CHAPTER 11 VECTORS AND MATRICES

11.1 Vectors and Dot Products (page 405)

A vector has length and **direction**. If $\mathbf{v}$ has components 6 and -8, its length is $|\mathbf{v}| = \mathbf{10}$ and its direction vector is $\mathbf{u} = .6\mathbf{i} - .8\mathbf{j}$. The product of $|\mathbf{v}|$ with $\mathbf{u}$ is $\mathbf{v}$. This vector goes from (0,0) to the point $x = \mathbf{6}, y = -\mathbf{8}$. A combination of the coordinate vectors $\mathbf{i} = (\mathbf{1}, \mathbf{0})$ and $\mathbf{j} = (\mathbf{0}, \mathbf{1})$ produces $\mathbf{v} = \mathbf{x}\,\mathbf{i} + \mathbf{y}\,\mathbf{j}$.

To add vectors we add their **components**. The sum of $(6, -8)$ and (1,0) is $(\mathbf{7}, -\mathbf{8})$. To see $\mathbf{v} + \mathbf{i}$ geometrically, put the **tail** of $\mathbf{i}$ at the **head** of $\mathbf{v}$. The vectors form a **parallelogram with diagonal** $\mathbf{v} + \mathbf{i}$. (The other diagonal is $\mathbf{v} - \mathbf{i}$). The vectors $2\mathbf{v}$ and $-\mathbf{v}$ are $(\mathbf{12}, -\mathbf{16})$ and $(-\mathbf{6}, \mathbf{8})$. Their lengths are **20** and **10**.

In a space without axes and coordinates, the tail of $\mathbf{V}$ can be placed **anywhere**. Two vectors with the same **components or the same length and direction** are the same. If a triangle starts with $\mathbf{V}$ and continues with $\mathbf{W}$, the third side is $\mathbf{V} + \mathbf{W}$. The vector connecting the midpoint of $\mathbf{V}$ to the midpoint of $\mathbf{W}$ is $\frac{1}{2}(\mathbf{V} + \mathbf{W})$. That vector is **half of** the third side. In this coordinate-free form the dot product is $\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}||\mathbf{W}| \cos\theta$.

Using components, $\mathbf{V} \cdot \mathbf{W} = \mathbf{V_1W_1} + \mathbf{V_2W_2} + \mathbf{V_3W_3}$ and $(1,2,1) \cdot (2,-3,7) = \mathbf{3}$. The vectors are perpendicular if $\mathbf{V} \cdot \mathbf{W} = \mathbf{0}$. The vectors are parallel if $\mathbf{V}$ **is a multiple of** $\mathbf{W}$. $\mathbf{V} \cdot \mathbf{V}$ is the same as $|\mathbf{V}|^2$. The dot product of $\mathbf{U} + \mathbf{V}$ with $\mathbf{W}$ equals $\mathbf{U} \cdot \mathbf{W} + \mathbf{V} \cdot \mathbf{W}$. The angle between $\mathbf{V}$ and $\mathbf{W}$ has $\cos\theta = \mathbf{V} \cdot \mathbf{W}/|\mathbf{V}||\mathbf{W}|$. When $\mathbf{V} \cdot \mathbf{W}$ is negative then θ is **greater than 90°**. The angle between $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$ is $\pi/3$ **with cosine** $\frac{1}{2}$. The Cauchy-Schwarz inequality is $|\mathbf{V} \cdot \mathbf{W}| \le |\mathbf{V}||\mathbf{W}|$, and for $\mathbf{V} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{i} + \mathbf{k}$ it becomes $\mathbf{1} \le \mathbf{2}$.

1 $(0,0,0); (5,5,5); 3; -3; \cos\theta = -1$ **3** $2\mathbf{i} - \mathbf{j} - \mathbf{k}; -\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}; 6; 1; \cos\theta = \frac{1}{6}$

5 $(v_2, -v_1); (v_2, -v_1, 0), (v_3, 0, -v_1)$ **7** (0,0);(0,0,0) **9** Cosine of θ; projection of $\mathbf{w}$ on $\mathbf{v}$

11 F;T;F **13** Zero; sum = 10 o'clock vector; sum = 8 o'clock vector times $\frac{1+\sqrt{3}}{2}$

15 45° **17** Circle $x^2 + y^2 = 4; (x-1)^2 + y^2 = 4$; vertical line $x = 2$; half-line $x \ge 0$

19 $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}, \mathbf{w} = 2\mathbf{i} - \mathbf{j}; \mathbf{i} = 4\mathbf{v} - \mathbf{w}$ **21** $d = -6; C = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

23 $\cos\theta = \frac{1}{\sqrt{3}}; \cos\theta = \frac{2}{\sqrt{6}}; \cos\theta = \frac{1}{3}$ **25** $\mathbf{A} \cdot (\mathbf{A} + \mathbf{B}) = 1 + \mathbf{A} \cdot \mathbf{B} = 1 + \mathbf{B} \cdot \mathbf{A} = \mathbf{B} \cdot (\mathbf{A} + \mathbf{B})$; equilateral, 60°

27 $a = \mathbf{A} \cdot \mathbf{I}, b = \mathbf{A} \cdot \mathbf{J}$ **29** $(\cos t, \sin t)$ and $(-\sin t, \cos t)$; $(\cos 2t, \sin 2t)$ and $(-2\sin 2t, 2\cos 2t)$

31 $\mathbf{C} = \mathbf{A} + \mathbf{B}, \mathbf{D} = \mathbf{A} - \mathbf{B}; \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{B} = r^2 - r^2 = 0$

33 $\mathbf{U} + \mathbf{V} - \mathbf{W} = (2,5,8), \mathbf{U} - \mathbf{V} + \mathbf{W} = (0,-1,-2), -\mathbf{U} + \mathbf{V} + \mathbf{W} = (4,3,6)$

35 c and $\sqrt{a^2 + b^2}$; b/a and $\sqrt{a^2 + b^2 + c^2}$

37 $\mathbf{M}_1 = \frac{1}{2}\mathbf{A} + \mathbf{C}, \mathbf{M}_2 = \mathbf{A} + \frac{1}{2}\mathbf{B}, \mathbf{M}_3 = \mathbf{B} + \frac{1}{2}\mathbf{C}; \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \frac{3}{2}(\mathbf{A} + \mathbf{B} + \mathbf{C}) = \mathbf{0}$

39 $8 \le 3 \cdot 3; 2\sqrt{xy} \le x + y$ **41** Cancel a^2c^2 and b^2d^2; then $b^2c^2 + a^2d^2 \ge 2abcd$ because $(bc - ad)^2 \ge 0$

43 F; T; T; F **45** all $2\sqrt{2}; \cos\theta = -\frac{1}{3}$

2 $\mathbf{V} + \mathbf{W} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}; 2\mathbf{V} - 3\mathbf{W} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}; |\mathbf{V}|^2 = 2; \mathbf{V} \cdot \mathbf{W} = 1; \cos\theta = \frac{1}{2}$

4 $\mathbf{V} + \mathbf{W} = (2,3,4,5); 2\mathbf{V} - 3\mathbf{W} = (-1,-4,-7,-10); |\mathbf{V}|^2 = 4; \mathbf{V} \cdot \mathbf{W} = 10; \cos\theta = \frac{10}{2\sqrt{30}}$

6 $(0,0,1)$ and $(1,-1,0)$

8 Unit vectors $\frac{1}{\sqrt{3}}(1,1,1); \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}); \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}); (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

10 $(\cos\theta, \sin\theta)$ and $(\cos\theta, -\sin\theta)$; $(r\cos\theta, r\sin\theta)$ and $(r\cos\theta, -r\sin\theta)$.

12 We want $\mathbf{V}\cdot(\mathbf{W}-c\mathbf{V})=0$ or $\mathbf{V}\cdot\mathbf{W}=c\mathbf{V}\cdot\mathbf{V}$. Then $c=\frac{6}{3}=2$ and $\mathbf{W}-c\mathbf{V}=(-1,0,1)$.

14 (a) Try two possibilities: keep clock vectors 1 through 5 or 1 through 6. The five add to $1+2\cos 30°+2\cos 60° = 2\sqrt{3} = 3.73$ (in the direction of 3:00). The six add to $2\cos 15° + 2\cos 45° + 2\cos 75° = 3.86$ which is longer (in the direction of 3:30). (b) The 12 o'clock vector (call it $\mathbf{j}$ because it is vertical) is subtracted from all twelve clock vectors. So the sum changes from $\mathbf{V}=0$ to $\mathbf{V}^* = -12\mathbf{j}$.

