Linear Algebra Online

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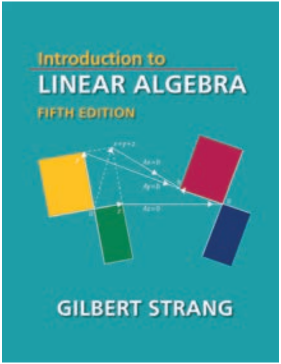
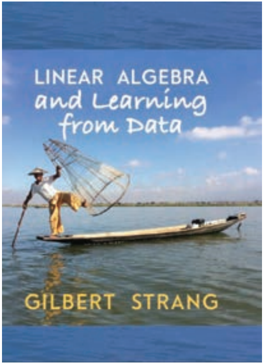
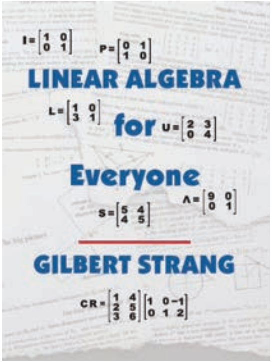
Textbooks by Gilbert Strang [math.mit.edu/∼gs](math.mit.edu/~gs)



(5th edition : 2016)



(1st edition : 2019)



(1st edition : 2020)

Please see their 3 websites on math.mit.edu

math.mit.edu/linearalgebra   math.mit.edu/learningfromdata   math.mit.edu/everyone

Video lectures   Math 18.06, 18.065, ocw.mit.edu/courses

For book orders : **math.mit.edu/weborder.php**

Key ideas of this lecture

1 The **nullspace** $\mathbf{N}(A)$ in $\mathbf{R}^n$ contains all solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$.

2 Elimination from A to R_0 to R does not change the nullspace.

3 $\boldsymbol{R_0} = \textbf{rref}(\boldsymbol{A})$ has $\boldsymbol{I}$ in r columns and $\boldsymbol{F}$ in $n - r$ columns.

4 Every column of F leads to a "*special solution*" to $A\boldsymbol{x} = \mathbf{0}$.

5 Every matrix factors into $\boldsymbol{A} = \boldsymbol{CR}$.

6 Every short wide matrix with $m < n$ has nonzero solutions to $A\boldsymbol{x} = \mathbf{0}$.

Example 1

$$R = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 6 \end{bmatrix} \qquad R\boldsymbol{x} = \mathbf{0} \ \text{ is } \quad \begin{aligned} x_1 + \quad 3x_3 + 5x_4 &= 0 \\ x_2 + 4x_3 + 6x_4 &= 0 \end{aligned}$$

Two "special solutions" are easy to find.

Set $x_3 = \mathbf{1}$ & $x_4 = \mathbf{0}$. Eqn 1 gives $x_1 = \mathbf{-3}$. Eqn 2 gives $x_2 = \mathbf{-4}$.

Set $x_3 = \mathbf{0}$ & $x_4 = \mathbf{1}$. Eqn 1 gives $x_1 = \mathbf{-5}$. Eqn 2 gives $x_2 = \mathbf{-6}$.

These two special solutions $\boldsymbol{s}_1 = (\mathbf{-3}, \mathbf{-4}, \mathbf{1}, \mathbf{0})$ and $\boldsymbol{s}_2 = (\mathbf{-5}, \mathbf{-6}, \mathbf{0}, \mathbf{1})$ are in the nullspace of R. They give $R\boldsymbol{s}_1 = \mathbf{0}$ and $R\boldsymbol{s}_2 = \mathbf{0}$.

Example 2

$$R_0 = \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_0\boldsymbol{x} = \mathbf{0} \text{ is} \qquad \begin{aligned} x_1 + 7x_2 + 0x_3 + 8x_4 &= 0 \\ x_3 + 9x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

I is in columns **1** and **3**. And row 3 is all zero.

The 1's in the identity matrix are still the first nonzeros in their rows.

Set $\boldsymbol{x_2} = \mathbf{1}$ & $\boldsymbol{x_4} = \mathbf{0}$. Eqn 1 gives $\boldsymbol{x_1} = \mathbf{-7}$. Eqn 2 gives $\boldsymbol{x_3} = \mathbf{0}$.

