem, we will assume the incident sunlight falls on projected surface area equal to a circle with the radius of the Earth ($A_{in} = \pi R_e^2$) while the outgoing radiation happens over the total surface area of the earth ($A_{out} = 4\pi R_e^2$). We were not given R_e but fortunately it cancels out in Eq. 7. Plugging in these values and rearranging to solve for T:

$$\dot{q}''_{solar}\cancel{\pi R_e^2}(1-\alpha) = 4\cancel{\pi R_e^2}\sigma\varepsilon T^4 \tag{8}$$

$$T = \left(\frac{\dot{q}''_{solar}(1-\alpha)}{4\sigma\varepsilon}\right)^{\frac{1}{4}} \tag{9}$$

Plugging in numbers:

$$T = \left(\frac{1400\ [\text{W/m}^2] * (1-0.3)}{4 * 5.67\text{x}10^{-8}\ [\text{W/m}^2\text{-K}^4] * 0.8}\right)^{\frac{1}{4}} = 271\ \text{K} = \boxed{-2.1\ ^\circ\text{C}} \tag{10}$$

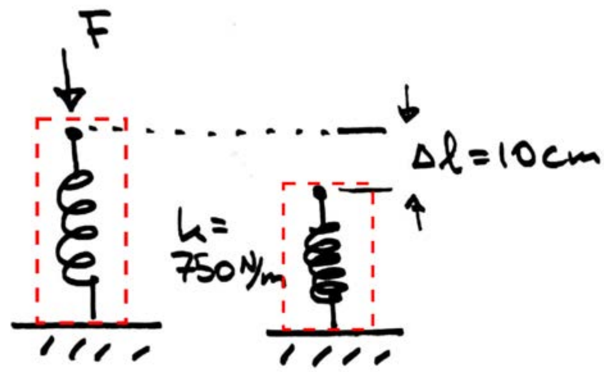

Figure 2: Diagram for Problem 4a.

(b) A more detailed computer simulation shows that Earth's surface temperature would be -18 °C without an atmosphere. Is your value higher or lower? Why might that be the case?

We assumed a smaller emissivity or a higher solar irradiance. If you plug in 1 for ε into Eq. 9, you will find that indeed the value is much closer to -18 °C. Note that for perfectly perpendicular incident light, $\varepsilon = (1 - \alpha)$; however, since our albedo is taken as an average here, which includes added reflectance of lower wavelengths at oblique angles (i.e. why sunsets and sunrises are more red than blue), the emissivity is higher than $(1 - \alpha)$. Also care must be taken to determine the effective insolation area. You could have said it was equal to half the surface area of sphere as well, for example, which would skew temperature even higher.

4. **Energy lost in a spring:** Let's say you have a linear spring with a spring constant of 750 N/m. This spring is non-ideal though and in fact dissipates some energy. You put in 5 J of work in compressing the spring such that its length changes by 10 cm.

(a) How much work would it have taken if the spring was ideal and did not dissipate any energy?

As always, start with a picture and define control volume. See Fig. 2. Writing the First Law for the control volume,

$$\Delta E_{CV} = \not{Q} - W \tag{11}$$

Using the constitutive relationship for a spring that relates the stored potential energy to the compressed length, Δl, we find that

$$W = -\frac{1}{2}k(\Delta l)^2 = -\frac{1}{2} * 750 \text{ [N/m]} * (0.1 \text{ [m]})^2 \tag{12}$$

$$W = \boxed{-3.75 \text{ J}} \tag{13}$$

where the negative sign indicates that work is being done on the control volume.

(b) How much thermal energy was generated in the process with the non-ideal spring?

Now the spring heats up under the compression. Again writing the first law, as before, we can separate out the components of the internal energy into sensible heat and spring potential as

$$\Delta E_{CV} = \Delta E_{TE} + \Delta E_{PE} = mc\Delta T + \frac{1}{2}k(\Delta l)^2 \tag{14}$$

We know how much work went into ΔE_{PE} from part (a) and the total work is given to us as 5 J. Therefore, we know the work that must have gone into the thermal energy, ΔE_{TE}, is simply $5 - 3.75 = \boxed{1.25 \text{ J}}$.

(c) If the spring is made of iron and has a mass of 5 kg, by how much does its temperature change during this process? (Iron has a specific heat of capacity of 444 J/kg-K).

$$\Delta E_{TE} = mc\Delta T = 1.25 \text{ J} \tag{15}$$

and thus

$$T = \frac{\Delta E_{TE}}{mc} = \frac{1.25 \text{ [J]}}{5 \text{ [kg]} * 444 \text{ [J/kg-K]}} = \boxed{0.00056 \text{ K}} \tag{16}$$

Not much!

5. **(Challenge) Atmospheric pressure:** Derive the Earth's atmospheric pressure as a function of height above sea level assuming that it is an ideal gas for cases where:

(a) The temperature profile is constant.