16 (a) The angle between these unit vectors is $\theta-\phi$ (or $\phi-\theta$), and the cosine is $\frac{\mathbf{u}_1\cdot\mathbf{u}_2}{1\cdot 1} = \cos\theta\cos\phi+\sin\theta\sin\phi$. (b) $\mathbf{u}_3 = (-\sin\phi, \cos\phi)$ is perpendicular to $\mathbf{u}_2$. Its angle with $\mathbf{u}_1$ is $\frac{\pi}{2}+\phi-\theta$, whose cosine is $-\sin(\theta-\phi)$. The cosine is also $\frac{\mathbf{u}_1\cdot\mathbf{u}_3}{1\cdot 1} = -\cos\theta\sin\phi+\sin\theta\cos\phi$. To get the formula $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$, take the further step of changing θ to $-\theta$.

18 (a) The points $t\mathbf{B}$ form **a line from the origin in the direction of B.** (b) $\mathbf{A}+t\mathbf{B}$ forms a **line from A in the direction of B.** (c) $s\mathbf{A}+t\mathbf{B}$ forms **a plane containing A and B.** (d) $\mathbf{v}\cdot\mathbf{A}=\mathbf{v}\cdot\mathbf{B}$ means $\frac{\cos\theta_1}{\cos\theta_2}$ = fixed number $\frac{|\mathbf{B}|}{|\mathbf{A}|}$ where θ_1 and θ_2 are the angles from $\mathbf{v}$ to $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{v}$ is on the plane through the origin that gives this fixed number. (If $|\mathbf{A}|=|\mathbf{B}|$ the plane bisects the angle between those vectors.)

20 The choice $\mathbf{Q}=(\frac{1}{2},\frac{1}{2})$ makes PQR a right angle because $Q-P=(\frac{1}{2},\frac{1}{2})$ is perpendicular to $R-Q=(-\frac{1}{2},\frac{1}{2})$. The other choices for Q lie on a **circle** whose diameter is PR. (From geometry: the diameter subtends a right angle from any point on the circle.) This circle has radius $\frac{1}{2}$ and center $\frac{1}{2}$; in Section 9.1 it was the circle $r=\sin\theta$.

22 If a boat has velocity $\mathbf{V}$ with respect to the water and the water has velocity $\mathbf{W}$ with respect to the land, then **the boat has velocity $\mathbf{V}+\mathbf{W}$ with respect to the land.** The speed is not $|\mathbf{V}|+|\mathbf{W}|$ but $|\mathbf{V}+\mathbf{W}|$.

24 For any triangle PQR the side PR is twice as long as the line AB connecting midpoints in Figure 11.4. (The triangle PQR is twice as big as the triangle AQB.) Similarly $|PR|=2|\mathbf{W}|$ based on the triangle PSR. Since $\mathbf{V}$ and $\mathbf{W}$ have equal length and are both parallel to PR, they are equal.

26 (a) $\mathbf{I}=(\cos\theta,\sin\theta)$ and $\mathbf{J}=(-\sin\theta,\cos\theta)$. (b) One answer is $\mathbf{I}=(\cos\theta,\sin\theta,0), \mathbf{J}=(-\sin\theta,\cos\theta,0)$ and $\mathbf{K}=\mathbf{k}$. A more general answer is $\mathbf{I}=\sin\phi(\cos\theta,\sin\theta,0), \mathbf{J}=\sin\phi(-\sin\theta,\cos\theta,0)$ and $\mathbf{K}=\cos\phi(0,0,1)$.

28 $\mathbf{I}\cdot\mathbf{J}=\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\cdot\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}=\frac{1-1}{2}=0$. Add $\mathbf{i}+\mathbf{j}=\sqrt{2}\mathbf{I}$ to $\mathbf{i}-\mathbf{j}=\sqrt{2}\mathbf{J}$ to find $\mathbf{i}=\frac{\sqrt{2}}{2}(\mathbf{I}+\mathbf{J})$. Substitute back to find $\mathbf{j}=\frac{\sqrt{2}}{2}(\mathbf{I}-\mathbf{J})$. Then $\mathbf{A}=2\mathbf{i}+3\mathbf{j}=\sqrt{2}(\mathbf{I}+\mathbf{J})+\frac{3\sqrt{2}}{2}(\mathbf{I}-\mathbf{J})=a\mathbf{I}+b\mathbf{J}$ with $a=\sqrt{2}+\frac{3\sqrt{2}}{2}$ and $b=\sqrt{2}-\frac{3\sqrt{2}}{2}$.

30 $|\mathbf{A}\cdot\mathbf{i}|^2+|\mathbf{A}\cdot\mathbf{j}|^2+|\mathbf{A}\cdot\mathbf{k}|^2=|\mathbf{A}|^2$. Check for $\mathbf{A}=(x,y,z): x^2+y^2+z^2=|\mathbf{A}|^2$.

32 The third figure has $PR=\mathbf{A}+\mathbf{B}$ and $QS=\mathbf{B}-\mathbf{A}$. Then $|PR|^2+|QS|^2=(\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B})+(\mathbf{B}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}-\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{A})$ which equals $2\mathbf{A}\cdot\mathbf{A}+2\mathbf{B}\cdot\mathbf{B}$ = sum of squares of the four side lengths.

34 The diagonals are $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}-\mathbf{A}$. Suppose $|\mathbf{A}+\mathbf{B}|^2=\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$ equals $|\mathbf{B}-\mathbf{A}|^2$ $=\mathbf{B}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$. After cancelling this is $4\mathbf{A}\cdot\mathbf{B}=0$ (note that $\mathbf{A}\cdot\mathbf{B}$ is the same as $\mathbf{B}\cdot\mathbf{A}$). The region is a **rectangle.**

36 $|\mathbf{A}+\mathbf{B}|^2=\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$. If this equals $\mathbf{A}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$ (and always $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$), then $2\mathbf{A}\cdot\mathbf{B}=0$. So $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

38 In Figure 11.4, the point P is $\frac{2}{3}$ of the way along all medians. For the vectors, this statement means $\mathbf{A}+\frac{2}{3}\mathbf{M}_3=\frac{2}{3}\mathbf{M}_2=-\mathbf{C}+\frac{2}{3}\mathbf{M}_1$. To prove this, substitute $-\mathbf{A}-\frac{1}{2}\mathbf{C}$ for $\mathbf{M}_3$ and $\mathbf{A}+\frac{1}{2}\mathbf{B}$ for $\mathbf{M}_2$ and $\mathbf{C}+\frac{1}{2}\mathbf{A}$ for $\mathbf{M}_1$. Then the statement becomes $\frac{1}{3}\mathbf{A}=\frac{1}{3}\mathbf{C}=\frac{2}{3}\mathbf{A}+\frac{1}{3}\mathbf{B}=-\frac{1}{3}\mathbf{C}+\frac{1}{3}\mathbf{A}$. This is true because

$\mathbf{B} = -\mathbf{A} - \mathbf{C}$.

40 Choose $\mathbf{W} = (1,1,1)$. Then $\mathbf{V} \cdot \mathbf{W} = V_1 + V_2 + V_3$. The Schwarz inequality $|\mathbf{V} \cdot \mathbf{W}|^2 \le |\mathbf{V}|^2|\mathbf{W}|^2$ is $(V_1 + V_2 + V_3)^2 \le 3(V_1^2 + V_2^2 + V_3^2)$.

42 $|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$ or $|\mathbf{C}| \le |\mathbf{A}| + |\mathbf{B}|$ says that any side length is less than **the sum of the other two side lengths.** Proof: $|A + B|^2 \le$ (using Schwarz for $A \cdot B$)$|A|^2 + 2|A||B| + |B|^2 = (|A| + |B|)^2$.

44 $|\mathbf{V} + \mathbf{W}| = |\mathbf{V}| + |\mathbf{W}|$ only if $\mathbf{V}$ and $\mathbf{W}$ are in the same direction: $\mathbf{W}$ is a multiple $c\mathbf{V}$ with $c \ge 0$. Given $\mathbf{V} = \mathbf{i} + 2\mathbf{k}$ this leads to $\mathbf{W} = c\,(\mathbf{i} + 2\,\mathbf{k})$ (for example $\mathbf{W} = 2\mathbf{i} + 4\mathbf{k}$).

46 (a) $\mathbf{V} = \mathbf{i} + \mathbf{j}$ has $\cos\theta = \frac{\mathbf{V} \cdot \mathbf{i}}{|\mathbf{V}||\mathbf{i}|} = \frac{1}{\sqrt{2}}$ (45° angle also with $\mathbf{j}$) (b) $\mathbf{V} = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$ has $\cos\theta = \frac{1}{2}$ (60° angle with $\mathbf{i}$ and $\mathbf{j}$) (c) $\mathbf{V} = \mathbf{i} + \mathbf{j} + c\mathbf{k}$ has $\cos\theta = \frac{1}{\sqrt{2+c^2}}$ which cannot be larger than $\frac{1}{\sqrt{2}}$ so an angle below 45° is impossible. (Alternative: If the angle from $\mathbf{i}$ to $\mathbf{V}$ is 30° and the angle from $\mathbf{V}$ to $\mathbf{j}$ is 30° then the angle from $\mathbf{i}$ to $\mathbf{j}$ will be $\le 60°$ which is false.)

