CHAPTER 5 INTEGRALS

5.1 The Idea of the Integral (page 181)

The problem of summation is to add $v_1 + \cdots + v_n$. It is solved if we find f's such that $v_j = \mathbf{f_j} - \mathbf{f_{j-1}}$. Then $v_1 + \cdots + v_n$ equals $\mathbf{f_n} - \mathbf{f_0}$. The cancellation in $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$ leaves only $\mathbf{f_n}$ and $-\mathbf{f_0}$. Taking sums is the **reverse (or inverse)** of taking differences.

The differences between 0, 1, 4, 9 are $v_1, v_2, v_3 = \mathbf{1, 3, 5}$. For $f_j = j^2$ the difference between f_{10} and f_9 is $v_{10} = \mathbf{19}$. From this pattern $1 + 3 + 5 + \cdots + 19$ equals **100**.

For functions, finding the integral is the reverse of **finding the derivative.** If the derivative of $f(x)$ is $v(x)$, then the **integral** of $v(x)$ is $f(x)$. If $v(x) = 10x$ then $f(x) = \mathbf{5x^2}$. This is the **area** of a triangle with base x and height $10x$.

Integrals begin with sums. The triangle under $v = 10x$ out to $x = 4$ has area **80**. It is approximated by four rectangles of heights 10, 20, 30, 40 and area **100.** It is better approximated by eight rectangles of heights $5, 10, \cdots, 40$ and area **90.** For n rectangles covering the triangle the area is the sum of $\mathbf{\frac{4}{n}(\frac{40}{n} + \frac{80}{n} + \cdots + 40)} =$ $\mathbf{80 + \frac{80}{n}}$. As $n \to \infty$ this sum should approach the number **80.** *That is the integral of* $v = 10x$ *from 0 to 4.*

1 $1, 3, 7, 15, 127$ **3** $-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} = \frac{1}{8} - 1$ **5** $f_j - f_0 = \frac{r^j - 1}{r-1}$ **7** $3x$ for $x \le 7, 7x - 4$ for $x \ge 1$
9 $\frac{1}{52} \frac{1}{\sqrt{52}}, \frac{2}{52}, \frac{1}{52}\sqrt{\frac{j}{52}}$ **11** Lower by 2 **13** Up, down; rectangle **15** $\sqrt{x + \Delta x} - \sqrt{x}; \Delta x; \frac{df}{dx}; \sqrt{x}$
17 6; 18; triangle **19** 18 rectangles **21** $6x - \frac{1}{2}x^2 - 10; 6 - x$ **23** $\frac{14}{27}$ **25** $x^2; x^2; \frac{1}{3}x^3$

2 (a) $2^5 - 2^4 = 16 = v_5$ (b) $1 + 2 + 4 + 8 + 16 = f_5 - f_0 = \mathbf{31}$

4 Any C can be added to $f(x)$ because the **derivative** of a constant is zero.
Any C can be added to $f_0, f_1, \cdots$ because the **difference** between f's is not changed.

6 $f_0 = \frac{1-1}{r-1} = 0; 1 + r + \cdots + r^n = f_n = \mathbf{\frac{r^n - 1}{r-1}}$.

8 The f's are $\mathbf{0, 1, -1, 2, -2}, \cdots$ Here $v_j = (-1)^{j+1}j$ or $v_j = \begin{cases} j & j \text{ odd} \\ -j & j \text{ even} \end{cases}$ and $f_j = \begin{cases} \frac{j+1}{2} & j \text{ odd} \\ \frac{-j}{2} & j \text{ even} \end{cases}$

10 Within each quarter the sum over 13 weeks is lower than the single value for the whole quarter.

12 The last rectangle for the pessimist has height $\sqrt{\frac{15}{4}}$. Since the optimist's last rectangle of area $\frac{1}{4}\sqrt{\frac{16}{4}} = \frac{1}{2}$ is missed, the total area is reduced by $\frac{1}{2}$.

