CHAPTER 8 APPLICATIONS OF THE INTEGRAL

8.1 Areas and Volumes by Slices (page 318)

The area between $y = x^3$ and $y = x^4$ equals the integral of $\mathbf{x^3 - x^4}$. If the region ends where the curves intersect, we find the limits on x by solving $\mathbf{x^3 = x^4}$. Then the area equals $\int_0^1 (\mathbf{x^3 - x^4})\mathbf{dx} = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. When the area between $y = \sqrt{x}$ and the y axis is sliced horizontally, the integral to compute is $\int \mathbf{y^2 dy}$.

In three dimensions the volume of a slice is its thickness dx times its **area**. If the cross-sections are squares of side $1 - x$, the volume comes from $\int (\mathbf{1 - x})^2\mathbf{dx}$. From $x = 0$ to $x = 1$, this gives the volume $\frac{1}{3}$ of a square **pyramid**. If the cross-sections are circles of radius $1 - x$, the volume comes from $\int \pi(\mathbf{1 - x})^2\mathbf{dx}$. This gives the volume $\frac{\pi}{3}$ of a circular **cone**.

For a solid of revolution, the cross-sections are **circles**. Rotating the graph of $y = f(x)$ around the x axis gives a solid volume $\int \pi(\mathbf{f(x)})^2\mathbf{dx}$. Rotating around the y axis leads to $\int \pi(\mathbf{f^{-1}(y)})^2\mathbf{dy}$. Rotating the area between $y = f(x)$ and $y = g(x)$ around the x axis, the slices look like **washers**. Their areas are $\pi(\mathbf{f(x)})^2 - \pi(\mathbf{g(x)})^2 = \mathbf{A(x)}$ so the volume is $\int \mathbf{A(x)dx}$.

Another method is to cut the solid into thin cylindrical **shells**. Revolving the area under $y = f(x)$ around the y axis, a shell has height $\mathbf{f(x)}$ and thickness dx and volume $\mathbf{2\pi x\, f(x)dx}$. The total volume is $\int \mathbf{2\pi x\, f(x)dx}$.

1 $x^2 - 3 = 1$ gives $x = \pm 2$; $\int_{-2}^{2}[(1 - (x^2 - 3)]dx = \frac{32}{3}$
3 $y^2 = x = 9$ gives $y = \pm 3$; $\int_{-3}^{3}[9 - y^2]dy = 36$
5 $x^4 - 2x^2 = 2x^2$ gives $x = \pm 2$ (or $x = 0$); $\int_{-2}^{2}[2x^2 - (x^4 - 2x^2)]dx = \frac{128}{15}$
7 $y = x^2 = -x^2 + 18x$ gives $x = 0, 9$; $\int_0^9[(-x^2 + 18x) - x^2]dx = 243$
9 $y = \cos\, x = \cos^2 x$ when $\cos x = 1$ or 0, $x = 0$ or $\frac{\pi}{2}$ or $\cdots$ $\int_0^{\pi/2}(\cos x - \cos^2 x)dx = 1 - \frac{\pi}{4}$
11 $e^x = e^{2x-1}$ gives $x = 1$; $\int_0^1[e^x - e^{2x-1}]dx = (e - 1) - (\frac{e-e^{-1}}{2})$
13 Intersections $(0,0), (1,3), (2,2)$; $\int_0^1[3x - x]dx + \int_1^2[4 - x - x]dx = 2$
15 Inside, since $1 - x^2 < \sqrt{1 - x^2}$; $\int_{-1}^{1}[\sqrt{1 - x^2} - (1 - x^2)]dx = \frac{\pi}{2} - \frac{4}{3}$
17 $V = \int_{-a}^{a} \pi y^2 dx = \int_{-a}^{a} \pi b^2(1 - \frac{x^2}{a^2})dx = \frac{4\pi b^2 a}{3}$; around y axis $V = \frac{4\pi a^2 b}{3}$; rotating $x = 2, y = 0$ around y axis gives a circle not in the first football
19 $V = \int_0^{\pi} 2\pi x \sin x\, dx = 2\pi^2$ **21** $\int_0^8 \pi(8 - x)^2 dx = \frac{512\pi}{3}$; $\int_0^8 2\pi x(8 - x)dx = \frac{512\pi}{3}$ (same cone tipped over)
23 $\int_0^1 \pi(x^4)^2 dx = \frac{\pi}{9}$; $\int_0^1 2\pi x\, x^4 dx = \frac{\pi}{3}$
25 $\pi(3)^2\, \frac{1}{3} + \int_{1/3}^{2} \pi(\frac{1}{x})^2 dx = \frac{11\pi}{2}$; $\pi(\frac{1}{3})^2 3 + \int_{1/3}^{2} 2\pi x \frac{1}{x} dx = \frac{11\pi}{3}$
27 $\int_0^1 \pi[(x^{2/3})^2 - (x^{3/2})^2]dx = \frac{5\pi}{28}$; $\int_0^1 2\pi x(x^{2/3} - x^{3/2})dx = \frac{5\pi}{28}$ (notice xy symmetry)
29 $x^2 = R^2 - y^2, V = \int_{R-h}^{R} \pi(R^2 - y^2)dy = \pi(Rh^2 - \frac{h^3}{3})$
31 $\int_{-a}^{a}(2\sqrt{a^2 - x^2})^2 dx = \frac{16}{3}a^3$ **33** $\int_0^1(2\sqrt{1 - y})^2 dy = 2$ **37** $\int A(x)dx$ or in this case $\int a(y)dy$
39 Ellipse; $\sqrt{1 - x^2} \tan\theta$; $\frac{1}{2}(1 - x^2) \tan\theta$; $\frac{2}{3} \tan\theta$
41 Half of $\pi r^2 h$; rectangles **43** $\int_1^3 \pi(5^2 - 2^2)dx = 42\pi$ **45** $\int_1^3 \pi(4^2 - 1^2)dx = 30\pi$
47 $\int_0^{b-a} \pi((b - y)^2 - a^2)dy = \frac{\pi}{3}(b^3 - 3a^2 b + 2a^3)$ **49** $\int_0^2 \pi(3 - x)^2 dx$; $\int_0^1 2\pi y(2)dy + \int_1^3 2\pi y(3 - y)dy$
51 $\int_a^b \pi(\frac{y}{m})^2 dy = \frac{\pi(b^3 - a^3)}{3m^2}$ **53** $960\ \pi$ cm **55** $\frac{\pi}{2}$ **57** $\frac{2\pi}{3}$
59 2π **61** $\int_0^4 2\pi y(2 - \sqrt{y})dy = \frac{32\pi}{5}$ **63** $3\pi e$ **65** Height 1; $\int_0^a 2\pi x\, dx = \pi a^2$; cylinder

67 Length of hole is $2\sqrt{b^2 - a^2} = 2$, so $b^2 - a^2 = 1$ and volume is $\frac{4\pi}{3}$ **69** F; T(?); F; T

2 Intersect at $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$; area $\int_{-\sqrt{2}}^{\sqrt{2}} [0 - (x^2 - 2)]dx = \mathbf{\frac{8\sqrt{2}}{3}}$.

4 Intersect when $y^2 = y + 2$ at $(1, -1)$ and $(4,2)$: area $= \int_{-1}^{2} [(y+2) - y^2]dy = \mathbf{\frac{9}{2}}$.

6 $y = x^{1/5}$ and $y = x^4$ intersect at (0,0) and (1,1): area $= \int_0^1 (x^{1/5} - x^4)dx = \frac{5}{6} - \frac{1}{5} = \mathbf{\frac{19}{30}}$.

8 $y = \frac{1}{x}$ meets $y = \frac{1}{x^3}$ at (1,1); upper limit $x = 3$: area $= \int_1^3 (\frac{1}{x} - \frac{1}{x^3})dx = [\frac{-1}{2x^2} + \frac{1}{3x^3}]_1^3 = -\frac{1}{18} + \frac{1}{81} + \frac{1}{2} - \frac{1}{3} = \mathbf{\frac{10}{81}}$.

10 $2x = \sin \pi x$ at $x = \frac{1}{2}$: area $= \int_0^{1/2} (\sin \pi x - 2x)dx = [-\frac{\cos \pi x}{\pi} - x^2]_0^{1/2} = \mathbf{\frac{1}{\pi} - \frac{1}{4}}$.

12 The region is a curved triangle between $x = -1$ (where $e^{-x} = e$) and $x = 1$ (where $e^x = e$). Vertical strips end at e^{-x} for $x < 0$ and at e^x for $x > 0$: Area $= \int_{-1}^{0} (e - e^{-x})dx + \int_0^1 (e - e^x)dx = \mathbf{2}$.

14 This region has $y = 1$ as its base. The top point is at $x = 9, y = 3$, where $12 - x = \sqrt{x}$. Strips go up to $y = \sqrt{x}$ between $x = 1$ and $x = 9$. Strips go up to $y = 12 - x$ between $x = 9$ and $x = 11$. Area $= \int_1^9 (\sqrt{x} - 1)dx + \int_9^{11} (12 - x - 1)dx = \frac{2}{3}(27 - 1) - 8 + 22 - 20 = \frac{52}{3} - 6 = \mathbf{\frac{34}{3}}$.