11.2 Planes and Projections (page 414)

A plane in space is determined by a point $P_0 = (x_0, y_0, z_0)$ and a **normal** vector $\mathbf{N}$ with components (a, b, c). The point $P = (x, y, z)$ is on the plane if the dot product of $\mathbf{N}$ with $\mathbf{P} - \mathbf{P_0}$ is zero. (*That answer was not P!*) The equation of this plane is $a(\mathbf{x} - \mathbf{x_0}) + b(\mathbf{y} - \mathbf{y_0}) + c(\mathbf{z} - \mathbf{z_0}) = 0$. The equation is also written as $ax + by + cz = d$, where d equals $\mathbf{a}\mathbf{x_0} + \mathbf{b}\mathbf{y_0} + \mathbf{c}\mathbf{z_0}$ or $\mathbf{N} \cdot \mathbf{P_0}$. A parallel plane has the same $\mathbf{N}$ and a different $\mathbf{d}$. A plane through the origin has $d = \mathbf{0}$.

The equation of the plane through $P_0 = (2,1,0)$ perpendicular to $\mathrm{N} = (3,4,5)$ is $\mathbf{3x + 4y + 5z = 10}$. A second point in the plane is $P = (0,0,\mathbf{2})$. The vector from P_0 to P is $(\mathbf{-2}, \mathbf{-1}, \mathbf{2})$, and it is **perpendicular** to N. (Check by dot product). The plane through $P_0 = (2,1,0)$ perpendicular to the z axis has $\mathbf{N} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ and equation $\mathbf{z} = \mathbf{0}$.

The component of $\mathbf{B}$ in the direction of $\mathbf{A}$ is $|\mathbf{B}|\cos\theta$, where θ is the angle between the vectors. This is $\mathbf{A} \cdot \mathbf{B}$ divided by $|\mathbf{A}|$. The projection vector $\mathbf{P}$ is $|\mathbf{B}|\cos\theta$ times a **unit** vector in the direction of $\mathbf{A}$. Then $\mathbf{P} = (|\mathbf{B}|\cos\theta)(\mathbf{A}/|\mathbf{A}|)$ simplifies to $(\mathbf{A} \cdot \mathbf{B})\mathbf{A}/|\mathbf{A}|^2$. When $\mathbf{B}$ is doubled, $\mathbf{P}$ is **doubled.** When $\mathbf{A}$ is doubled, $\mathbf{P}$ **is not changed.** If $\mathbf{B}$ reverses direction, then $\mathbf{P}$ **reverses direction.** If $\mathbf{A}$ reverses direction, then $\mathbf{P}$ **stays the same.**

When $\mathbf{B}$ is a velocity vector, $\mathbf{P}$ represents the **velocity in the A direction.** When $\mathbf{B}$ is a force vector, $\mathbf{P}$ is **the force component along A.** The component of $\mathbf{B}$ perpendicular to $\mathbf{A}$ equals $\mathbf{B} - \mathbf{P}$. The shortest distance from (0,0,0) to the plane $ax + by + cz = d$ is along the **normal** vector. The distance from the origin is $|\mathrm{d}|/\sqrt{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2}$ and the point on the plane closest to the origin is $P = (\mathrm{da}, \mathrm{db}, \mathrm{dc})/(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2)$. The distance from $\mathbf{Q} = (x_1, y_1, z_1)$ to the plane is $|\mathrm{d} - \mathbf{a}\mathbf{x_1} - \mathbf{b}\mathbf{y_1} - \mathbf{c}\mathbf{z_1}|/\sqrt{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2}$.

1 (0,0,0) and $(2,-1,0)$; $\mathrm{N} = (1,2,3)$ **3** $(0,5,6)$ and (0,6,7); N = (1,0,0)

5 (1,1,1) and (1,2,2); N = (1,1,–1) **7** $x + y = 3$ **9** $x + 2y + z = 2$

11 Parallel if $\mathbf{N} \cdot \mathbf{V} = 0$; perpendicular if $\mathbf{V}$ = multiple of $\mathbf{N}$

13 $\mathbf{i}+\mathbf{j}+\mathbf{k}$ (vector between points) is not perpendicular to $\mathbf{N}$; $\mathbf{V}\cdot\mathbf{N}$ is not zero; plane through first three is $x+y+z=1$; $x+y-z=3$ succeeds; right side must be zero

15 $ax+by+cz=0; a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ **17** $\cos\theta=\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{3},\frac{1}{3}$

19 $\frac{2}{36}\mathbf{A}$ has length $\frac{1}{3}$ **21** $\mathbf{P}=\frac{1}{2}\mathbf{A}$ has length $\frac{1}{2}|\mathbf{A}|$ **23** $\mathbf{P}=-\mathbf{A}$ has length $|\mathbf{A}|$ **25** $\mathbf{P}=\mathbf{O}$

27 Projection on $\mathbf{A}=(1,2,2)$ has length $\frac{5}{3}$; force down is 4; mass moves in the direction of $\mathbf{F}$

29 $|\mathbf{P}|_{\min}=\frac{5}{|\mathbf{N}|}$ = distance from plane to origin **31** Distances $\frac{1}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$ both reached at $(\frac{1}{3},\frac{1}{3},-\frac{1}{3})$

33 $\mathbf{i}+\mathbf{j}+\mathbf{k}; t=-\frac{4}{3}; (\frac{2}{3},-\frac{1}{3},-\frac{1}{3}); \frac{4}{\sqrt{3}}$

35 Same $\mathbf{N}=(2,-2,1)$; for example $\mathbf{Q}=(0,0,1)$; then $\mathbf{Q}+\frac{2}{9}\mathbf{N}=(\frac{4}{9},-\frac{4}{9},\frac{11}{9})$ is on second plane; $\frac{2}{9}|\mathbf{N}|=\frac{2}{3}$

37 $3\mathbf{i}+4\mathbf{j}$; $(3t,4t)$ is on the line if $3(3t)+4(4t)=10$ or $t=\frac{10}{25}$; $P=(\frac{30}{25},\frac{40}{25}), |P|=2$

39 $2x+2(\frac{10}{4}-\frac{3}{4}x)(-\frac{3}{4})=0$ so $x=\frac{30}{25}=\frac{6}{5}$; $3x+4y=10$ gives $y=\frac{8}{5}$

41 Use equations (8) and (9) with $\mathbf{N}=(a,b)$ and $\mathbf{Q}=(x_1,y_1)$ **43** $t=\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}$; $\mathbf{B}$ onto $\mathbf{A}$

45 $aVL=\frac{1}{2}\mathbf{L}_I-\frac{1}{2}\mathbf{L}_{III}; aVF=\frac{1}{2}\mathbf{L}_{II}+\frac{1}{2}\mathbf{L}_{III}$

47 $\mathbf{V}\cdot\mathbf{L}_I=2-1; \mathbf{V}\cdot\mathbf{L}_{II}=-3-1, \mathbf{V}\cdot\mathbf{L}_{III}=-3-2$; thus $\mathbf{V}\cdot 2\mathbf{i}=1, \mathbf{V}\cdot(\mathbf{i}-\sqrt{3}\mathbf{j})=-4$, and $\mathbf{V}=\frac{1}{2}\mathbf{i}+\frac{3\sqrt{3}}{2}\mathbf{j}$

2 $P=(6,0,0)$ and $P_0=(0,0,2)$ are on the plane, and $\mathbf{N}=(1,2,3)$ is normal. Check $\mathbf{N}\cdot(P-P_0)=(1,2,3)\cdot(6,0,-2)=0$.

4 $P=(1,1,2)$ and $P_0=(0,0,0)$ give $P-P_0$ perpendicular to $\mathbf{N}=\mathbf{i}+\mathbf{j}-\mathbf{k}$. (The plane is $x+y-z=0$ and P lies on this plane.)

6 The plane $y-z=0$ contains the given points (0,0,0) and (1,0,0) and (0,1,1). The normal vector is $\mathbf{N}=\mathbf{j}-\mathbf{k}$. (Certainly $P=(0,1,1)$ and $P_0=(0,0,0)$ give $\mathbf{N}\cdot(P-P_0)=0$.)