14 The optimist's rectangles contain the curve. The pessimist's rectangles lie under the curve.

16 Under the $\sqrt{x}$ curve, the first triangle has base 1, height 1, area $\frac{1}{2}$. To its right is a rectangle of area 3.
Above the rectangle is a triangle of base 3, height 1, area $\frac{3}{2}$. The total area $\frac{1}{2} + 3 + \frac{3}{2} = 5$ is below the curve.

18 The total rectangular area is 21.

20 The rectangles have area 2 times 5, 2 times 3, and 2 times 1, adding to 18. This is exactly correct because each overestimate is compensated by an equal underestimate.

22 The region is a right triangle with height $6 - x$ and base $6 - x$ and area $\frac{1}{2}(6 - x)^2$. This has derivative $x - 6$, which is $-v(x)$ (minus sign because area decreases as x increases).

24 The areas under $\sqrt{x}$ and under x^2 add to 1. The same is true for the areas under x^3 and $x^{1/3}$.

Reason: Area under inverse function equals area above original function (provided $f(0) = 0$).

26 $A \approx 5.3313556$

5.2 Antiderivatives (page 186)

Integration yields the **area** under a curve $y = v(x)$. It starts from rectangles with the base $\mathbf{\Delta x}$ and heights $v(x)$ and areas $\mathbf{v(x)\Delta x}$. As $\Delta x \to 0$ the area $v_1\Delta x + \cdots + v_n\Delta x$ becomes the **integral** of $v(x)$. The symbol for the indefinite integral of $v(x)$ is $\int \mathbf{v(x)dx}$.

The problem of integration is solved if we find $f(x)$ such that $\frac{\mathbf{df}}{\mathbf{dx}} = \mathbf{v(x)}$. Then f is the **antiderivative** of v, and $\int_2^6 v(x)dx$ equals $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. The limits of integration are **2 and 6**. This is a **definite** integral, which is a **number** and not a function $f(x)$.

The example $v(x) = x$ has $f(x) = \mathbf{\frac{1}{2}x^2}$. It also has $f(x) = \mathbf{\frac{1}{2}x^2 + 1}$. The area under $v(x)$ from 2 to 6 is **16**. The constant is canceled in computing the difference $\mathbf{f(6)}$ minus $\mathbf{f(2)}$. If $v(x) = x^8$ then $f(x) = \mathbf{\frac{1}{9}x^9}$.

The sum $v_1 + \cdots + v_n = f_n - f_0$ leads to the Fundamental Theorem $\int_a^b v(x)dx = \mathbf{f(b) - f(a)}$. The **indefinite** integral is $f(x)$ and the **definite** integral is $f(b) - f(a)$. Finding the **area** under the v-graph is the opposite of finding the **slope** of the f-graph.

1 $x^5 + \frac{2}{3}x^6; \frac{5}{3}$ **3** $2\sqrt{x}; 2$ **5** $\frac{3}{4}x^{4/3}(1 + 2^{1/3}); \frac{3}{4}(1 + 2^{1/3})$ **7** $-2\cos x - \frac{1}{2}\cos 2x; \frac{5}{2} - 2\cos 1 - \frac{1}{2}\cos 2$
9 $x\sin x + \cos x; \sin 1 + \cos 1 - 1$ **11** $\frac{1}{2}\sin^2 x; \frac{1}{2}\sin^2 1$ **13** $f = C; 0$ **15** $f(b) - f(a); f_8 - f_3$
17 $8 + \frac{8}{N}$ **19** $\frac{\pi}{3}(1 + \sqrt{3}); \frac{\pi}{6}(3 + \sqrt{3}); 2$ **21** $\frac{5}{2}; \frac{205}{36}; \infty$ **23** $f(x) = 2\sqrt{x}$ **25** $\frac{1}{2}$, below -1; $\frac{1}{4}, \frac{5}{4}$
27 Increase - decrease; increase - decrease - increase
29 Area under B − area under D; time when $B = D$; time when $B - D$ is largest **33** T; F; F; T; F