16 The triangle with base from $x = -1$ to $x = 1$ and vertex at (0,1) fits inside the circle and parabola. Its area is $\frac{1}{2}(2)(1) = 1$. General method: If the vertex is at $(t, \sqrt{1 - t^2})$ on the circle or at $(t, 1 - t^2)$ on the parabola, the area is $\sqrt{1 - t^2}$ or $1 - t^2$. Maximum $= 1$ at $t = 0$.

18 Volume $= \int_0^{\pi} \pi \sin^2 x dx = [\pi(\frac{x - \sin x \cos x}{2})]_0^{\pi} = \mathbf{\frac{\pi^2}{2}}$.

20 Shells around the y axis have radius x and height $2 \sin x$ and volume $(2\pi x) 2 \sin x dx$. Integrate for the volume of the galaxy: $\int_0^{\pi} 4\pi x \sin x dx = [4\pi(\sin x - x \cos x)]_0^{\pi} = \mathbf{8\pi^2}$.

22 (a) Volume $= \int_0^1 \pi(1 + e^x)^2 dx = \pi(-\frac{3}{2} + 2e + \frac{e^2}{2})$ (b) Volume $= \int_0^1 2\pi x(1 + e^x)dx = [\pi x^2 + 2\pi(xe^x - e^x)]_0^1 = 3\pi$.

24 (a) Volume $= \int_0^{\pi/4} \pi \sin^2 x dx + \int_{\pi/4}^{\pi/2} \pi \cos^2 x dx = [\frac{\pi x}{2} - \frac{\pi \sin 2x}{4}]_0^{\pi/4} + [\frac{\pi x}{2} + \frac{\pi \sin 2x}{4}]_{\pi/4}^{\pi/2} = \frac{\pi^2}{8} - \frac{\pi}{4} + \frac{\pi^2}{4} - \frac{\pi^2}{8} - \frac{\pi}{4} = \mathbf{\frac{\pi^2}{4} - \frac{\pi}{2}}$. (b) Volume $= \int_0^{\pi/4} 2\pi x \sin x dx + \int_{\pi/4}^{\pi/2} 2\pi x \cos x dx = [2\pi(\sin x - x \cos x)]_0^{\pi/4} + [2\pi(\cos x + x \sin x)]_{\pi/4}^{\pi/2} = \mathbf{\pi^2(1 - \frac{1}{\sqrt{2}})}$.

26 The region is a curved triangle, with base between $x = 3, y = 0$ and $x = 9, y = 0$. The top point is where $y = \sqrt{x^2 - 9}$ meets $y = 9 - x$; then $x^2 - 9 = (9 - x)^2$ leads to $x = 5, y = 4$. (a) Around the x axis: Volume $= \int_3^5 \pi(x^2 - 9)dx + \int_5^9 \pi(9 - x)^2 dx = \mathbf{36\pi}$. (b) Around the y axis: Volume $= \int_3^5 2\pi x \sqrt{x^2 - 9}dx + \int_5^9 2\pi x(9 - x)dx = [\frac{2\pi}{3}(x^2 - 9)^{3/2}]_3^5 + [9\pi x^2 - \frac{2\pi x^3}{3}]_5^9 = \frac{2\pi}{3}(64) + 9\pi(9^2 - 5^2) - \frac{2\pi}{3}(9^3 - 5^3) = \mathbf{144\pi}$.

28 The region is a circle of radius 1 with center (2,1). (a) Rotation around the x axis gives a torus with no hole: it is Example 10 with $a = b = 1$ and volume $\mathbf{2\pi^2}$. The integral is $\pi \int_1^3 [(1 + \sqrt{1 - (x-2)^2}) - (1 - \sqrt{1 - (x-2)^2}]dx = 4\pi \int_1^3 \sqrt{1 - (x-2)^2}dx = 4\pi \int_{-1}^{1} \sqrt{1 - x^2}dx = 2\pi^2$. (b) Rotation around the y axis also gives a torus. The center now goes around a circle of radius 2 so by Example 10 $V = \mathbf{4\pi^2}$. The volume by shells is $\int_1^3 2\pi x[(1 + \sqrt{1 - (x-2)^2}) - (1 - \sqrt{1 - (x-2)^2})]dx = 4\pi \int_1^3 x\sqrt{1 - (x-2)^2}dx = 4\pi \int_{-1}^{1} (x + 2)\sqrt{1 - x^2}dx =$ (odd integral is zero) $8\pi \int_{-1}^{1} \sqrt{1 - x^2}dx = 4\pi^2$.

30 (a) The slice at height y is a square of side $\mathbf{\frac{6-y}{3}}$ (then side $= 2$ when $y = 0$ and side $= 0$ when $y = 6$). The volume up to height 3 is $\int_0^3 (\frac{6-y}{3})^2 dy = [-\frac{1}{9}\frac{(6-y)^3}{3}]_0^3 = \frac{6^3 - 3^3}{9 \cdot 3} = \mathbf{7}$. (b) The big pyramid has volume $\frac{1}{3}$ (base area) (height) $= \frac{1}{3}(4)(6) = 8$. The pyramid from $y = 3$ to the top has volume $\frac{1}{3}(1)(3) = 1$. Subtract to find $8 - 1 = \mathbf{7}$.

32 Volume by slices $= \int_{-1}^{1} (1 - x^2)^2 dx = \int_{-1}^{1} (1 - 2x^2 + x^4)dx = \mathbf{\frac{16}{15}}$.

34 The area of a semicircle is $\frac{1}{2}\pi r^2$. Here the diameter goes from the base $y = 0$ to the top edge $y = 1 - x$ of the triangle. So the semicircle radius is $r = \frac{1-x}{2}$. The volume by slices is $\int_0^1 \frac{\pi}{2}(\frac{1-x}{2})^2 dx = [-\frac{\pi}{8}\frac{(1-x)^3}{3}]_0^1 = \mathbf{\frac{\pi}{24}}$.

36 The tilted cylinder has circular slices of area πr^2 (at all heights from 0 to h). So the volume is $\int_0^h \pi r^2 dy = \mathbf{\pi r^2 h}$. This equals the volume of an *untilted* cylinder (Cavalieri's principle: same slice areas give same volume).

38 (Work with $\frac{1}{8}$ region in figure.) The horizontal slice at height y is a square with side length $\sqrt{a^2 - y^2}$. The area is $a^2 - y^2$. So the volume is $\int_0^a (a^2 - y^2)dy = \frac{2}{3}a^3$. Multiply by 8 to find the total volume $\mathbf{\frac{16}{3}a^3}$.

40 (a) The slices are **rectangles.** (b) The slice area is $\mathbf{2\sqrt{1-y^2}}$ **times y** $\tan\theta$. (c) The volume is $\int_0^1 2\sqrt{1-y^2}y\tan\theta dy = [-\frac{2}{3}(1-y^2)^{3/2}\tan\theta]_0^1 = \frac{2}{3}\tan\theta$. (d) Multiply radius by r and volume by $\mathbf{r^3}$.

42 The area is the base length $2\sqrt{r^2-x^2}$ times the height $\frac{h(r-x)}{2r}$. The volume is $\int_{-r}^{r} 2\sqrt{r^2-x^2}\frac{h(r-x)}{2r}dx =$ (odd integral is zero) $\int_{-r}^{r} 2\sqrt{r^2-x^2}\frac{h}{2}dx = \mathbf{h\frac{\pi r^2}{2}}$. This is half the volume of the glass!

44 Slices are washers with outer radius $x=3$ and inner radius $x=1$ and area $\pi(3^2-1^2)=8\pi$. Volume $= \int_2^5 8\pi dy = \mathbf{24\,\pi}$.

46 Rotation produces a **cylinder with a cone removed.** (Rotation of the unit square produces the circular cylinder; rotation of the standard unit triangle produces the cone; our triangle is the unit square minus the standard triangle.) The volume of cylinder minus cone is $\pi(1^2)(1) - \frac{1}{3}\pi(1^2)(1) = \frac{2\pi}{3}$. Check by washers: $\int_0^1 \pi(1^2-(1-x)^2)dx = \int_0^1 \pi(2x-x^2)dx = \frac{2\pi}{3}$.

47 Note: Boring a hole of radius a removes a circular cylinder and two spherical caps. Use Problem 29 (volume of cap) to check Problem 47.

48 The volume common to two spheres is *two caps* of height h. By Problem 29 this volume is $\mathbf{2\pi(rh^2 - \frac{h^3}{3})}$.

50 Volume by shells $= \int_0^2 2\pi x(8-x^3)dx = [8\pi x^2 - \frac{2\pi}{5}x^5]_0^2 = 32\pi - \frac{64\pi}{5} = \frac{96\pi}{5}$; volume by horizontal disks $= \int_0^8 \pi(y^{1/3})^2 dy = [\frac{3\pi}{5}y^{5/3}]_0^8 = \frac{3\pi}{5}32 = \frac{96\pi}{5}$.