8 $P=(x,y,z)$ lies on the plane if $\mathbf{N}\cdot(P-P_0)=1(x-1)+2(y-2)-1(z+1)=0$ or $\mathbf{x}+\mathbf{2y}-\mathbf{z}=\mathbf{4}$.

10 $x+y+z=x_0+y_0+z_0$ or $(x-x_0)+(y-y_0)+(z-z_0)=0$.

12 (a) **No**: the line where the planes (or walls) meet is not perpendicular to itself. (b) A third plane perpendicular to the first plane could make **any angle** with the second plane.

14 The normal vector to $3x+4y+7z-t=0$ is $\mathbf{N}=(3,4,7,-1)$. The points $P=(1,0,0,3)$ and $Q=(0,1,0,4)$ are on the hyperplane. Check $(P-Q)\cdot\mathbf{N}=(1,-1,0,-1)\cdot(3,4,7,-1)=0$.

16 A curve in 3D is the intersection of **two** surfaces. A line in 4D is the intersection of **three** hyperplanes.

18 If the vector $\mathbf{V}$ makes an angle θ with a plane, it makes an angle $\frac{\pi}{2}-\theta$ with the normal $\mathbf{N}$. Therefore $\frac{\mathbf{V}\cdot\mathbf{N}}{|\mathbf{V}||\mathbf{N}|}=\cos(\frac{\pi}{2}-\theta)=\sin\theta$. The normal to the xy plane is $\mathbf{N}=\mathbf{k}$, so $\sin\theta=\frac{\sqrt{2}}{\sqrt{4}}=\frac{\sqrt{2}}{2}$ and $\theta=\frac{\pi}{4}$.

20 The projection $\mathbf{P}=\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}$ is $\frac{2}{2}\mathbf{A}=(1,-1,0)$. Its length is $|\mathbf{P}|=\sqrt{2}$. Here the projection onto $\mathbf{A}$ equals $\mathbf{A}$!

22 If $\mathbf{B}$ makes a $60°$ angle with $\mathbf{A}$ then the length of $\mathbf{P}$ is $|\mathbf{B}|\cos 60°=2\cdot\frac{1}{2}=1$. Since $\mathbf{P}$ is in the direction of $\mathbf{A}$ it must be $\frac{\mathbf{A}}{|\mathbf{A}|}$.

24 The projection is $\mathbf{P}=\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}A=\frac{1}{2}(\mathbf{i}+\mathbf{j})$. Its length is $|\mathbf{P}|=\frac{\sqrt{2}}{2}$.

26 $\mathbf{A}$ is along $\mathbf{N}=(1,-1,1)$ so the projection of $\mathbf{B}=(1,1,5)$ is $\mathbf{P}=\frac{\mathbf{N}\cdot\mathbf{B}}{|\mathbf{N}|^2}\mathbf{N}=\frac{5}{3}(1,-1,1)$.

28 $\mathbf{P}=\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}$ and the perpendicular projection is $\mathbf{B}-\mathbf{P}$. The dot product $\mathbf{P}\cdot(\mathbf{B}-\mathbf{P})$ or $\mathbf{P}\cdot\mathbf{B}-\mathbf{P}\cdot\mathbf{P}$ is zero: $\frac{(\mathbf{A}\cdot\mathbf{B})^2}{|\mathbf{A}|^2}-\frac{(\mathbf{A}\cdot\mathbf{B})^2}{|\mathbf{A}|^4}\mathbf{A}\cdot\mathbf{A}=0$.

30 We need the **angle** between the jet's direction and the wind direction. If this angle is θ, the speed over land is $\mathbf{500}+\mathbf{50}\cos\theta$.

32 The points at distance 1 from the plane $x+2y+2z=3$ fill two parallel planes $\mathbf{x}+\mathbf{2y}+\mathbf{2z}=\mathbf{6}$ and

$\mathbf{x} + \mathbf{2y} + \mathbf{2z} = \mathbf{0}$. Check: The point (0,0,0) on the last plane is a distance $\frac{|d|}{|\mathbf{N}|} = \frac{3}{3} = 1$ from the plane $x + 2y + 2z = 3$.

34 The plane through (1, 1, 1) perpendicular to $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is $x + 2y + 2z = 5$. Its distance from (0, 0, 0) is $\frac{|d|}{|\mathbf{N}|} = \mathbf{\frac{5}{3}}$.

36 The distance is zero because **the two planes meet.** They are not parallel; their normal vectors (1, 1, 5) and (3, 2, 1) are in different directions.

38 The point $P = Q + t\mathbf{N} = (3 + t, 3 + 2t)$ lies on the line $x + 2y = 4$ if $(3 + t) + 2(3 + 2t) = 4$ or $9 + 5t = 4$ or $t = \mathbf{-1}$. Then $P = (\mathbf{2}, \mathbf{1})$.

40 The drug runner takes $\frac{1}{2}$ second to go the 4 meters. You have 5 meters to travel in the same $\frac{1}{2}$ second. Your speed must be **10 meters per second.** The projection of your velocity (a vector) onto the drug runner's velocity equals **the drug runner's velocity.**

42 The equation $ax + by + cz = d$ is equivalent to $\frac{\mathbf{a}}{\mathbf{d}}\mathbf{x} + \frac{\mathbf{b}}{\mathbf{d}}\mathbf{y} + \frac{\mathbf{c}}{\mathbf{d}}\mathbf{z} = \mathbf{1}$. So the three numbers $e = \frac{a}{d}, f = \frac{b}{d}, g = \frac{c}{d}$ determine the plane. (Note: We say that three points determine a plane. But that makes 9 coordinates! We only need the 3 numbers e, f, g determined by those 9 coordinates.)

44 Two planes $ax + by + cz = d$ and $ex + fy + gz = h$ are (a) parallel if the normal vector (a, b, c) is a multiple of (e, f, g) (b) perpendicular if the normal vectors are perpendicular (c) at a 45° angle if the normal vectors are at a 45° angle: $\frac{N_1 \cdot N_2}{|N_1||N_2|} = \frac{\sqrt{2}}{2}$.

46 The aVR lead is in the direction of $\mathbf{A} = -\mathbf{i} + \mathbf{j}$. The projection of $\mathbf{V} = 2\mathbf{i} - \mathbf{j}$ in this direction is $\mathbf{P} = \frac{\mathbf{A} \cdot \mathbf{V}}{|\mathbf{A}|^2}\mathbf{A} = \frac{-3}{2}(-\mathbf{i} + \mathbf{j}) = (\mathbf{\frac{3}{2}}, -\mathbf{\frac{3}{2}})$. The length of $\mathbf{P}$ is $\mathbf{\frac{3\sqrt{2}}{2}}$.

48 If $\mathbf{V}$ is perpendicular to $\mathbf{L}$, the reading on that lead is **zero**. If $\int \mathbf{V}(t)dt$ is perpendicular to $\mathbf{L}$ then $\int \mathbf{V}(t) \cdot \mathbf{L}dt = 0$. This is the area under $\mathbf{V}(t) \cdot \mathbf{L}$ (which is proportional to the reading on lead $\mathbf{L}$).

11.3 Cross Products and Determinants (page 423)

The cross product $\mathbf{A} \times \mathbf{B}$ is a **vector** whose length is $|\mathbf{A}||\mathbf{B}| \sin \theta$. Its direction is **perpendicular** to $\mathbf{A}$ and $\mathbf{B}$. That length is the area of a **parallelogram**, whose base is $|\mathbf{A}|$ and whose height is $|\mathbf{B}| \sin \theta$. When $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, the area is $|\mathbf{a_1b_2} - \mathbf{a_2b_1}|$. This equals a 2 by 2 **determinant**. In general $|\mathbf{A} \cdot \mathbf{B}|^2 + |\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2$.

The rules for cross products are $\mathbf{A} \times \mathbf{A} = \mathbf{0}$ and $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$ and $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$. In particular $\mathbf{A} \times \mathbf{B}$ needs the **right**-hand rule to decide its direction. If the fingers curl from $\mathbf{A}$ towards $\mathbf{B}$ (not more than 180°), then $\mathbf{A} \times \mathbf{B}$ points **along the right thumb**. By this rule $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.

The vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ have cross product $(\mathbf{a_2b_3} - \mathbf{a_3b_2})\mathbf{i} + (\mathbf{a_3b_1} - \mathbf{a_1b_3})\mathbf{j} + (\mathbf{a_1b_2} - \mathbf{a_2b_1})\mathbf{k}$. The vectors $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j}$ have $\mathbf{A} \times \mathbf{B} = -\mathbf{i} + \mathbf{j}$. (*This is also the 3 by 3 determinant* $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$.) Perpendicular to the plane containing (0,0,0), (1,1,1), (1,1,0) is the normal vector $\mathbf{N} = -\mathbf{i} + \mathbf{j}$. The area of the triangle with those three vertices is $\frac{1}{2}\sqrt{2}$, which is half the area of the parallelogram with fourth vertex at **(2, 2, 1)**.