Set $\boldsymbol{x_2} = \mathbf{0}$ & $\boldsymbol{x_4} = \mathbf{1}$. Eqn 1 gives $\boldsymbol{x_1} = \mathbf{-8}$. Eqn 2 gives $\boldsymbol{x_3} = \mathbf{-9}$.

Special solutions $\boldsymbol{s}_1 = (\mathbf{-7}, \mathbf{1}, \mathbf{0}, \mathbf{0})$ and $\boldsymbol{s}_2 = (\mathbf{-8}, \mathbf{0}, \mathbf{-9}, \mathbf{1})$

$r, m, n = 2, 2, 4$	**Simplest case** $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$	as in $\begin{bmatrix} \mathbf{1} & 0 & 3 & 5 \\ 0 & \mathbf{1} & 4 & 6 \end{bmatrix}$
$r, m, n = 2, 3, 4$	**General case** $\boldsymbol{R_0} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{P}$	as in $\begin{bmatrix} \mathbf{1} & 7 & 0 & 8 \\ 0 & 0 & \mathbf{1} & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

R_0 has $m - r$ rows of zeros. I has r columns. F has $n - r$ columns.

$$P = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

exchanges columns 2 and 3. Then I goes into columns 1 and 3 of R_0 and R.

Column-row factorization $A = CR$

$$A = CR = C \begin{bmatrix} I & F \end{bmatrix} P = \begin{bmatrix} C & CF \end{bmatrix} P$$

$$= [\,\textbf{Independent cols} \quad \textbf{Dependent cols}\,]\ \textbf{Permute cols}$$

Dependent cols of A are combinations CF of independent cols in C.

Basis for the column space of A: Columns of C

Basis for the row space of A: Rows of R

Steps of Elimination

1. **Subtract** a multiple of one row from another row (above or below **!**)
2. **Multiply** a row by any nonzero number
3. **Exchange** any rows.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{11} & \mathbf{17} \\ \mathbf{3} & \mathbf{7} & \mathbf{37} & \mathbf{57} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{11} & \mathbf{17} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} & \mathbf{6} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} & \mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} & \mathbf{6} \end{bmatrix} = \boldsymbol{R}$$

$$\begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{I} & \boldsymbol{W}^{-1}\boldsymbol{H} \end{bmatrix}$$

What did elimination do? **Inverted leading $\mathbf{2 \times 2}$ matrix** $\boldsymbol{W} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{7} \end{bmatrix}$.

First $\boldsymbol{r}$ rows W at the start of A became I at the start of R.

Multiply $\boldsymbol{W^{-1}A = W^{-1}} \begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \end{bmatrix}$ **for** $\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{W^{-1}H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$.

Dependent columns $\boldsymbol{H} = \begin{bmatrix} 11 & 17 \\ 37 & 57 \end{bmatrix} =$ Independent columns $\boldsymbol{W} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ times $\boldsymbol{F} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

However you compute $\boldsymbol{R}$ from $\boldsymbol{A}$, you always reach the same $\boldsymbol{R}$.

1 First r independent cols of A locate the cols of R containing I

2 Remaining columns F in R are determined by $H = WF$:
(Dependent columns of A) = (Independent columns of A) times F

3 The last $m - r$ rows of R_0 are **rows of zeros**. Delete in R.

Second example produces a zero row in $\boldsymbol{R_0}$

$$\boldsymbol{A}=\begin{bmatrix}1 & 7 & 3 & 35\\ 2 & 14 & 6 & 70\\ 2 & 14 & 9 & 97\end{bmatrix}\to\begin{bmatrix}1 & 7 & 3 & 35\\ 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 27\end{bmatrix}\to\begin{bmatrix}1 & 7 & 0 & 8\\ 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 27\end{bmatrix}\to\begin{bmatrix}\mathbf{1} & 7 & \mathbf{0} & 8\\ \mathbf{0} & 0 & \mathbf{1} & 9\\ 0 & 0 & 0 & 0\end{bmatrix}=\boldsymbol{R_0}$$

$$\boldsymbol{C} \text{ times } \boldsymbol{F} = \begin{bmatrix}1 & 3\\ 2 & 6\\ 2 & 9\end{bmatrix}\begin{bmatrix}7 & 8\\ 0 & 9\end{bmatrix} = \begin{bmatrix}7 & 35\\ 14 & 70\\ 14 & 97\end{bmatrix} = \begin{matrix}\textbf{dependent}\\ \textbf{columns}\\ \textbf{2 and 4 of } \boldsymbol{A}\end{matrix}$$