52 Substituting $y=f(x)$ changes $\int_0^6 \pi(f^{-1}(y))^2 dy$ to $\int_1^0 \pi x^2 f'(x)dx$. Integrate by parts with $u=\pi x^2$ and $dv = f'(x)dx$: volume $= [\pi x^2 f(x)]_1^0 - \int_1^0 2\pi x f(x)dx =$ zero $+ \int_0^1 2\pi x f(x)dx =$ volume by shells.

56 $\int_1^{100} 2\pi x(\frac{1}{x})dx = 2\pi(99) = \mathbf{198\pi}$. 58 $\int_0^3 2\pi x(\frac{1}{1+x^2})dx = [\pi\ln(1+x^2)]_0^3 = \boldsymbol{\pi}\,\mathbf{ln\ 10}$.

60 $\int_0^1 2\pi x(\frac{1}{\sqrt{1-x^2}})dx = [-2\pi\sqrt{1-x^2}]_0^1 = \mathbf{2\pi}$.

62 Shells around x axis: volume $= \int_{y=0}^{1} 2\pi y(1)dy + \int_{y=1}^{e} 2\pi y(1-\ln y)dy = [\pi y^2]_0^1 + [\pi y^2 - 2\pi\frac{y^2}{2}\ln y + 2\pi\frac{y^2}{4}]_1^e$ $= \pi + \pi e^2 - \pi e^2 + 2\pi\frac{e^2}{4} - \pi + 0 - 2\pi\frac{1}{4} = \frac{\pi}{2}(\mathbf{e^2-1})$. Check disks: $\int_0^1 \pi(e^x)^2 dx = [\pi\frac{e^{2x}}{2}]_0^1 = \frac{\pi}{2}(e^2-1)$.

64 (a) Volume by shells $= \int_0^1 2\pi x(x-x^2)dx = 2\pi(\frac{1}{3}-\frac{1}{4}) = \frac{\pi}{6}$; volume by washers $= \int_0^1 \pi(\sqrt{y}^{\,2} - y^2)dy = \pi(\frac{1}{2}-\frac{1}{3}) = \frac{\pi}{6}$.

66 (a) The top of the hole is at $y=\sqrt{b^2-a^2}$.
(b) The volume is $\int$ (area of washer) $dy = \int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \pi(b^2-y^2-a^2)dy = \frac{4\pi}{3}(b^2-a^2)^{3/2}$.

68 Note: The distance h is the *vertical* separation between planes. (a) The volume of a circular cylinder (flat top and bottom) is $\pi r^2 h$. Remove a wedge from the bottom and put it on the top to produce the solid between planes slicing at angle α. (b) Tilt so the top and bottom are flat. The base is an **ellipse** with area π times r times $\frac{r}{\sin\alpha}$. The height is $H = \mathbf{h}\sin\alpha$. The volume is again $\boldsymbol{\pi}\,\mathbf{r^2\,h}$.

8.2 Length of a Plane Curve (page 324)

The length of a straight segment (Δx across, Δy up) is $\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Between two points on the graph of $y(x), \Delta y$ is approximately dy/dx times Δx. The length of that piece is approximately $\sqrt{(\Delta x)^2 + (dy/dx)^2(\Delta x)^2}$. An infinitesimal piece of the curve has length $ds = \sqrt{1+(dy/dx)^2}\,dx$. Then the arc length integral is $\int ds$.

For $y = 4-x$ from $x=0$ to $x=3$ the arc length is $\int_0^3 \sqrt{2}\,dx = 3\sqrt{2}$. For $y=x^3$ the arc length integral is $\int\sqrt{1+9x^4}\,dx$.

The curve $x=\cos t, y=\sin t$ is the same as $x^2+y^2=1$. The length of a curve given by $x(t), y(t)$ is

$\int\sqrt{(\mathbf{dx/dt})^2+(\mathbf{dy/dt^2})}dt$. For example $x=\cos t, y=\sin t$ from $t=\pi/3$ to $t=\pi/2$ has length $\int_{\pi/3}^{\pi/2}\mathbf{dt}$. The speed is $ds/dt=\mathbf{1}$. For the special case $x=t, y=f(t)$ the length formula goes back to $\int\sqrt{\mathbf{1}+(\mathbf{f'(x)})^2}dx$.

1 $\int_0^1\sqrt{1+(\frac{3}{2}x^{1/2})^2}dx=\frac{8}{27}[(\frac{13}{4})^{3/2}-1]=\frac{13\sqrt{13}-8}{27}$ **3** $\int_0^1\sqrt{1+x^2(x^2+2)}dx=\int_0^1(1+x^2)dx=\frac{4}{3}$

5 $\int_1^3\sqrt{1+(x^2-\frac{1}{4x^2})^2}dx=\int_1^3(x^2+\frac{1}{4x^2})dx=\frac{53}{6}$

7 $\int_1^4\sqrt{1+(x^{1/2}-\frac{1}{4}x^{-1/2})^2}dx=\int_1^4(x^{1/2}+\frac{1}{4}x^{-1/2})dx=\frac{31}{6}$

9 $\int_0^{\pi/2}\sqrt{9\cos^4 t\ \sin^2 t+9\sin^4 t\ \cos^2 t}\ dt=\int_0^{\pi/2}3\cos t\sin t\ dt=\frac{3}{2}$

11 $\int_0^{\pi/2}\sqrt{\sin^2 t+(1-\cos t)^2}dt=\int_0^{\pi/2}\sqrt{2-2\cos t}\ dt=\int_0^{\pi/2}2\sin\frac{t}{2}\ dt=4-2\sqrt{2}$

13 $\int_0^1\sqrt{t^2+2t+1}\ dt=\int_0^1(t+1)dt=\frac{3}{2}$ **15** $\int_0^{\pi}\sqrt{1+\cos^2 x}\ dx=3.820$ **17** $\int_1^e\sqrt{1+\frac{1}{x^2}}dx=2.003$

19 Graphs are flat toward (1,0) then steep up to (1,1); limiting length is 2

21 $\frac{ds}{dt}=\sqrt{36\sin^2 3t+36\cos^2 3t}=6$ **23** $\int_0^1\sqrt{26}\ dy=\sqrt{26}$

25 $\int_{-1}^1\sqrt{\frac{1}{4}(e^y-e^{-y})^2+1}\ dy=\int_{-1}^1\frac{1}{2}(e^y+e^{-y})dy=\frac{1}{2}(e^y-e^{-y})]_{-1}^1=e-\frac{1}{e}$.

Using $x=\cosh y$ this is $\int\sqrt{1+\sinh^2 y}\ dy=\int\cosh y\ dy=\sinh y]_{-1}^1=2\sinh 1$

27 Ellipse; two y's for the same x **29** Carpet length $2\neq$ straight distance $\sqrt{2}$

31 $(ds)^2=(dx)^2+(dy)^2+(dz)^2; ds=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}dt;$

$ds=\sqrt{\sin^2 t+\cos^2 t+1}dt=\sqrt{2}dt; 2\pi\sqrt{2}$; curve = helix, shadow = circle

33 $L=\int_0^1\sqrt{1+4x^2}dx; \int_0^2\sqrt{1+x^2}dx=\int_0^1\sqrt{1+4u^2}\ 2du=2L$; stretch xy plane by 2 ($y=x^2$ becomes $\frac{y}{2}=(\frac{x}{2})^2$)

2 $y=x^{2/3}$ has $\frac{dy}{dx}=\frac{2}{3}x^{-1/3}$ and length $=\int_0^1(1+\frac{4}{9}x^{-2/3})^{1/2}dx$. (a) This is the mirror image of the curve $y=x^{3/2}$ in Problem 1. So the length is the same. (b) Substitute $u=\frac{4}{9}+x^{2/3}$ and $du=\frac{2}{3}x^{-1/3}dx$ to get $\int_{4/9}^{13/9}u^{1/2}du(\frac{3}{2})=[u^{3/2}]_{4/9}^{13/9}=\mathbf{\frac{13^{3/2}-4^{3/2}}{27}}$.

4 $y=\frac{1}{3}(x^2-2)^{3/2}$ has $\frac{dy}{dx}=x(x^2-2)^{1/2}$ and length $=\int_2^4\sqrt{1+x^2(x^2-2)}dx=\int_2^4(x^2-1)dx=\mathbf{\frac{50}{3}}$.

6 $y=\frac{x^4}{4}+\frac{1}{8x^2}$ has $\frac{dy}{dx}=x^3-\frac{1}{4x^3}$ and length $=\int_1^2(1+(x^3-\frac{1}{4x^3})^2)^{1/2}dx=\int_1^2(x^6+\frac{1}{2}+\frac{1}{16x^6})^{1/2}dx=\int_1^2(x^3+\frac{1}{4x^3})dx=\mathbf{\frac{123}{32}}$.