2 $f(x) = \frac{1}{2}x^2 + 4x^3; f(1) - f(0) = \mathbf{4\frac{1}{2}}$. **4** $f(x) = \frac{2}{5}x^{5/2}; f(1) - f(0) = \mathbf{\frac{2}{5}}$.
6 $\frac{x^{1/3}}{x^{2/3}} = x^{-1/3}$ which has antiderivative $f(x) = \frac{3}{2}x^{2/3}; f(1) - f(0) = \mathbf{\frac{3}{2}}$.
8 $f(x) = \tan x + x; f(1) - f(0) = \mathbf{\tan 1 + 1}$. **10** $f(x) = \sin x - x\cos x; f(1) - f(0) = \mathbf{\sin 1 - \cos 1}$
12 $f(x) = \frac{1}{3}\sin^3 x; f(1) - f(0) = \mathbf{\frac{1}{3}(\sin 1)^3}$.
14 $f(x) = -x$ plus any constant C; $f(1) - f(0) = -1 + C - C = \mathbf{-1}$.
16 The sum of v's is multiplied by $\mathbf{\Delta x}$. The difference of f's is divided by $\mathbf{\Delta x}$.
18 Areas 0, 1, 2, 3 add to $A_4 = \mathbf{6}$. Each rectangle misses a triangle of base $\frac{4}{N}$ and height $\frac{4}{N}$. There are N triangles of total area $N \cdot \frac{1}{2}(\frac{4}{N})^2 = \frac{8}{N}$. So the N rectangles have area $\mathbf{8 - \frac{8}{N}}$.
20 Example: Under $y = x^2$ the rectangles with heights $0, (.8)^2, (.9)^2$ and bases .8, .1, .1 have area .145. The two rectangles with heights 0 and $(.7)^2$ and bases .7 and .3 have larger area .147.
22 Two rectangles have base $\frac{1}{2}$ and heights 2 and 1, with area $\mathbf{\frac{3}{2}}$. Four rectangles have base $\frac{1}{4}$ and heights 4, 3, 2, 1 with area $\frac{10}{4} = \mathbf{\frac{5}{2}}$. Eight rectangles have area $\frac{7}{2}$. The limiting area under $y = \frac{1}{x}$ is **infinite.**
24 $\frac{1}{3}x^3$ is an antiderivative of x^2. So the area under x^2 from 0 to 4 is $\frac{1}{3}4^3 = \mathbf{\frac{64}{3}}$. The area under $\sqrt{x}$ is $\frac{16}{3}$. Those areas do **not** combine to give a rectangle.
26 Choose $v(x)$ to be **positive** until $x = 1$, **zero** to $x = 2$, then **negative** to $x = 3$. For total area 1,

take $v(x) = 2$ then 0 then -1.

28 The area $f(4) - f(3)$ is $-\frac{1}{2}$, and $f(3) - f(2)$ is -1, and $f(2) - f(1)$ is $\frac{1}{2}(\frac{2}{3})(2) - \frac{1}{2}(\frac{1}{3})(1)$. Total -1. The graph of f_4 is x^2 to $x = 1$.

30 $y_4(x)$ equals 2 up to $x = 1$, then -3, then 0, then 1. **32** $12 =$ area of complete rectangle.

5.3 Summation Versus Integration (page 194)

The Greek letter $\sum$ indicates summation. In $\sum_1^n v_j$ the dummy variable is **j**. The limits are **j = 1** and **j = n**, so the first term is $\mathbf{v_1}$ and the last term is $\mathbf{v_n}$. When $v_j = j$ this sum equals $\frac{1}{2}\mathbf{n(n+1)}$. For $n = 100$ the leading term is $\frac{1}{2}\mathbf{100^2 = 5000}$. The correction term is $\frac{1}{2}\mathbf{n = 50}$. The leading term equals the integral of $v = x$ from 0 to 100, which is written $\int_0^{100} \mathbf{x\,dx}$. The sum is the total **area** of 100 rectangles. The correction term is the area between the **sloping line** and the **rectangles**.