Vectors **A, B, C** from the origin determine a **box**. Its volume $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ comes from a 3 by 3 **determinant**. There are six terms, **three** with a plus sign and **three** with minus. In every term each row and **column** is represented once. The rows (1,0,0), (0,0,1), and (0,1,0) have determinant $=-\mathbf{1}$. That box is a **cube**, but its sides form a **left-handed** triple in the order given.

If **A, B, C** lie in the same plane then $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is **zero**. For $\mathbf{A}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ the first row contains the letters **x,y,z**. So the plane containing **B** and **C** has the equation $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=0$. When $\mathbf{B}=\mathbf{i}+\mathbf{j}$ and $\mathbf{C}=\mathbf{k}$ that equation is $\mathbf{x}-\mathbf{y}=\mathbf{0}$. $\mathbf{B}\times\mathbf{C}$ is $\mathbf{i}-\mathbf{j}$.

A 3 by 3 determinant splits into **three** 2 by 2 determinants. They come from rows 2 and 3, and are multiplied by the entries in row 1. With **i, j, k** in row 1, this determinant equals the **cross** product. Its **j** component is $-(\mathbf{a_1b_3}-\mathbf{a_3b_1})$, including the **minus** sign which is easy to forget.

1 $\mathbf{0}$ **3** $3\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ **5** $-2\mathbf{i}+3\mathbf{j}-5\mathbf{k}$ **7** $27\,\mathbf{i}+12\,\mathbf{j}-17\,\mathbf{k}$

9 **A** perpendicular to **B**; **A, B, C** mutually perpendicular **11** $|\mathbf{A}\times\mathbf{B}|=\sqrt{2}, \mathbf{A}\times\mathbf{B}=\mathbf{j}-\mathbf{k}$ **13** $\mathbf{A}\times\mathbf{B}=\mathbf{0}$

15 $|\mathbf{A}\times\mathbf{B}|^2=(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2=(a_1b_2-a_2b_1)^2; \mathbf{A}\times\mathbf{B}=(a_1b_2-a_2b_1)\mathbf{k}$

17 T; T; F; T **19** $\mathbf{N}=(2,1,0)$ or $2\mathbf{i}+\mathbf{j}$ **21** $x-y+z=2$ so $\mathbf{N}=\mathbf{i}-\mathbf{j}+\mathbf{k}$

23 $[(1,2,1)-(2,1,1)]\times[(1,1,2)-(2,1,1)]=\mathbf{N}=\mathbf{i}+\mathbf{j}+\mathbf{k}; x+y+z=4$

25 $(1,1,1)\times(a,b,c)=\mathbf{N}=(c-b)\mathbf{i}+(a-c)\mathbf{j}+(b-a)\mathbf{k}$; points on a line if $a=b=c$ (many planes)

27 $\mathbf{N}=\mathbf{i}+\mathbf{j}$, plane $x+y=$ constant **29** $\mathbf{N}=\mathbf{k}$, plane $z=$ constant

31 $\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = x-y+z=0$ **33** $\mathbf{i}-3\mathbf{j}; -\mathbf{i}+3\mathbf{j}; -3\mathbf{i}-\mathbf{j}$ **35** $-1, 4, -9$

39 $+c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

41 $\text{area}^2=(\frac{1}{2}ab)^2+(\frac{1}{2}ac)^2+(\frac{1}{2}bc)^2=(\frac{1}{2}|\mathbf{A}\times\mathbf{B}|)^2$ when $\mathbf{A}=a\mathbf{i}-b\mathbf{j}, \mathbf{B}=a\mathbf{i}-c\mathbf{k}$

43 $A=\frac{1}{2}(2\cdot 1-(-1)1)=\frac{3}{2}$; fourth corner can be (3,3)

45 $a_1\mathbf{i}+a_2\mathbf{j}$ and $b_1\mathbf{i}+b_2\mathbf{j}$; $|a_1b_2-a_2b_1|$; $\mathbf{A}\times\mathbf{B}=\cdots+(a_1b_2-a_2b_1)\mathbf{k}$

47 $\mathbf{A}\times\mathbf{B}$; from Eq. (6), $(\mathbf{A}\times\mathbf{B})\times\mathbf{i}=-(a_3b_1-a_1b_3)\mathbf{k}+(a_1b_2-a_2b_1)\mathbf{j}; (\mathbf{A}\cdot\mathbf{i})\mathbf{B}-(\mathbf{B}\cdot\mathbf{i})\mathbf{A}=$
$a_1(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})-b_1(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})$

49 $\mathbf{N}=(Q-P)\times(R-P)=\mathbf{i}+\mathbf{j}+\mathbf{k}$; area $\frac{1}{2}\sqrt{3}$; $x+y+z=2$

2 $(\mathbf{i}\times\mathbf{j})\times\mathbf{i}=\mathbf{k}\times\mathbf{i}=\mathbf{j}$. **4** $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 2 & 3 & -1 \end{vmatrix} = \mathbf{i}(-6)+\mathbf{j}(+4)+\mathbf{k}(0)=-6\mathbf{i}+4\mathbf{j}$.

6 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i}(0)+\mathbf{j}(-2)+\mathbf{k}(-2)=-2\mathbf{j}-2\mathbf{k}$.

8 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \end{vmatrix} = 0\mathbf{i}+0\mathbf{j}+\mathbf{k}(-\cos^2\theta-\sin^2\theta)=-\mathbf{k}$.

10 (a) **True** ($\mathbf{A}\times\mathbf{B}$ is a vector, $\mathbf{A}\cdot\mathbf{B}$ is a number) (b) **True** (Equation (1) becomes $0=|\mathbf{A}|^2|\mathbf{B}|^2$ so $\mathbf{A}=\mathbf{0}$ or $\mathbf{B}=\mathbf{0}$) (c) **False**: $\mathbf{i}\times(\mathbf{j})=\mathbf{i}\times(\mathbf{i}+\mathbf{j})$

12 Equation (1) gives $|\mathbf{A}\times\mathbf{B}|^2+0^2=(2)(2)$ or $|\mathbf{A}\times\mathbf{B}|=\mathbf{2}$. Check: $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\1&-1&0\end{vmatrix}=-2\mathbf{k}.$

14 Equation (1) gives $|\mathbf{A}\times\mathbf{B}|^2+1^2=(2)(2)$ or $|\mathbf{A}\times\mathbf{B}|=\sqrt{\mathbf{3}}$. Check: $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&1&1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}.$

16 $|\mathbf{A}\times\mathbf{B}|^2=|\mathbf{A}|^2|\mathbf{B}|^2-(\mathbf{A}\cdot\mathbf{B})^2$ which is $(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$. Multiplying and simplifying leads to $(a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2+(a_2b_3-a_3b_2)^2$ which confirms $|\mathbf{A}\times\mathbf{B}|$ in eq. (6).

18 (a) In $\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$, set $\mathbf{B}$ equal to $\mathbf{A}$. Then $\mathbf{A}\times\mathbf{A}=-(\mathbf{A}\times\mathbf{A})$ and $\mathbf{A}\times\mathbf{A}$ must be zero. (b) The *converse*: Suppose the cross product of any vector with itself is zero. Then $(\mathbf{A}+\mathbf{B})\times(\mathbf{A}+\mathbf{B})=\mathbf{A}\times\mathbf{A}+\mathbf{B}\times\mathbf{A}+\mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{B}$ reduces to $0=\mathbf{B}\times\mathbf{A}+\mathbf{A}\times\mathbf{B}$ or $\mathbf{A}\times\mathbf{B}=-(\mathbf{B}\times\mathbf{A})$.

20 $\mathbf{N}=(3,0,4)$. **22** $\mathbf{N}=(1,1,1)\times(1,1,2)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\1&1&2\end{vmatrix}=\mathbf{i}-\mathbf{j}.$

24 These three points are on a line! The direction of the line is (1, 1, 1), so the plane has normal vector perpendicular to (1,1,1). Example: $\mathbf{N}=(1,-2,1)$ and plane $x-2y+z=0$.

26 The plane has normal $\mathbf{N}=(\mathbf{i}+\mathbf{j})\times\mathbf{k}=\mathbf{i}\times\mathbf{k}+\mathbf{j}\times\mathbf{k}=-\mathbf{j}+\mathbf{i}$. So the plane is $x-y=d$. If the plane goes through the origin, its equation is $x-y=0$.