8 Length $=\int_0^1\sqrt{1+4x^2}dx=2\int_0^1\sqrt{x^2+(\frac{1}{2})^2}dx=[x\sqrt{x^2+\frac{1}{4}}+\frac{1}{4}\ln|x+\sqrt{x^2+\frac{1}{4}}|]_0^1=\sqrt{\frac{5}{4}}+\frac{1}{4}(\ln(1+\sqrt{\frac{5}{4}})-\ln\sqrt{\frac{1}{4}})=\mathbf{\frac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5})}$.

10 $\frac{dx}{dt}=\cos t-\sin t$ and $\frac{dy}{dt}=-\sin t-\cos t$ and $(\frac{dx}{dt})^2+(\frac{dy}{dt})^2=2$. So length $=\int_0^{\pi}\sqrt{2}dt=\mathbf{\sqrt{2}\pi}$. The curve is a **half** of a circle of radius $\sqrt{2}$ because $x^2+y^2=2$ and t stops at π.

12 $\frac{dx}{dt}=\cos t-t\sin t$ and $\frac{dy}{dt}=\sin t+t\cos t$ and $(\frac{dx}{dt})^2+(\frac{dy}{dt})^2=1+t^2$. Then length $=\int\sqrt{\mathbf{1+t^2}}\mathbf{dt}$. (Note: the parabola $y=\frac{1}{2}x^2$ also leads to this length integral: Compare Problem 8.)

14 $\frac{dx}{dt}=(1-\frac{1}{2}\cos 2t)(-\sin t)+\sin 2t\cos t=\frac{3}{2}\sin t\cos 2t$. Note: first rewrite $\sin 2t\cos t=2\sin t\cos^2 t=\sin t(1+\cos 2t)$. Similarly $\frac{dy}{dt}=\frac{3}{2}\cos t\cos 2t$. Then $(\frac{dx}{dt})^2+(\frac{dy}{dt})^2=(\frac{3}{2}\cos 2t)^2$. So length $=\int_0^{\pi/4}\frac{3}{2}\cos 2tdt=\mathbf{\frac{3}{4}}$. This is the only arc length I have ever personally discovered; the problem was meant to have an asterisk.

16 Exact integral; $\int_0^1\sqrt{1+e^{2x}}dx=\int_1^e\sqrt{1+u^2}\frac{du}{u}=$ (by integral 22 on last page) $[\sqrt{u^2+1}-\ln\frac{1+\sqrt{u^2+1}}{u}]_1^e=\sqrt{1+e^2}-\sqrt{2}-\ln\frac{1+\sqrt{1+e^2}}{e(1+\sqrt{2})}\approx\mathbf{2.01}$.

18 $\frac{dx}{dt}=-\sin t$ and $\frac{dy}{dt}=3\cos t$ so length $=\int_0^{2\pi}\sqrt{\sin^2 t+9\cos^2 t}\ dt=$ perimeter of ellipse. This integral has no closed form. Match it with a table of "elliptic integrals" by writing it as $4\int_0^{\pi/2}\sqrt{9-8\sin^2 t}dt=12\int_0^{\pi/2}\sqrt{1-\frac{8}{9}\sin^2 t}dt$. The table with $k^2=\frac{8}{9}$ gives 1.14 for this integral or 12 (1.14) = 13.68 for the perimeter. Numerical integration is the expected route to this answer.

20 The straight line must be shortest.

22 Substitute $\mathbf{x = t^2}$ in $\int_0^4 \sqrt{1+\frac{9}{4}x}\,dx = \int_{t=0}^2 \sqrt{1+\frac{9}{4}t^2}\,2t\,dt = \int_0^2 \sqrt{4t^2+9t^4}\,dt$.

24 The curve $x = y^{3/2}$ is the **mirror image** of $y = x^{3/2}$ in Problem 1: same length $\frac{13^{3/2}-4^{3/2}}{27}$ (also Problem 2).

26 The curve $x = g(y)$ has length $\int \sqrt{\mathbf{1 + g'(y)^2}}\,\mathbf{dy}$.

28 (a) Length integral $= \int_0^\pi \sqrt{4\cos^2 t\sin^2 t + 4\cos^2 t\sin^2 t}\,dt = \int_0^\pi 2\sqrt{2}|\cos t\sin t|dt = \mathbf{2\sqrt{2}}$. (Notice that $\cos t$ is *negative* beyond $t = \frac{\pi}{2}$: split into $\int_0^{\pi/2} + \int_{\pi/2}^\pi$. (b) All points have $x + y = \cos^2 t + \sin^2 t = 1$. (c) The path from (1,0) reaches (0,1) when $t = \frac{\pi}{2}$ and returns to (1,0) at $t = \pi$. Two trips of length $\sqrt{2}$ give $2\sqrt{2}$.

30 The strip around the ellipse does have area $\approx \pi(a+b)\Delta$. But **its width is not everywhere** Δ (the width is measured perpendicular to the ellipse.) So it is false that the length of the strip is $\pi(a+b)$.

34 Length of parabola $= \int_0^b \sqrt{1+4x^2}\,dx =$ (by the solution to Problem 8) $b\sqrt{b^2+\frac{1}{4}}+\frac{1}{4}\ln|b+\sqrt{b^2+\frac{1}{4}}|-\frac{1}{4}\ln\sqrt{\frac{1}{4}}$. Length of straight line $= \sqrt{b^2+b^4} = b\sqrt{b^2+1}$. The ln term approaches infinity as $b \to \infty$ so the length difference also goes to infinity.

8.3 Area of a Surface of Revolution (page 327)

A surface of revolution comes from revolving a **curve** around **an axis** (a line). This section computes the **surface area**. When the curve is a short straight piece (length Δs), the surface is a **cone**. Its area is $\Delta S = \mathbf{2\pi r\Delta s}$. In that formula (Problem 13) r is the radius of **the circle traveled by the middle point**. The line from (0,0) to (1,1) has length $\mathbf{\Delta s} = \sqrt{\mathbf{2}}$, and revolving it produces area $\pi\sqrt{\mathbf{2}}$.

When the curve $y = f(x)$ revolves around the x axis, the area of the surface of revolution is the integral $\int \mathbf{2\pi f(x)}\sqrt{\mathbf{1 + (df/dx)^2}}\mathbf{dx}$. For $y = x^2$ the integral to compute is $\int \mathbf{2\pi x^2\sqrt{1+4x^2}dx}$. When $y = x^2$ is revolved around the y axis, the area is $S = \int \mathbf{2\pi x}\sqrt{\mathbf{1+(df/dx)^2}}\mathbf{dx}$. For the curve given by $x = 2t, y = t^2$, change ds to $\sqrt{\mathbf{4+4t^2}}\mathbf{dt}$.

1 $\int_2^6 2\pi\sqrt{x}\sqrt{1+(\frac{1}{2\sqrt{x}})^2}dx = \int_2^6 2\pi\sqrt{x+\frac{1}{4}}dx = \frac{49\pi}{3}$ **3** $2\int_0^1 2\pi(7x)\sqrt{50}dx = 14\pi\sqrt{50}$

5 $\int_{-1}^1 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}dx = \int_{-1}^1 4\pi dx = 8\pi$ **7** $\int_0^2 2\pi x\sqrt{1+(2x)^2}dx = \frac{\pi}{6}(1+4x^2)^{3/2}]_0^2 = \frac{\pi}{6}[17^{3/2}-1]$

9 $\int_0^3 2\pi x\sqrt{2}dx = 9\pi\sqrt{2}$ **11** Figure shows radius s times angle θ = arc $2\pi R$

13 $2\pi r\Delta s = \pi(R+R')(s-s') = \pi Rs - \pi R's'$ because $R's - Rs' = 0$

15 Radius a, center at $(0,b)$; $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = a^2$, surface area $\int_0^{2\pi} 2\pi(b + a\sin t)a\,dt = 4\pi^2 ab$

17 $\int_1^2 2\pi x\sqrt{1+\frac{(1-x)^2}{2x-x^2}}dx = \int_1^2 \frac{2\pi x\,dx}{\sqrt{2x-x^2}} = \pi^2 + 2\pi$ (write $2x - x^2 = 1-(x-1)^2$ and set $x - 1 = \sin\theta$)

19 $\int_{1/2}^1 2\pi x\sqrt{1+\frac{1}{x^4}}dx$ (can be done)

21 Surface area $= \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > \int_1^\infty \frac{2\pi dx}{x} = 2\pi\ln x]_1^\infty = \infty$ but volume $= \int_1^\infty \pi(\frac{1}{x})^2 dx = \pi$

23 $\int_0^\pi 2\pi\sin t\sqrt{2\sin^2 t+\cos^2 t}\,dt = \int_0^\pi 2\pi\sin t\sqrt{2-\cos^2 t}\,dt = \int_{-1}^1 2\pi\sqrt{2-u^2}du =$
$\pi u\sqrt{2-u^2} + 2\pi\sin^{-1}\frac{u}{\sqrt{2}}]_{-1}^1 = 2\pi + \pi^2$

2 Area $= \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx = [\frac{\pi}{27}(1+9x^4)^{3/2}]_0^1 = \frac{\pi}{\mathbf{27}}(\mathbf{10^{3/2}-1})$

4 Area $= \int_0^2 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_0^2 4\pi dx = \mathbf{8\pi}$

6 Area $= \int_0^1 2\pi\cosh x\sqrt{1+\sinh^2 x}\,dx = \int_0^1 2\pi\cosh^2 x dx = \int_0^1 \frac{\pi}{2}(e^{2x}+2+e^{-2x})dx = [\frac{\pi}{2}(\frac{e^{2x}}{2}+2x+\frac{e^{-2x}}{-2})]_0^1 =$
$\frac{\pi}{2}(\frac{e^2}{2}+2+\frac{e^{-2}}{-2}-1) = \frac{\pi}{\mathbf{2}}(\frac{\mathbf{e^2-e^{-2}}}{\mathbf{2}}+\mathbf{1})$.