The sum $\sum_{i=3}^6 i^2$ is the same as $\sum_{j=1}^4 (\mathbf{j+2})^2$ and equals **86**. The sum $\sum_{i=4}^5 v_i$ is the same as $\sum_{\mathbf{i=0}}^{\mathbf{1}} v_{i+4}$ and equals $\mathbf{v_4 + v_5}$. For $f_n = \sum_{j=1}^n v_j$ the difference $f_n - f_{n-1}$ equals $\mathbf{v_n}$.

The formula for $1^2 + 2^2 + \cdots + n^2$ is $f_n = \frac{1}{6}\mathbf{n(n+1)(2n+1)} = \frac{1}{3}\mathbf{n^3} + \frac{1}{2}\mathbf{n^2} + \frac{1}{6}\mathbf{n}$. To prove it by mathematical induction, check $f_1 = \mathbf{1}$ and check $f_n - f_{n-1} = \mathbf{n^2}$. The area under the parabola $v = x^2$ from $x = 0$ to $x = 9$ is $\frac{1}{3}\mathbf{9^3}$. This is close to the area of $\mathbf{9/\Delta x}$ rectangles of base Δx. The correction terms approach zero very slowly.

1 $\frac{25}{12}; 16$ **3** $127; 2^{n+1} - 1$ **5** $\sum_{j=1}^{50} 2j = 2550; \sum_{i=1}^{50}(2i-1) = 2500; \sum_{k=1}^{4}(-1)^{k+1}/k = \frac{7}{12}$

7 $\sum_{k=0}^{n} a_k x^k; \sum_{j=1}^{n} \sin\frac{2\pi j}{n}$ **9** $5.18738; 7.48547$ **11** $2(a_i^2 + b_i^2)$ **13** $2^{n+1} - 1; \frac{1}{11} - \frac{1}{1}$ **15** F; T

17 $\frac{df}{dx} + C; f_9 - f_8 - f_1 + f_0$ **19** $f_1 = 1; n^2 + (2n+1) = (n+1)^2$

21 $a + b + c = 1, 2a + 4b + 8c = 5, 3a + 9b + 27c = 14$; sum of squares **23** $S_{400} = 80200; E_{400} = .0025 = \frac{1}{n}$

25 $S_{100,1/3} \approx 350, E_{100,1/3} \approx .00587; S_{100,3} = 25502500, E_{100,3} = .0201$ **27** v_1 and v_2 have the same sign

29 $v_1 = 9, v_2 = 12, \Sigma\Sigma = 21$ **31** At $N = 1, 2^{N-2}$ is not 1 **33** $0; \frac{1}{n}(v_1 + \cdots + v_n)$

35 $\Delta x \sum_{j=1}^{n} v(j\Delta x)$ **37** $f(1) - f(0) = \int_0^1 \frac{df}{dx} dx$

2 $8; 1 - \frac{1}{2^n}$ **4** The sums are $-1, 1, -2, 2, \cdots$ and the sum up to $n = 6$ is 3.

6 $\sum_{j=1}^4 (-1)^{j+1} v_j;$ $\sum_{i=1}^n v_i w_i;$ $\sum_{i=1}^3 v_{2i-1}$. **8** $(a+b)^n = \sum_{j=1}^n \binom{n}{j} a^{n-j} b^j$.

10 The first sum is close to $e^{-1} = .36788$; the second is close to $e = 2.71828$; the product is extremely near 1.

12 Choose all a's and b's equal to 1. Then $n^2 \neq n$.

14 $f_n - f_0$ and $f_{13} - f_3$ (by telescoping: the other terms cancel).

16 $\sum_{i=1}^n v_i = \sum_{j=0}^{n-1} \mathbf{v_{j+1}}$ and $\sum_{i=0}^6 i^2 = \sum_{i=2}^8 (\mathbf{i-2})^2$.