Starting with the mass form of the ideal gas law:

$$P = \rho \widetilde{R} T \tag{17}$$

We know from hydrostatics that the pressure in the atmosphere varies linearly with height:

$$\frac{dP}{dz} = -\rho g \tag{18}$$

(equivalent to the more common form $P = \rho g z$). Solving for density ρ and plugging into Eq. 17, rearranging to separate variables, and integrating:

$$P = -\frac{dP}{dz} \frac{\widetilde{R} T}{g} \tag{19}$$

$$\int_{P_0}^{P} \frac{dP'}{P'} = -\int_0^z \frac{g\, dz'}{\widetilde{R} T} \tag{20}$$

$$\ln \frac{P}{P_0} = -\frac{gz}{\widetilde{R} T} \tag{21}$$

giving us the final barometric formula equation for constant temperature:

$$\boxed{P = P_0 \exp\left(-\frac{gz}{\widetilde{R} T}\right)} \tag{22}$$

where $\widetilde{R} = R/M$ and M is the weighted average molecular weight of air.

(b) The temperature profile decreases linearly from 300 K at the surface to 200 K at the *mesopause*, 85 km above sea level.

We do the same as in part (a) but now instead of constant T, we are given that it is decreasing linearly. Expressed mathematically,

$$T = 300 \text{ K} - \frac{(300 - 200) \text{ K}}{85,000 \text{ m}} * z = 300 - 0.0012\, z \tag{23}$$

Plugging this relationship into Eq. 20 before the integration:

$$\int_{P_0}^{P} \frac{dP'}{P'} = -\int_0^z \frac{g\,dz'}{\widetilde{R}(300 - 0.0012\,z')} \tag{24}$$

and integrating:

$$\ln \frac{P}{P_0} = \frac{g}{-0.0012\,\widetilde{R}} \ln \frac{300}{300 - 0.0012\,z} = \ln \left(\frac{300}{300 - 0.0012\,z} \right)^{\frac{-g}{0.0012\,\widetilde{R}}} \tag{25}$$

Giving us the final equation

$$\boxed{P = P_0 \left(\frac{300}{300 - 0.0012\,z} \right)^{\frac{-g}{0.0012\,\widetilde{R}}}} \tag{26}$$

(c) How much work is done by a 1 kg packet of air rising from sea level to the mesopause (ignoring gravitational potential energy)? Assume that at all altitudes, the pressure of the air packet is equal to atmospheric pressure at that altitude. Use the equation you derived for the linear temperature profile case.

(*Hint for parts (a) and (b): start with the ideal gas law and assume the pressure is hydrostatic - i.e. linearly proportional to height. Remember though that the density will be changing with altitude!*)

Note: We will ignore gravitational potential energy here. The definition of work for the gas in our control volume is

$$W = \int_{V_0}^{V} P\,dV' \tag{27}$$

We need to get the pressure and volume of the gas in terms of the height above the ground, z. Assuming we know nothing about the heat transfer occurring, we can safely assume that the air packet still obeys the ideal gas law, giving us a relationship for the volume of the gas:

$$V(z) = \frac{m\widetilde{R}T(z)}{P(z)} \tag{28}$$

into which we can plug our known $P(z)$ and $T(z)$:

$$V(z) = \frac{m\widetilde{R}(T_0 + \beta z)}{P_0 \left(\frac{T_0}{T_0+\beta z}\right)^{g/\widetilde{R}\beta}} \tag{29}$$

Taking the derivative of this equation to get dV in terms of dz:

$$dV = \frac{m\widetilde{R}\beta}{P_0} \left(\frac{g}{\widetilde{R}\beta} + 1\right) \left(\frac{T_0 + \beta z}{T_0}\right)^{g/\widetilde{R}\beta} dz \tag{30}$$

Plugging into Eq. 27:

$$W = \int_0^{z_f} \cancel{P_0} \cancel{\left(\frac{T_0}{T_0 + \beta z}\right)^{g/\widetilde{R}\beta}} \frac{m\widetilde{R}\beta}{\cancel{P_0}} \left(\frac{g}{\widetilde{R}\beta} + 1\right) \cancel{\left(\frac{T_0 + \beta z}{T_0}\right)^{g/\widetilde{R}\beta}} dz \tag{31}$$

$$= m\widetilde{R}\beta \left(\frac{g}{\widetilde{R}\beta} + 1\right) \int_0^{z_f} dz \tag{32}$$

$$\tag{33}$$

Taking $\beta = -0.0012$ K/m, $\widetilde{R} = 287$ J/kg-K, $g = 9.81$ m/s^2, and $T_0 = 300$ K, the final value after working through this integral yields:

$$W = \boxed{804 \text{ kJ}} \tag{34}$$

As a bonus challenge, how much heat must be transferred over this process? *Hint, we know the initial and final internal energy.*

Resource: Thermodynamics and Climate Change
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