8 Area $= \int_0^1 2\pi x\sqrt{1+x^2}\,dx = [\frac{2\pi}{3}(1+x^2)^{3/2}]_0^1 = \frac{\mathbf{2\pi}}{\mathbf{3}}(\mathbf{2^{3/2}-1})$

10 Area $= \int_0^1 2\pi x\sqrt{1+\frac{1}{9}x^{-4/3}}dx$. This is unexpectedly difficult (rotation around the x axis is easier). Substitute $u = 3x^{2/3}$ and $du = 2x^{-1/3}dx$ and $x = (\frac{u}{3})^{3/2}$: Area $= \int_0^3 2\pi(\frac{u}{3})^{3/2}\sqrt{1+\frac{1}{u^2}}\frac{du}{2}(\frac{u}{3})^{1/2} =$ $\int_0^3 \frac{\pi}{9}u\sqrt{u^2+1}du = [\frac{\pi}{27}(u^2+1)^{3/2}]_0^3 = \frac{\pi}{\mathbf{27}}(\mathbf{10^{3/2}} - \mathbf{1})$. An equally good substitution is $u = x^{4/3} + \frac{1}{9}$.

12 The surface area of the band is the surface area of the larger cone minus the surface area of the smaller cone.

14 (a) $dS = 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}dx = \mathbf{2\pi dx}$. (b) The area between $x = a$ and $x = a+h$ is $\mathbf{2\pi h}$. All slices of thickness h have this area, whether the slice goes near the center or near the outside. (c) $\frac{1}{4}$ of the Earth's area is above latitude $\mathbf{30°}$ where the height is $R\sin 30° = \frac{R}{2}$. The slice from the Equator up to 30° has the same area (and so do two more slices below the Equator).

16 Rotate a quarter-circle to produce half a sphere. The surface area is $\int_0^{\pi/2} 2\pi R\cos t\sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt =$ $\int_0^{\pi/2} 2\pi R^2 \cos t dt = \mathbf{2\pi R^2}$. Note the limits $0 \le t \le \frac{\pi}{2}$.

18 The cylinder has side area $2\pi rh = 2\pi(\frac{1}{4})(\frac{1}{3}) = \frac{\pi}{6}$. The light bulb is a slice of a sphere, and its area is also $2\pi rh$($r = 1$ for the basketball in Problem 14, now $r = \frac{1}{2}$). The slice thickness is $h = \frac{1}{2} + \frac{\sqrt{3}}{4}$ (check triangle with sides $\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}$), so $2\pi rh = \pi(\frac{1}{2} + \frac{\sqrt{3}}{4})$. Adding the cylinder yields total area $\pi(\mathbf{\frac{2}{3}} + \mathbf{\frac{\sqrt{3}}{4}})$.

20 Area $= \int_{1/2}^1 2\pi x\sqrt{1+\frac{1}{x^4}}dx = \int_{1/2}^1 2\pi\frac{\sqrt{x^4+1}}{x^4}x^3dx$. Substitute $u = \sqrt{x^4+1}$ and $du = 2x^3dx/u$ to find $\int_{\sqrt{17}/4}^{\sqrt{2}} \frac{\pi u^2 du}{u^2-1} = [\pi u - \frac{\pi}{2}\ln\frac{u+1}{u-1}]_{\sqrt{17}/4}^{\sqrt{2}} = \pi(\sqrt{2} - \frac{\sqrt{17}}{4} - \frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2}\ln\frac{\sqrt{17}+4}{\sqrt{17}-4}) \approx 5.0$.

22 It seems reasonable that the strips of tape should be placed side by side (parallel) to best cover the disk. The proof follows the hint: Each strip of tape is the xy projection of a slice of the sphere. Since the strip has width $h = \frac{1}{2}$, the slice has surface area $2\pi h = \pi$ by Problem 14. (Less area if the slice is far to the side and partly off the sphere.) The four slices have total area 4π, which is the area of the sphere. To cover the sphere the slices *must not overlap*. So the slices are parallel with spacing $\frac{1}{2}$.

24 A first estimate is $4\pi r^2$ (pretend the egg is a sphere). Somewhat better is $4\pi ab \approx 60\text{ cm}^2$ for a medium egg (a and b are half-axes of an ellipse). Really serious is to rotate the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $y = \frac{b}{a}\sqrt{a^2-x^2}$. Then the surface area is $\int_{-a}^a 2\pi\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}dx$ (use table of integrals).

8.4 Probability and Calculus (page 334)

Discrete probability uses counting, **continuous** probability uses calculus. The function $p(x)$ is the probability **density.** The chance that a random variable falls between a and b is $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{p(x)dx}$. The total probability is $\int_{-\infty}^{\infty} p(x)dx = \mathbf{1}$. In the discrete case $\sum p_n = \mathbf{1}$. The mean (or expected value) is $\mu = \int \mathbf{xp(x)dx}$ in the continuous case and $\mu = \sum np_n$ in the **discrete case.**

The Poisson distribution with mean λ has $p_n = \mathbf{\lambda^n e^{-\lambda}/n!}$. The sum $\sum p_n = 1$ comes from the **exponential** series. The exponential distribution has $p(x) = e^{-x}$ or $2e^{-2x}$ or $\mathbf{ae^{-ax}}$. The standard Gaussian (or **normal**) distribution has $\sqrt{2\pi}p(x) = e^{-x^2/2}$. Its graph is the well-known **bell-shaped** curve. The chance that the variable falls below x is $F(x) = \int_{-\infty}^{\mathbf{x}} \mathbf{p(x)dx}$. F is the **cumulative** density function. The difference $F(x+dx) - F(x)$ is about $\mathbf{p(x)dx}$, which is the chance that X is between x and $x + dx$.

The *variance,* which measures the spread around μ, is $\sigma^2 = \int \mathbf{(x-\mu)^2p(x)dx}$ in the continuous case and $\sigma^2 = \sum \mathbf{(n-\mu)^2p_n}$ in the discrete case. Its square root σ is the **standard deviation.** The normal distribution has $p(x) = \mathbf{e^{-(x-\mu)^2/2\sigma^2}}/\sqrt{2\pi}\sigma$. If $\overline{X}$ is the **average** of N samples from any population with mean μ and variance σ^2, the Law of Averages says that $\overline{X}$ will approach **the mean** μ. The Central Limit Theorem says that the distribution for $\overline{X}$ approaches a **normal distribution**. Its mean is μ and its variance is σ^2/N.

In a yes-no poll when the voters are 50-50, the mean for one voter is $\mu = 0(\frac{1}{2}) + 1(\frac{1}{2}) = \frac{1}{2}$. The variance is $(0-\mu)^2 p_0 + (1-\mu)^2 p_1 = \frac{1}{4}$. For a poll with $N = 100, \overline{\sigma}$ is $\sigma/\sqrt{N} = \frac{1}{20}$. There is a 95% chance that $\overline{X}$ (the fraction saying yes) will be between $\mu - 2\overline{\sigma} = \frac{1}{2} - \frac{1}{10}$ and $\mu + 2\overline{\sigma} = \frac{1}{2} + \frac{1}{10}$.