The position of $\boldsymbol{I}$ in $\boldsymbol{R_0}$ locates the column matrix $\boldsymbol{C}$ in $\boldsymbol{A}$.

$$\boldsymbol{A} = \boldsymbol{C}\boldsymbol{R} \text{ is } \begin{bmatrix} 1 & 7 & 3 & 35 \\ 2 & 14 & 6 & 70 \\ 2 & 14 & 9 & 97 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

$$m \times n \qquad\qquad m \times r \qquad\qquad r \times n$$

The two special solutions to $\boldsymbol{Ax = 0}$

$$\boldsymbol{Rs_1 = 0} \qquad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{-7} \\ \mathbf{1} \\ 0 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put **1** and **0** in positions 2 and 4

$$\boldsymbol{Rs_2 = 0} \qquad \begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} \mathbf{-8} \\ \mathbf{0} \\ \mathbf{-9} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put **0** and **1** in positions 2 and 4

Special solutions to $\begin{bmatrix} I & F \end{bmatrix} \boldsymbol{x} = \mathbf{0}$ are columns of $\begin{bmatrix} -F \\ I \end{bmatrix}$ in Example 1

Special solutions to $\begin{bmatrix} I & F \end{bmatrix} P \boldsymbol{x} = \mathbf{0}$ are cols of $P^{\mathrm{T}} \begin{bmatrix} -F \\ I \end{bmatrix}$ in Example 2

$$\begin{bmatrix} I & F \end{bmatrix} P \text{ times } P^{\mathrm{T}} \begin{bmatrix} -F \\ I \end{bmatrix} \text{ reduces to } \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix} \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

Suppose $A\boldsymbol{x} = \mathbf{0}$ has more unknowns than equations ($\boldsymbol{n} > \boldsymbol{m}$).

There must be **at least $\boldsymbol{n} - \boldsymbol{m}$ free columns in $\boldsymbol{F}$**

$\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has nonzero solutions in the nullspace of A

$$\boldsymbol{A}=\begin{bmatrix}1 & 2\\3 & 8\end{bmatrix} \quad \boldsymbol{B}=\begin{bmatrix}A\\2A\end{bmatrix}=\begin{bmatrix}1 & 2\\3 & 8\\2 & 4\\6 & 16\end{bmatrix} \quad \boldsymbol{M}=\begin{bmatrix}A & 2A\end{bmatrix}=\begin{bmatrix}1 & 2 & 2 & 4\\3 & 8 & 6 & 16\end{bmatrix}.$$

Row space dimension = rank $\boldsymbol{r} = 2$

Nullspace dimension = rank $\boldsymbol{n} - \boldsymbol{r}$

Elimination by block multiplication

$P_R A P_C = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{H} \\ \boldsymbol{J} & \boldsymbol{K} \end{bmatrix} \qquad C = \begin{bmatrix} W \\ J \end{bmatrix}$ & $B = \begin{bmatrix} W & H \end{bmatrix}$ have full rank r

Multiply r top rows by W^{-1} to get $W^{-1}B = \begin{bmatrix} I & W^{-1}H \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \end{bmatrix}$

Subtract $J\begin{bmatrix} I & W^{-1}H \end{bmatrix}$ from $m - r$ lower rows $\begin{bmatrix} J & K \end{bmatrix}$ to get $\begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix}$

$$\boxed{\boldsymbol{P_R A P_C} = \begin{bmatrix} W & H \\ J & K \end{bmatrix} \rightarrow \begin{bmatrix} I & W^{-1}H \\ J & K \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{I} & \boldsymbol{W^{-1}H} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \boldsymbol{R_0}}$$

Resource: A 2020 Vision of Linear Algebra
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