28 $\mathbf{N}=\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$ makes a 60° angle with $\mathbf{i}$ and $\mathbf{j}$. (Note: A *plane* can't make 60° angles with those vectors, because $\mathbf{N}$ would have to make 30° angles. By Problem 11.1.46 this is impossible.)

30 $\frac{1}{2}$; $\frac{1}{6}$; $\frac{1}{24}$ **32** Right-hand triple: $\mathbf{i}$, $\mathbf{i}+\mathbf{j}$, $\mathbf{i}+\mathbf{j}+\mathbf{k}$; left-hand triple: $\mathbf{k}$, $\mathbf{j}+\mathbf{k}$, $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

34 $\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B})=b_1(a_2b_3-a_3b_2)+b_2(a_3b_1-a_1b_3)+b_3(a_1b_2-a_2b_1)=0.$

36 $\mathbf{A}\times\mathbf{B}=(\mathbf{A}+\mathbf{B})\times\mathbf{B}$ (because the extra $\mathbf{B}\times\mathbf{B}$ is zero); also $\frac{1}{2}(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B})=\frac{1}{2}\mathbf{A}\times\mathbf{A}-\frac{1}{2}\mathbf{B}\times\mathbf{A}+\frac{1}{2}\mathbf{A}\times\mathbf{B}-\frac{1}{2}\mathbf{B}\times\mathbf{B}=\mathbf{0}+\mathbf{A}\times\mathbf{B}-\mathbf{0}=\mathbf{A}\times\mathbf{B}$.

38 The six terms $-b_1a_2c_3+b_1a_3c_2+b_2a_1c_3-b_2a_3c_1-b_3a_1c_2+b_3a_2c_1$ equal the determinant.

40 Add up three parts: $(\mathbf{B}-\mathbf{A})\cdot(\mathbf{A}\times\mathbf{B})=0$ because $\mathbf{A}\times\mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{B}$; for the same reason $(\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}\times\mathbf{C})=-\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and $(\mathbf{B}-\mathbf{A})\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$. Add to get zero because $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ equals $\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$.

Changing the letters $\mathbf{A},\mathbf{B},\mathbf{C}$ to $\mathbf{B},\mathbf{C},\mathbf{A}$ and to $\mathbf{C},\mathbf{A},\mathbf{B}$, the vector $(\mathbf{A}\times\mathbf{B})+(\mathbf{B}\times\mathbf{C})+(\mathbf{C}\times\mathbf{A})$ stays the same. So this vector is perpendicular to $\mathbf{C}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$ as well as $\mathbf{B}-\mathbf{A}$.

42 The two sides going out from (a_1,b_1) are $(a_2-a_1)\mathbf{i}+(b_2-b_1)\mathbf{j}$ and $(a_3-a_1)\mathbf{i}+(b_3-b_1)\mathbf{j}$. The cross product of those sides gives the area of the parallelogram as $|(a_2-a_1)(b_3-b_1)-(a_3-a_1)(b_2-b_1)|$. Divide by 2 for the area of the triangle.

44 Area of triangle $=\frac{1}{2}\begin{vmatrix}a_1&b_1&1\\a_2&b_2&1\\a_3&b_3&1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2&1&1\\4&2&1\\1&2&1\end{vmatrix}=\frac{1}{2}(4+8+1-2-4-4)=\frac{\mathbf{3}}{\mathbf{2}}$. Note that expanding the first determinant produces the formula already verified in Problem 42.

46 (a) $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&-4\\-1&1&0\end{vmatrix}=4\mathbf{i}+4\mathbf{j}+2\mathbf{k}$. The inner products with $\mathbf{i},\mathbf{j},\mathbf{k}$ are $4,4,2$. (b) Square and add to find $|\mathbf{A}\times\mathbf{B}|^2=4^2+4^2+2^2=36$. This is the square of the parallelogram area.

48 The triple vector product in Problem 47 is $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$. Take the dot product with $\mathbf{D}$. The right side is easy: $(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})$. The left side is $((\mathbf{A}\times\mathbf{B})\times\mathbf{C})\cdot\mathbf{D}$ and the vectors $\mathbf{A}\times\mathbf{B},\mathbf{C},\mathbf{D}$ can be put in any cyclic order (see "useful facts" about volume of a box, after Theorem 11G). We choose $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$.

50 For a parallelogram choose S so that $S-R=Q-P$. Then $S=(2,3,3)$. The area is the length of the cross-product $(Q-P)\times(R-P)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&2\\1&2&2\end{vmatrix}=-2\mathbf{i}+2\mathbf{j}-\mathbf{k}$. Its length is $\sqrt{2^2+2^2+1^2}=3$.

One way to produce a box is to choose $T=P+S$ and $U=Q+S$ and $V=R+S$. (Then $STUV$ comes from shifting $OPQR$ by the vector S.) In that case the three edges from the origin are OP and OQ and OS. Find the determinant $\begin{vmatrix}1&0&-1\\1&1&1\\2&3&3\end{vmatrix}=3+0-3+2-3-0=-1$. Then the volume is the absolute value 1. Another box has edges OP,OQ,OR with the *same volume*.

11.4 Matrices and Linear Equations (page 433)

The equations $3x+y=8$ and $x+y=6$ combine into the vector equation $\mathbf{x}\begin{bmatrix}3\\1\end{bmatrix}+\mathbf{y}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}8\\6\end{bmatrix}=\mathbf{d}$. The left side is $A\mathbf{u}$ with coefficient matrix $A=\begin{bmatrix}3&1\\1&1\end{bmatrix}$ and unknown vector $\mathbf{u}=\begin{bmatrix}x\\y\end{bmatrix}$. The determinant of A is **2**, so this problem is not **singular**. The row picture shows two intersecting **lines**. The column picture shows $x\mathbf{a}+y\mathbf{b}=\mathbf{d}$, where $\mathbf{a}=\begin{bmatrix}3\\1\end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix}1\\1\end{bmatrix}$. The inverse matrix is $A^{-1}=\frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}$. The solution is $\mathbf{u}=A^{-1}\mathbf{d}=\begin{bmatrix}1\\5\end{bmatrix}$.

A matrix-vector multiplication produces a vector of dot **products** from the rows, and also a combination of the **columns**:

$$\begin{bmatrix}\mathbf{A}\\\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{u}\end{bmatrix}=\begin{bmatrix}\mathbf{A}\cdot\mathbf{u}\\\mathbf{B}\cdot\mathbf{u}\end{bmatrix},\quad\begin{bmatrix}\mathbf{a}&\mathbf{b}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\mathbf{a}+y\mathbf{b}\end{bmatrix},\quad\begin{bmatrix}3&1\\1&1\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}8\\6\end{bmatrix}.$$

If the entries are a, b, c, d, the determinant is $D=\mathbf{ad}-\mathbf{bc}$. A^{-1} is $\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ divided by D. Cramer's Rule shows components of $\mathbf{u}=A^{-1}\mathbf{d}$ as ratios of determinants: $x=(\mathbf{b_2d_1}-\mathbf{b_1d_2})/D$ and $y=(\mathbf{a_1d_2}-\mathbf{a_2d_1})/D$.

A matrix-matrix multiplication MV yields a matrix of dot products, from the rows of M and the columns of $\mathbf{V}$:

$$\begin{bmatrix}\mathbf{A}\\\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1&\mathbf{v}_2\end{bmatrix}=\begin{bmatrix}\mathbf{A}\cdot\mathbf{v}_1&\mathbf{A}\cdot\mathbf{v}_2\\\mathbf{B}\cdot\mathbf{v}_1&\mathbf{B}\cdot\mathbf{v}_2\end{bmatrix}\qquad\begin{bmatrix}3&1\\1&1\end{bmatrix}\begin{bmatrix}1&2\\5&6\end{bmatrix}=\begin{bmatrix}8&12\\6&8\end{bmatrix}$$

$$\begin{bmatrix}3&1\\1&1\end{bmatrix}\begin{bmatrix}1/2&-1/2\\-1/2&3/2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\qquad\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}A\end{bmatrix}=\begin{bmatrix}A\end{bmatrix}.$$

The last line contains the **identity** matrix, denoted by I. It has the property that $IA=AI=\mathbf{A}$ for every matrix A, and $I\mathbf{u}=\mathbf{u}$ for every vector $\mathbf{u}$. The inverse matrix satisfies $A^{-1}A=\mathbf{I}$. Then $A\mathbf{u}=\mathbf{d}$ is solved by multiplying both sides by $\mathbf{A^{-1}}$, to give $\mathbf{u}=\mathbf{A^{-1}d}$. There is no inverse matrix when $\det\mathbf{A}=0$.