1 $P(X<4) = \frac{7}{8}, P(X=4) = \frac{1}{16}, P(X>4) = \frac{1}{16}$ **3** $\int_0^\infty p(x)dx$ is not 1; $p(x)$ is negative for large x
5 $\int_2^\infty e^{-x}dx = \frac{1}{e^2}; \int_1^{1.01} e^{-x}dx \approx (.01)\frac{1}{e}$ **7** $p(x) = \frac{1}{\pi}; F(x) = \frac{x}{\pi}$ for $0 \le x \le \pi$ ($F = 1$ for $x > \pi$)
9 $\mu = \frac{1}{7}\cdot 1 + \frac{1}{7}\cdot 2 + \cdots + \frac{1}{7}\cdot 7 = 4$ **11** $\int_0^\infty \frac{2x\,dx}{\pi(1+x^2)} = \frac{1}{\pi}\ln(1+x^2)]_0^\infty = +\infty$
13 $\int_0^\infty axe^{-ax}dx = [-xe^{-ax}]_0^\infty + \int_0^\infty e^{-ax}dx = \frac{1}{a}$
15 $\int_0^x \frac{2dx}{\pi(1+x^2)} = \frac{2}{\pi}\tan^{-1}x; \int_0^x e^{-x}dx = 1-e^{-x}; \int_0^x ae^{-ax}dx = 1-e^{-ax}$ **17** $\int_{10}^\infty \frac{1}{10}e^{-x/10}dx = -e^{-x/10}]_{10}^\infty = \frac{1}{e}$
19 *Exponential* better than Poisson: 60 years $\to \int_0^{60} .01e^{-.01x}\,dx = 1 - e^{-.6} = .45$
21 $y = \frac{x-\mu}{\sigma}$; three areas $\approx \frac{1}{3}$ each because $\mu - \sigma$ to μ is the same as μ to $\mu + \sigma$ and areas add to 1
23 $-2\mu\int xp(x)dx + \mu^2\int p(x)dx = -2\mu\cdot\mu + \mu^2\cdot 1 = -\mu^2$
25 $\mu = 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 2\cdot\frac{1}{3} = 1$; $\sigma^2 = (0-1)^2\cdot\frac{1}{3} + (1-1)^2\cdot\frac{1}{3} + (2-1)^2\cdot\frac{1}{3} = \frac{2}{3}$.
Also $\sum n^2 p_n - \mu^2 = 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 4\cdot\frac{1}{3} - 1 = \frac{2}{3}$
27 $\mu = \int_0^\infty \frac{xe^{-x/2}dx}{2} = 2; 1 - \int_0^4 \frac{e^{-x/2}dx}{2} = 1 + [e^{-x/2}]_0^4 = e^{-2}$
29 Standard deviation (yes – no poll) $\le \frac{1}{2\sqrt{N}} = \frac{1}{2\sqrt{900}} = \frac{1}{60}$ Poll showed $\frac{870}{900} = \frac{29}{30}$ peaceful.
95% confidence interval is from $\frac{29}{30} - \frac{2}{60}$ to $\frac{29}{30} + \frac{2}{60}$, or 93% to 100% peaceful.
31 95% confidence of unfair if more than $\frac{2\sigma}{\sqrt{N}} = \frac{1}{\sqrt{2500}} = 2\%$ away from 50% heads.
2% of 2500 = 50. So unfair if more than 1300 or less than 1200.
33 55 is 1.5σ below the mean, and the area up to $\mu - 1.5\sigma$ is about 8% so 24 students fail.
A grade of 57 is 1.3σ below the mean and the area up to $\mu - 1.3\sigma$ is about 10%.
35 .999; $.999^{1000} = (1 - \frac{1}{1000})^{1000} \approx \frac{1}{e}$ because $(1-\frac{1}{n})^n \to \frac{1}{e}$.

2 The probability of an odd $X = 1,3,5,\cdots$ is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$. The probabilities $p_n = (\frac{1}{3})^n$ **do not add to 1.** They add to $\frac{1}{3} + \frac{1}{9} + \cdots = \frac{1}{2}$ so the adjusted $p_n = 2(\frac{1}{3})^n$ add to 1.
4 $P(X=2) + P(X=3) + P(X=5) = \frac{1}{4} + \frac{1}{8} + \frac{1}{32} = \frac{13}{32}$, so the probability of a prime is greater than $\frac{13}{32} = \frac{6.5}{16}$. The sum $P(X=6) + P(X=7) + \cdots = \frac{1}{64} + \frac{1}{128} + \cdots$ equals $\frac{1}{32}$. Most of these are not prime so the probability of a prime is below $\frac{13}{32} + \frac{1}{32} = \frac{7}{16}$.
6 $\int_1^\infty \frac{C}{x^3}dx = -\frac{C}{2x^2}]_1^\infty = \frac{C}{2} = 1$ when $C = 2$. Then Prob $(X \le 2) = \int_1^2 \frac{2\,dx}{x^3} = -\frac{1}{x^2}]_1^2 = \frac{3}{4}$.
8 $\mu = \frac{1}{2}(0) + \frac{1}{4}(1) + \frac{1}{4}(2) = \frac{3}{4}$. **10** $\mu = \frac{1}{e}(0) + \frac{1}{e}(1) + \frac{1}{2e}(2) + \frac{1}{6e}(3) + \cdots = \frac{1}{e}(1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots) = \frac{e}{e} = 1$.
12 $\mu = \int_0^\infty xe^{-x}dx = uv - \int v\,du = -xe^{-x}]_0^\infty + \int_0^\infty e^{-x}dx = 1$.
14 Substitute $u = \frac{x}{\sqrt{2}\sigma}$ and $du = \frac{dx}{\sqrt{2}\sigma}$. The limits are still $-\infty$ and $+\infty$. The integral $\int_{-\infty}^\infty e^{-u^2}du = \sqrt{\pi}$ is computed on page 531.
16 Poisson $p_n = \frac{2^n}{n!}e^{-2}$. Probability of a bump is $p_0 + p_1 = e^{-2} + 2e^{-2} = 3e^{-2} \approx .40$.
18 Prob $(X<3) = \int_0^3 e^{-x}dx = 1 - e^{-3} \approx .95$.
20 (a) Heads and tails are still equally likely. (b) The coin is still fair so the expected fraction of heads during the second N tosses is $\frac{1}{2}$ and the expected fraction overall is $\frac{1}{2}(\alpha + \frac{1}{2})$; which is the average.
22 $\mu = 0(1-p)^2 + 1(2p-2p^2) + 2p^2 = 2p$. Then $\sigma^2 = (0-2p)^2(1-p)^2 + (1-2p)^2(2p-2p^2) + (2-2p)^2p^2 = 2p(1-p)$ after much simplification. (First factor out p and $1-p$.) With N voters, $\mu = Np$ and $\sigma^2 = Np(1-p)$.
24 $\mu = \int xp(x) = \int_0^1 x\,dx = \frac{1}{2}$. Then $\sigma^2 = \int_0^1 (x-\frac{1}{2})^2 1\,dx = \frac{1}{3}(x-\frac{1}{2})^3]_0^1 = \frac{1}{12}$. Also $\int_0^1 x^2dx - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.
26 $\int x^2p(x)dx = \int_0^\infty x^2(2e^{-2x})dx = [-x^2e^{-2x}]_0^\infty + \int_0^\infty 2xe^{-2x}dx = [-xe^{-2x}]_0^\infty + \int_0^\infty e^{-2x}dx = \frac{1}{2}$. Then $\sigma^2 = \frac{1}{2} - \mu^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

28 $\mu = (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + p_4 + \cdots) + (p_3 + p_4 + \cdots) + \cdots = (1) + (\frac{1}{2}) + (\frac{1}{4}) + \cdots = \mathbf{2}.$
30 p equals $\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}$ in four tosses. It looks more bell-shaped with 16 tosses.
32 $2000 \pm 2\sigma$ gives **1700 to 2300** as the 95% confidence interval.
34 The average has mean $\bar{\mu} = 30$ and deviation $\bar{\sigma} = \frac{8}{\sqrt{N}} = 1$. An actual average of $\frac{2000}{64} = 31.25$ is $1.25\,\bar{\sigma}$ above the mean. The probability of exceeding $1.25\,\bar{\sigma}$ is about **.1** from Figure 8.12b. With $N = 256$ we still have $\frac{8000}{256} = 31.25$ but now $\bar{\sigma} = \frac{8}{\sqrt{256}} = \frac{1}{2}$. To go $2.5\,\bar{\sigma}$ above the mean has **probability** $< .01$.
36 (a) $\mathbf{.001(.999)^{999}} \approx .001(1 - \frac{1}{1000})^{1000} \approx \mathbf{.001\frac{1}{e}}$. (b) Multiply the answer to (a) by 1000 (**which gives** $\mathbf{\frac{1}{e}}$) since any of the 1000 players could have been the one to win. (c) The probability p_n of exactly n winners is "1000 choose n" times $(.001)^n(.999)^{1000-n}$. This counts all combinations of n players times the chance that the first n players are the winners. But "1000 choose n" $= \frac{1000(999)\cdots(1000-n+1)}{1(2)\cdots(n)} \approx \frac{1000^n}{n!}$. Multiplying by $(.001)^n \frac{1}{e}$ gives $p_n \approx \mathbf{\frac{1}{n!}\frac{1}{e}}$ which is Poisson (= fish in French) with $\lambda = 1$. With λ times 1000 players, the chance of n winners is about $\frac{\lambda^n}{n!}e^{-\lambda}$.

8.5 Masses and Moments (page 340)

If masses m_n are at distances x_n, the total mass is $M = \sum \mathbf{m_n}$. The total moment around $x = 0$ is $M_y = \sum \mathbf{m_n x_n}$. The center of mass is at $\bar{x} = \mathbf{M_y/M}$. In the continuous case, the mass distribution is given by the **density** $\rho(x)$. The total mass is $M = \int \rho(\mathbf{x})\mathbf{dx}$ and the center of mass is at $\bar{x} = \int \mathbf{x}\rho(\mathbf{x})\mathbf{dx/M}$. With $\rho = x$, the integrals from 0 to L give $M = \mathbf{L^2/2}$ and $\int x\rho(x)dx = \mathbf{L^3/3}$ and $\bar{x} = \mathbf{2L/3}$. The total moment is the same as if the whole mass M is placed at $\bar{\mathbf{x}}$.

In a plane with masses m_n at the points (x_n, y_n), the moment around the y axis is $\sum \mathbf{m_n x_n}$. The center of mass has $\bar{x} = \sum \mathbf{m_n x_n} / \sum \mathbf{m_n}$ and $\bar{y} = \sum \mathbf{m_n y_n} / \sum \mathbf{m_n}$. For a plate with density $\rho = 1$, the mass M equals the **area.** If the plate is divided into vertical strips of height $y(x)$, then $M = \int y(x)dx$ and $M_y = \int \mathbf{x y(x)}dx$. For a square plate $0 \le x, y \le L$, the mass is $M = \mathbf{L^2}$ and the moment around the y axis is $M_y = \mathbf{L^3/2}$. The center of mass is at $(\bar{x}, \bar{y}) = (\mathbf{L/2}, \mathbf{L/2})$. This point is the **centroid,** where the plate balances.

A mass m at a distance x from the axis has moment of inertia $I = \mathbf{mx^2}$. A rod with $\rho = 1$ from $x = a$ to $x = b$ has $I_y = \mathbf{b^3/3 - a^3/3}$. For a plate with $\rho = 1$ and strips of height $y(x)$, this becomes $I_y = \int \mathbf{x^2 y(x)dx}$. The torque T is **force** times **distance.**

1 $\bar{x} = \frac{10}{6}$ **3** $\bar{x} = \frac{4}{4}$ **5** $\bar{x} = \frac{3.5}{3}$ **7** $\bar{x} = \frac{2}{3} = \bar{y}$ **9** $\bar{x} = \frac{7/2}{7} = \bar{y}$ **11** $\bar{x} = \frac{1/3}{\pi/4} = \bar{y}$ **13** $\bar{x} = \frac{1/4}{1/2}, \bar{y} = \frac{1/8}{1/2}$
15 $\bar{x} = \frac{0}{3\pi} = \bar{y}$ **21** $I = \int x^2\rho\,dx - 2t\int x\rho\,dx + t^2\int\rho\,dx$; $\frac{dI}{dt} = -2\int x\rho\,dx + 2t\int\rho\,dx = 0$ for $t = \bar{x}$
23 South Dakota **25** $2\pi^2a^2b$ **27** $M_x = 0, M_y = \frac{\pi}{2}$ **29** $\frac{2}{\pi}$ **31** Moment
33 $I = \sum m_n r_n^2$; $\frac{1}{2}\sum m_n r_n^2\omega_n^2$; 0 **35** $14\pi\ell\frac{r^2}{2}$; $14\pi\ell\frac{r^4}{4}$; $\frac{1}{2}$
37 $\frac{2}{3}$; solid ball, solid cylinder, hallow ball, hollow cylinder **39** No
41 $T \approx \sqrt{1+J}$ by Problem **40** so $T \approx \sqrt{1.4}, \sqrt{1.5}, \sqrt{5/3}, \sqrt{2}$

2 $M = 3 + 3 + 3 + 3 = \mathbf{12}$; $M_y = 3(0 + 1 + 2 + 6) = \mathbf{27}$; $\bar{x} = \frac{27}{12} = \mathbf{\frac{9}{4}}$.
4 $M = \int_0^L x^2dx = \mathbf{\frac{L^3}{3}}$; $M_y = \int_0^L x^3dx = \mathbf{\frac{L^4}{4}}$; $\bar{x} = \frac{L^4/4}{L^3/3} = \mathbf{\frac{3L}{4}}$.
6 $M = \int_0^\pi \sin x\,dx = \mathbf{2}$; $M_y = \int_0^\pi x\sin x\,dx = [\sin x - x\cos x]_0^\pi = \pi$; $\bar{x} = \mathbf{\frac{\pi}{2}}$.
8 $M = 1 + 4 = \mathbf{5}$; $M_y = 1(1) + 4(0) = \mathbf{1}$, $M_x = 1(0) + 4(1) = \mathbf{4}$; $\bar{x} = \mathbf{\frac{1}{5}}$ and $\bar{y} = \mathbf{\frac{4}{5}}$.
10 $M = \mathbf{3(\frac{1}{2}ab)}$; $M_y = \int_0^a 3xb(1 - \frac{x}{a})dx = [\frac{3x^2b}{2} - \frac{x^3b}{a}]_0^a = \mathbf{\frac{a^2b}{2}}$ and by symmetry $M_x = \mathbf{\frac{b^2a}{2}}$; $\bar{x} = \frac{a^2b/2}{3ab/2} = \mathbf{\frac{a}{3}}$

and $\overline{y} = \frac{\mathbf{b}}{\mathbf{3}}$. Note that the centroid of the triangle is at $(\frac{a}{3}, \frac{b}{3}.)$

12 Area $M = \int_0^1 x dx + \int_1^2 (2-x)dx = \mathbf{1}$ which is $\frac{1}{2}$ (base) (height); $M_y = \int_0^1 x^2 dx + \int_1^2 x(2-x)dx = 1$ so that $\overline{x} = \frac{1}{1} = \mathbf{1}$; $M_x = \int y$ (strip length at height y)$dy = \int_0^1 y(2-2y)dy = \frac{1}{3}$ and $\overline{y} = \frac{1/3}{1} = \frac{\mathbf{1}}{\mathbf{3}}$. Check: centroid of triangle is $(1, \frac{1}{3})$.

14 Area $M = \int_0^1 (x - x^2)dx = \frac{\mathbf{1}}{\mathbf{6}}$; $M_y = \int_0^1 x(x-x^2)dx = \frac{1}{12}$ and $\overline{x} = \frac{1/12}{1/6} = \frac{\mathbf{1}}{\mathbf{2}}$ (also by symmetry); $M_x = \int_0^1 y(\sqrt{y} - y)dy = \frac{1}{15}$ and $\overline{y} = \frac{1/15}{1/6} = \frac{\mathbf{2}}{\mathbf{5}}$.

16 Area $M = \frac{1}{2}(\pi(2)^2 - \pi(0)^2) = \frac{\mathbf{3\pi}}{\mathbf{2}}$; $M_y = 0$ and $\overline{x} = \mathbf{0}$ by symmetry; M_x for halfcircle of radius 2 minus M_x for halfcircle of radius 1 = (by Example 4) $\frac{2}{3}(2^3 - 1^3) = \frac{14}{3}$ and $\overline{y} = \frac{14/3}{3\pi/2} = \frac{\mathbf{28}}{\mathbf{9\pi}}$.

18 $I_y = \int_{-a/2}^{a/2} x^2$ (strip height) $dx = \int_{-a/2}^{a/2} x^2 a dx = \frac{a^4}{12}$.

20 $I_y = \int_{-a}^{a} x^2(2\sqrt{a^2 - x^2})dx =$ (integral 34 on last page) $[\frac{x}{4}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{4}\sin^{-1}\frac{x}{a}]_{-a}^{a} = \frac{\pi a^4}{4}$.

22 Around $x = c$ the moment of inertia is $I = \int (x-c)^2$ (strip height) $dx = \int x^2$ (strip height) $dx-$ $2c\int x$ (strip height) $dx + c^2 \int$ (strip height) $dx = I_y - 0 + (c^2)$ (area). This is smallest when $c = 0$; the moment of inertia I is smallest around the centroid.

24 Pappus cut the solid into **shells** (radius of shell $= y$, length of shell = strip width at height y). Then $V = 2\pi\overline{y}M$. This is the same volume as if the whole mass is concentrated in a shell of radius $\overline{y}$.

26 The triangle with sides $x = 0, y = 0, y = 4 - 2x$ has $M = 4$ and $\overline{y} = \frac{4}{3}$ by Example 3. Then Pappus says that the volume of the cone is $V = 2\pi(\frac{4}{3})(4) = \frac{\mathbf{32\pi}}{\mathbf{3}}$. This agrees with $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(4)^2(2)$.

28 Rotating a horizontal wire along $y = 3$ produces a **cylinder** of radius 3 and length L. Certainly $\overline{y} = 3$. The surface area is $\mathbf{2\pi(3)(L)}$ (correct for a cylinder: $A = 2\pi rh$). Rotating a vertical wire produces a **washer**: inner radius 1, outer radius $L + 1$, $A = \pi((L+1)^2 - 1^2) = \pi(L^2 + 2L)$. Pappus has $\overline{y} = \frac{L}{2} + 1$ and area $= 2\pi(\frac{L}{2} + 1)L = \pi(L^2 + 2L)$ which agrees.

30 The surface is a cone with area $2\pi\overline{y}M = \mathbf{2\pi(\frac{m}{2})\sqrt{1 + m^2}}$ (by Pappus). This agrees with Section 8.3: area of cone = side length ($s = \sqrt{1 + m^2}$) times middle circumference ($2\pi r = \pi m$). Problem 11 in Section 8.3 gives the same answer.