The combination $x\mathbf{a}+y\mathbf{b}$ is the projection of $\mathbf{d}$ when the error $\mathbf{d}-\mathbf{xa}-\mathbf{yb}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. If

$\mathbf{a} = (1,1,1)$, $\mathbf{b} = (1,2,3)$, and $\mathbf{d} = (0,8,4)$, the equations for x and y are $\mathbf{3x + 6y = 12}$ **and** $\mathbf{6x + 14y = 28}$. Solving them also gives the closest **line** to the data points (1,0), **(2,8)**, and (3,4). The solution is $x = 0, y = 2$, which means the best line is **horizontal**. The projection is $0\mathbf{a} + 2\mathbf{b} = \mathbf{(2,4,6)}$. The three error components are $(\mathbf{-2,4,-2})$. Check perpendicularity: $(\mathbf{1,1,1}) \cdot (\mathbf{-2,4,-2}) = 0$ and $(\mathbf{1,2,3}) \cdot (\mathbf{-2,4,-2}) = 0$. Applying calculus to this problem, x and y minimize the sum of squares $E = (\mathbf{-x-y})^2 + (\mathbf{8-x-2y})^2 + (\mathbf{4-x-3y})^2$.

1 $x = 5, y = 2, D = -2, \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **3** $x = 3, y = 1, \begin{bmatrix} 8 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix}, D = -8$

5 $x = 2y, y =$ anything, $D = 0, 2y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **7** no solution, $D = 0$

9 $x = \frac{1}{D}\begin{vmatrix} 8 & -1 \\ 0 & -3 \end{vmatrix} = \frac{-24}{-8} = 3, y = \frac{1}{D}\begin{bmatrix} 3 & 8 \\ 1 & 0 \end{bmatrix} = \frac{-8}{-8} = 1$ **11** $\frac{0}{0}$

15 $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad - bc = 1$ **17** Are parallel; multiple; the same; infinite

19 Multiples of each other; in the same direction as the columns; infinite

21 $d_1 = .34, d_2 = 4.91$ **23** $.96x + .02y = .58, .04x + .98y = 4.92; D = .94, x = .5, y = 5$

25 $a = 1$ gives any $x = -y; a = -1$ gives any $x = y$

27 $D = -2, A^{-1} = -\frac{1}{2}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix}; D = -8, (2A)^{-1} = \frac{1}{2}A^{-1}; D = \frac{1}{-2}, (A^{-1})^{-1} =$ original A;
$D = -2$ (not $+2$), $(-A)^{-1} = -A^{-1}; D = 1, I^{-1} = I$

29 $AB = \begin{bmatrix} 7 & 5 \\ 5 & 1 \end{bmatrix}, BA = \begin{bmatrix} 5 & 11 \\ 3 & 3 \end{bmatrix}, BC = \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}, CB = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$

31 $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}, \begin{matrix} aecf + aedh & + bgcf + bgdh \\ -afce - afdg & - bhce - bhdg \end{matrix} = (ad - bc)(eh - fg)$

33 $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & \frac{1}{2} \end{bmatrix}, B^{-1} = \begin{bmatrix} \frac{1}{2} & -1 \\ 0 & 1 \end{bmatrix}$ **35** Identity; $B^{-1}A^{-1}$ **37** Perpendicular; $\mathbf{u} = \mathbf{v} \times \mathbf{w}$

39 Line $4 + t$, errors $-1, 2, -1$ **41** $d_1 - 2d_2 + d_3 = 0$ **43** A^{-1} can't multiply **O** and produce **u**

2 $x = 5, y = 1; 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}; \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1.$ **4** Parallel lines (no solution); $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0.$

6 $x = 0, y = 1; 0\begin{bmatrix} 10 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{vmatrix} 10 & 1 \\ 1 & 1 \end{vmatrix} = 9.$

8 The solution is $x = \frac{d-b}{ad-bc}, y = \frac{a-c}{ad-bc}$ (ok to use Cramer's Rule) (solution breaks down if $ad = bc$);
$\frac{d-b}{ad-bc}\begin{bmatrix} a \\ c \end{bmatrix} + \frac{a-c}{ad-bc}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

10 In Problem 4, $x = \dfrac{\begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \dfrac{-2}{0}$ and $y = \dfrac{\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \dfrac{1}{0}$ (no solution)

12 With $A = I$ the equations are $\begin{matrix} 1x + 0y = d_1 \\ 0x + 1y = d_2 \end{matrix}$. Then $x = \dfrac{\begin{vmatrix} d_1 & 0 \\ d_2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = d_1$ and $y = \dfrac{\begin{vmatrix} 1 & d_1 \\ 0 & d_2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = d_2.$

14 Row picture: $10x + y = 1$ and $x + y = 1$ intersect at $(0, 1)$. Column picture: Add $0\,[\begin{smallmatrix} 10 \\ 1 \end{smallmatrix}]$ and $1\,[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}]$.

16 If $ad-bc=1$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

18 $x = \dfrac{\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}} = \dfrac{-2}{0}$ (no solution); $x = \dfrac{\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}} = \dfrac{0}{0}$ and $y = \dfrac{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}} = \dfrac{0}{0}$. In Cramer's Rule this $\dfrac{0}{0}$ signals that a solution might (or might not) exist.

20 $x-y=d_1$ and $9x-9y=d_2$ can be solved if $\mathbf{d_2 = 9d_1}$.

22 Problem 21 is $\begin{array}{l} .96x+.02y=d_1 \\ .04x+.98y=d_2. \end{array}$ The sums down the columns of A are $.96+.04=1$ and $.02+.98=1$. Reason: Everybody has to be accounted for. Nobody is lost or gained. Then $x+y$ (total population before move) equals d_1+d_2 (total population after move).

24 Determinant of $A = \begin{vmatrix} .96 & .02 \\ .04 & .98 \end{vmatrix} = \mathbf{.94}$; $A^{-1} = \frac{1}{.94}\begin{bmatrix} .98 & -.02 \\ -.04 & .96 \end{bmatrix}$ (columns still add to 1); $A^{-1}A = I$.

26 $x=0, y=0$ always solves $ax+by=0$ and $cx+dy=0$ (these lines always go through the origin). There are other solutions if the two lines are **the same**. This happens if $\mathbf{ad = bc}$.

28 Determinant of $A^{-1} = \frac{d}{ad-bc}\frac{a}{ad-bc} - \frac{(-b)}{ad-bc}\frac{(-c)}{ad-bc} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc}$. Therefore $\det A^{-1} = \frac{1}{\det A}$.

30 (a) $|A|=-9$; $|B|=2$; $|AB|=-18$; $|BA|=-18$.

(b) (determinant of BC) equals (determinant of B) times (determinant of C).

32 $\begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$ does not equal $\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$ Example of equality: $\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix}$.

34 $AB = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$ has determinant 4 so $(AB)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$. Check that this is also $B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$.

36 $C^{-1}B^{-1}A^{-1}ABC$ equals the **identity** matrix (because it collapses to $C^{-1}B^{-1}BC$ which is $C^{-1}C$ which is I). Then the inverse of ABC is $\mathbf{C^{-1}B^{-1}A^{-1}}$.

38 (a) Find x and y from the normal equations. First compute $\mathbf{a}\cdot\mathbf{a}=3$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}=0$ and $\mathbf{b}\cdot\mathbf{b}=2$ and $\mathbf{a}\cdot\mathbf{d}=12$ and $\mathbf{b}\cdot\mathbf{d}=2$. The normal equations $\begin{array}{l} 3x+0y=12 \\ 0x+2y=2 \end{array}$ give $x=\mathbf{4}, y=\mathbf{1}$.

(b) The projection $\mathbf{p}=x\mathbf{a}+y\mathbf{b}$ equals $4(1,1,1)+1(-1,0,1)=(3,4,5)$. Error $\mathbf{d}-\mathbf{p}=(-1,2,-1)$. Check perpendicularity of error: $(-1,2,-1)\cdot(1,1,1)=0$ and $(-1,2,-1)\cdot(-1,0,1)=0$.

40 Compute $\mathbf{a}\cdot\mathbf{a}=3$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}=2$ and $\mathbf{b}\cdot\mathbf{b}=6$ and $\mathbf{a}\cdot\mathbf{d}=5$ and $\mathbf{b}\cdot\mathbf{d}=6$. The normal equation (14) is $\begin{array}{l} 3x+2y=5 \\ 2x+6y=6 \end{array}$ with solution $x=\mathbf{\frac{18}{14}}=\mathbf{\frac{9}{7}}$ and $y=\mathbf{\frac{8}{14}}=\mathbf{\frac{4}{7}}$. The nearest combination $x\mathbf{a}+y\mathbf{b}$ is $\mathbf{p}=(\frac{5}{7},\frac{13}{7},\frac{17}{7})$. The vector of three errors is $\mathbf{d}-\mathbf{p}=(\frac{2}{7},-\frac{6}{7},\frac{4}{7})$. It is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. The best straight line is $f=x+yt=\frac{9}{7}+\frac{4}{7}t$.