32 Torque $= F - 2F + 3F - 4F \cdots + 9F - 10F = \mathbf{-5F}$.

34 The *polar* moment of inertia is $I_0 = \int (x^2 + y^2)dA$, which is $I_x + I_y$. For a disk this is $\frac{\pi a^4}{4} + \frac{\pi a^4}{4} = \frac{\pi a^4}{2}$. The radius of gyration is $\overline{r} = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{\pi a^4/2}{\pi a^2}} = \frac{\mathbf{a}}{\sqrt{\mathbf{2}}}$. The rotational energy is $\frac{1}{2}I_0\omega^2 = \frac{\pi a^4 \omega^2}{4}$. This is also $\frac{1}{2}M\overline{r}^2\omega^2 = \frac{1}{2}(\pi a^2)(\frac{a^2}{2})\omega^2$, when the whole mass M turns at radius $\overline{r}$.

36 $J = \frac{I}{mr^2}$ is smaller for a solid ball than a solid cylinder because the ball has its mass **nearer the center**.

38 Get most of the mass close to the center but keep the radius large.

40 The velocity is $v^2 = \frac{2gy}{1+J}$ after a drop of $h = y$ (this is equation (11) or (12): kinetic energy = loss of potential energy). Take square roots $v = c\sqrt{y}$ with $c = \sqrt{\frac{2g}{1+J}}$; multiply by $\sin\alpha$ for vertical velocity $\frac{dy}{dt}$. Integrate $\frac{dy}{dt} = c\sqrt{y}\sin\alpha$ or $\frac{dy}{\sqrt{y}} = c\sin\alpha dt$ to find $2\sqrt{y} = c(\sin\alpha)t$ or $T = \frac{2\sqrt{h}}{c\sin\alpha}$ at the bottom $y = h$.

42 (a) **False** (a solid ball goes faster than a hollow ball) (b) **False** (if the density is varied, the center of mass moves) (c) **False** (you reduce I_x but you increase I_y : the y direction is upward) (d) **False** (imagine the jumper as an arc of a circle going just over the bar: the center of mass of the arc stays below the the bar).

8.6 Force, Work, and Energy (page 346)

Work equals **force times distance.** For a spring the force $F = \mathbf{kx}$ is proportional to the extension x (this is **Hooke's** law). With this variable force, the work in stretching from 0 to x is $W = \int \mathbf{kx\,dx} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{kx^2}$. This equals the increase in the **potential** energy V. Thus W is a **definite** integral and V is the corresponding **indefinite** integral, which includes an arbitrary **constant**. The derivative dV/dx equals **the force**. The force of gravity is

$F = \mathbf{GMm/x^2}$ and the potential is $V = \mathbf{-GMm/x}$.

In falling, V is converted to **kinetic** energy $K = \mathbf{\frac{1}{2}mv^2}$. The total energy $K + V$ is **constant** (this is the law of **conservation of energy** when there is no external force).

Pressure is force per unit **area.** Water of density w in a pool of depth h and area A exerts a downward force $F = \mathbf{whA}$ on the base. The pressure is $p = \mathbf{wh}$. On the sides the **pressure** is still wh at depth h, so the total force is $\int whl\, dh$, where l is **the side length at depth h.** In a cubic pool of side s, the force on the base is $F = \mathbf{ws^3}$, the length around the sides is $l = \mathbf{4\pi s}$, and the total force on the four sides is $F = \mathbf{2\pi ws^3}$. The work to pump the water out of the pool is $W = \int whA\, dh = \mathbf{\frac{1}{2}ws^4}$.

1 2.4 ft lb; 2.424 ... ft lb **3** 24000 lb/ft; $83\frac{1}{3}$ ft lb **5** $10x$ ft lb; $10x$ ft lb **7** 25000 ft lb; 20000 ft lb
9 864,000 Nkm **11** $5.6 \cdot 10^7$ Nkm **13** $k = 10$ lb/ft; $W = 25$ ft lb **15** $\int 60wh\, dh = 48000w, 12000w$
17 $\frac{1}{2}wAH^2; \frac{3}{8}wAH^2$ **19** $9600w$ **21** $(1 - \frac{v^2}{c^2})^{-3/2}$ **23** (800) (9800) kg **25** $\pm$ force

2 (a) Spring constant $k = \frac{75 \text{ pounds}}{3 \text{ inches}} = \mathbf{25}$ **pounds per inch**
(b) Work $W = \int_0^3 kxdx = 25(\frac{9}{2}) = \mathbf{\frac{225}{2}}$ **inch-pounds** or $\frac{225}{24}$ foot-pounds (integral starts at no stretch)
(c) Work $W = \int_3^6 kxdx = 25(\frac{36-9}{2}) = \mathbf{\frac{675}{2}}$ **inch-pounds.**

4 $W = \int_0^2 (20x - x^3)dx = [10x^2 - \frac{x^4}{4}]_0^2 = \mathbf{36}; V(2) - V(0) = 36$ so $V(2) = \mathbf{41}; k = \frac{dF}{dx} = 20 - 3x^2 = \mathbf{8}$ at $x = 2$.

6 (a) At height h the burnt fuel weighs $100(\frac{h}{25}) = 4h$ so mass of fuel left $= \mathbf{100 - 4h}$ **kg**
(b) Work $= \int Fdx = \int_0^{25}(100 - 4h)gdh =$ **(1250) (9.8) Newton–km** = 12,250,000 joules.

8 The side length at height h is $800(1 - \frac{h}{500}) = 800 - \frac{8}{5}h$ so the area is $A = \mathbf{(800 - \frac{8}{5}h)^2}$. The work is $W = \int whAdh = \int_0^{500} 100h(800 - \frac{8}{5}h)^2 dh = 100[(800)^2(\frac{500}{2})^2 - 1600(\frac{8}{5})\frac{(500)^3}{3} + (\frac{8}{5})^2\frac{(500)^4}{4}] = 10^{10}[\frac{8^2 5^2}{2} - 16(\frac{8}{3})5^2 + \frac{8^2 5^2}{4}] = \mathbf{\frac{4}{3}10^{12}}$ **ft-lbs.**

10 The change in $V = -\frac{GmM}{x}$ is $\Delta V = GmM(\frac{1}{R-10} - \frac{1}{R+10}) = GmM\frac{20}{R^2 - 10^2} = \frac{20GmM}{R^2}\frac{R^2}{R^2 - 10^2}$. The first factor is the distance (20 feet) times the force (30 pounds). The second factor is the correction (practically 1.)

12 If the rocket starts at R and reaches x, its potential energy increases by $GMm(\frac{1}{R} - \frac{1}{x})$. This equals $\frac{1}{2}mv^2$ (gain in potential = loss in kinetic energy) so $\mathbf{\frac{1}{R} - \frac{1}{x} = \frac{v^2}{2GM}}$ and $\mathbf{x = (\frac{1}{R} - \frac{v^2}{2GM})^{-1}}$. If the rocket reaches $x = \infty$ then $\frac{1}{R} = \frac{v^2}{2GM}$ or $v = \sqrt{\frac{2GM}{R}} = 25{,}000$ mph.

14 A horizontal slice with radius 1 foot, height h feet, and density ρ lbs/ft^3 has potential energy $\pi(1)^2 h\rho dh$. Integrate from $h = 0$ to $h = 4$: $\int_0^4 \pi\rho h dh = \mathbf{8\pi\rho}$.

16 (a) Pressure $= \mathbf{wh} = 62$ h lbs/ft^2 for water. (b) $\frac{\ell}{h} = \frac{80}{30}$ so $\ell = \mathbf{\frac{8}{3}h}$ (c) Total force $F = \int wh\ell dh = \int_0^{30} 62h(\frac{8}{3}h)dh = \frac{(62)(8)}{9}(30)^3 = \mathbf{1{,}488{,}000}$ **ft-lbs.**

18 (a) Work to empty a full tank: $W = \frac{1}{2}wAH^2 = \frac{1}{2}(62)(25\pi)(20)^2 = \mathbf{310{,}000\pi}$ **ft-lbs** = 973,000 ft-lbs
(b) Work to empty a half-full tank: $W = \int_{H/2}^{H} wAhdh = \frac{3}{8}wAH^2 = \mathbf{232{,}500\pi}$ **ft-lbs** = 730,000 ft-lbs.

20 Work to empty a cone-shaped tank: $W = \int wAhdh = \int_0^H w\pi r^2 \frac{h^3}{H^2}dh = \mathbf{w\pi r^2 \frac{H^2}{4}}$. For a cylinder (Problem 17) $W = \frac{1}{2}wAH^2 = w\pi r^2\frac{H^2}{2}$. So the work for a cone is half of the work for a cylinder, even though the volume is only one third. (The cone-shaped tank has more water concentrated near the bottom.)

22 The cross-section has length 10 meters and depth 2 meters at one end and 1 meter at the other end. Its area is 10 times $1\frac{1}{2} = 15$ m^2; multiply by the width 4m to find the total volume 60m^3. This is $\frac{3}{4}$ of the box volume $(10)(2)(4) = 80$, so $\mathbf{\frac{1}{4}}$ of the volume is saved. The force is **perpendicular** to the bottom of the pool. (Extra question: How much work to empty this trapezoidal pool?)