42 $M=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

44 Suppose $\mathbf{u}\neq\mathbf{0}$ but $A\mathbf{u}=\mathbf{0}$. Then A^{-1} can't exist. It would multiply $\mathbf{0}$ (the zero vector) and produce $\mathbf{u}$.

11.5 Linear Algebra (page 443)

Three equations in three unknowns can be written as $A\mathbf{u}=\mathbf{d}$. The **vector u** has components x, y, z and A is a **3 by 3 matrix**. The row picture has a **plane** for each equation. The first two planes intersect in a **line**, and all three planes intersect in a **point**, which is **u**. The column picture starts with vectors **a**,**b**,**c** from the columns of **A** and combines them to produce $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. The vector equation is $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$.

The determinant of A is the triple product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. This is the volume of a box, whose edges from the origin are $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If det $A = \mathbf{0}$ then the system is **singular**. Otherwise there is an **inverse** matrix such that $A^{-1}A = \mathbf{I}$ (the **identity** matrix). In this case the solution to $A\mathbf{u} = \mathbf{d}$ is $\mathbf{u} = \mathbf{A^{-1}d}$.

The rows of A^{-1} are the cross products $\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}, \mathbf{a} \times \mathbf{b}$, divided by D. The entries of A^{-1} are 2 by 2 **determinants**, divided by D. The upper left entry equals $(\mathbf{b_2c_3} - \mathbf{b_3c_2})/\mathbf{D}$. The 2 by 2 determinants needed for a row of A^{-1} do not use the corresponding **column** of A.

The solution is $\mathbf{u} = A^{-1}\mathbf{d}$. Its first component x is a ratio of determinants, $|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|$ divided by $|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|$. Cramer's Rule breaks down when det $A = \mathbf{0}$. Then the columns **a**,**b**,**c** lie in the same **plane**. There is no solution to $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$, if **d** is not on that **plane**. In a singular row picture, the intersection of planes 1 and 2 is **parallel** to the third plane.

In practice **u** is computed by **elimination**. The algorithm starts by subtracting a multiple of row 1 to eliminate x from **the second equation**. If the first two equations are $x - y = 1$ and $3x + z = 7$, this elimination step leaves $\mathbf{3y + z = 4}$. Similarly x is eliminated from the third equation, and then **y** is eliminated. The equations are solved by back **substitution**. When the system has no solution, we reach an impossible equation like $\mathbf{1} = \mathbf{0}$. The example $x - y = 1, 3x + z = 7$ has no solution if the third equation is $\mathbf{3y + z = 5}$.

1 $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 5 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$ **3** $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

5 det $A = 0$, add 3 equations $\rightarrow 0 = 1$ **7** $5\mathbf{a} + 1\mathbf{b} + 0\mathbf{c} = \mathbf{d}$, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

9 $\mathbf{b} \times \mathbf{c}; \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = 0$; determinant is zero **11** 6, 2, 0; product of diagonal entries

13 $A^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, B^{-1} = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ 0 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **15** Zero; same plane; D is zero

17 $\mathbf{d} = (1, -1, 0); \mathbf{u} = (1, 0, 0)$ or (7,3,1) **19** $AB = \begin{bmatrix} 8 & 4 & 1 \\ 40 & 26 & 0 \\ 18 & 12 & 0 \end{bmatrix}$, det $AB = 12 =$ (det A) times (det B)

21 $A + C = \begin{bmatrix} 2 & 3 & -3 \\ -1 & 4 & 6 \\ 0 & -1 & 6 \end{bmatrix}$, det$(A + C)$ is not det A + det C

23 $p = \frac{(2)(3)-(0)(6)}{6} = 1, q = \frac{-(4)(3)+(0)(0)}{6} = -2$ **25** $(A^{-1})^{-1}$ is always A

27 $-1, -1, 1, 1, ; (y, x, z), (z, y, x), (y, z, x), (z, x, y)$ **29** $2! = 2, 4! = 24$

31 $z = \frac{1}{2}, y = -\frac{3}{2}, x = 3; z = \frac{7}{2}, y = \frac{3}{2}, x = -\frac{1}{2}$

33 New second equation $3z = 0$ doesn't contain y; exchange with third equation; there is a solution

35 Pivots 1,2,4, $D = 8$; pivots $1, -1, 2, D = -2$ **37** $a_{12} = 1, a_{21} = 0, \sum a_{ij}b_{jk} =$ row i, column k in AB

39 $a_{11}a_{22} - a_{12}a_{21} \neq 0; D = 0$

2 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **4** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$

6 By inspiration $(x, y, z) = (1, -1, 1)$. By Cramer's Rule: det $A = -1$ and then

$$x = \tfrac{1}{-1} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1, y = \tfrac{1}{-1} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1, z = \tfrac{1}{-1} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

8 $\begin{matrix} x + 2y + 2z = 0 \\ 2x + 3y + 5z = 0 \\ 2y + 2z = 8 \end{matrix} \rightarrow \begin{matrix} x + 2y + 2z = 0 \\ -y + z = 0 \\ 2y + 2z = 8 \end{matrix} \rightarrow \begin{matrix} x + 2y + 2z = 0 \\ -y + z = 0 \\ 4z = 8 \end{matrix} \rightarrow \begin{matrix} x = -8 \\ y = 2 \\ z = 2 \end{matrix}$

10 The plane $a_1x + b_1y + c_1z = d_1$ is perpendicular to $\mathbf{N_1} = (\mathbf{a_1}, \mathbf{b_1}, \mathbf{c_1})$. The second plane has $\mathbf{N_2} = (\mathbf{a_2}, \mathbf{b_2}, \mathbf{c_2})$. The planes meet in a line parallel to the **cross product** $\mathbf{N_1} \times \mathbf{N_2}$. If this line is parallel to the third plane the system is **singular**. The matrix has no **inverse**: $(\mathbf{N_1} \times \mathbf{N_2}) \cdot \mathbf{N_3} = 0$.

12 $\mathbf{a} \times \mathbf{b} = 2\mathbf{i}, \mathbf{a} \times \mathbf{c} = 6\mathbf{j} - 2\mathbf{k}, \mathbf{b} \times \mathbf{c} = 4\mathbf{j} - \mathbf{k}$.

14 $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ when $\begin{matrix} x = 1 \\ y = 0 \\ z = 0 \end{matrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 6 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ when $\begin{matrix} x = 0 \\ y = 0 \\ z = 1 \end{matrix}$.

The product $A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ automatically gives the first column of A^{-1}.

16 $\begin{matrix} x - y - 3z = 0 \\ -x + 2y = 0 \\ -y + 3z = 0 \end{matrix} \rightarrow \begin{matrix} x - y - 3z = 0 \\ y - 3z = 0 \\ -y + 3z = 0 \end{matrix} \rightarrow \begin{matrix} \mathbf{x = 6c} \\ \mathbf{y = 3c} \\ \mathbf{z = c} \end{matrix}$

18 Choose $\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as right side. The same steps as in Problem 16 end with $y - 3z = 0$ and $-y + 3z = 1$. Addition leaves $\mathbf{0 = 1}$. *No solution. Note:* The left sides of the three equations add to zero. There is a solution only if the right sides (components of **d**) also add to zero.

20 $BC = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -6 \\ 2 & 2 & -18 \end{bmatrix}$ and $CB = \begin{bmatrix} -20 & -13 & 1 \\ 4 & 2 & -1 \\ 16 & 11 & 0 \end{bmatrix}$. It is CB whose columns add to zero (they are combinations of columns of C, and those add to zero). BC and CB are singular because C is.

22 $2A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 4 & 12 \\ 0 & 0 & 6 \end{bmatrix}$ has determinant 48 which is **8** times det A. If an n by n matrix is multiplied by 2, the determinant is multiplied by 2^n. Here $2^3 = 8$.

24 The 2 by 2 determinants from the first two rows of B are $\mathbf{-1, -2}$, and $\mathbf{-1}$. These go into the third **column** of B^{-1}, after dividing by **det B = 2** and changing the sign of $\frac{-2}{2}$.

26 The inverse of AB is $\mathbf{B^{-1}A^{-1}}$. The inverses come in **reverse** order (last in – first out: shoes first!)

28 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ These are "**even**"

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ These are "**odd**"

30 The matrix PA has the same **rows** as A, permuted by P. The matrix AP has the same **columns** as A, permuted by P. Using P in Problem 27, the first two rows of A are exchanged in PA (two columns in AP.)