18 $f_1 = \frac{1}{6}(1)(2)(3) = 1; f_n - f_{n-1} = \frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}(n-1)(n)(2n-1) = n^2$.
20 $f_1 = \frac{1}{4}(1)^2(2)^2 = 1; f_n - f_{n-1} = \frac{1}{4}n^2(n+1)^2 - \frac{1}{4}(n-1)^2n^2 = \frac{1}{4}n^2(4n) = n^3$.
22 $q = \frac{1}{9}$ (emphasize the comparison with $\int x^8 dx = \frac{1}{9}x^9$).
24 $S_{50} = 42925; I_{50} = 41666\frac{2}{3}; D_{50} = 1258\frac{1}{3}; E_{50} = 0.0302$; E_n is approximately $\frac{1.5}{n}$ and exactly $\frac{1.5}{n} + \frac{1}{2n^2}$.
26 $E_{n,p} \approx \frac{p+1}{2n}$. Reason: A closer sum S includes only half of the last term n^p (trapezoidal rule: Section 5.8). Then $\frac{1}{2}n^p/I = \frac{p+1}{2n}$.
28 $xS = x + x^2 + x^3 + \cdots$ equals $S - 1$. Then $S = \frac{1}{1-x}$. If $x = 2$ the sums are $S = \infty$.
30 $(w_{2,1} + w_{2,2} + w_{2,3}); (w_{1,3} + w_{2,3})$; the sum is the same whether i or j comes first.
32 $4v_1 + 4v_2 + 4v_3 = 4(v_1 + v_2 + v_3); (u_1v_1 + u_1v_2 + u_1v_3) + (u_2v_1 + u_2v_2 + u_2v_3) = (u_1 + u_2)(v_1 + v_2 + v_3)$.
34 $14^2 = 196 \le (13)(17) = 221; (a_1b_1 + a_2b_2)^2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ because cancellation leaves $2a_1b_1a_2b_2 \le a_1^2b_2^2 + a_2^2b_1^2$ and this can be rewritten as $0 \le (a_1b_2 - a_2b_1)^2$ which is true.
36 The rectangular area is $\Delta x \sum_{j=1}^{1/\Delta x} v((j-1)\Delta x)$ or $\Delta x \sum_{i=0}^{(1/\Delta x)-1} v(i\Delta x)$.

5.4 Indefinite Integrals and Substitutions (page 200)

Finding integrals by substitution is the reverse of the **chain** rule. The derivative of $(\sin x)^3$ is $\mathbf{3(\sin x)^2\cos x}$. Therefore the antiderivative of $\mathbf{3(\sin x)^2\cos x}$ is $\mathbf{(\sin x)^3}$. To compute $\int (1+\sin x)^2 \cos x\, dx$, substitute $u = \mathbf{1 + \sin x}$. Then $du/dx = \mathbf{\cos x}$ so substitute $du = \mathbf{\cos x\, dx}$. In terms of u the integral is $\int \mathbf{u^2\, du} = \mathbf{\frac{1}{3}u^3}$. Returning to x gives the final answer.

The best substitutions for $\int \tan(x+3)\sec^2(x+3)dx$ and $\int (x^2+1)^{10}x\, dx$ are $u = \mathbf{\tan(x+3)}$ and $u = \mathbf{x^2 + 1}$. Then $du = \mathbf{\sec^2(x+3)dx}$ and $\mathbf{2x\, dx}$. The answers are $\mathbf{\frac{1}{2}\tan^2(x+3)}$ and $\mathbf{\frac{1}{22}(x^2+1)^{11}}$. The antiderivative of $v\, dv/dx$ is $\mathbf{\frac{1}{2}v^2}$. $\int 2x\, dx/(1+x^2)$ leads to $\int \mathbf{\frac{du}{u}}$, which we don't yet know. The integral $\int dx/(1+x^2)$ is known immediately as $\mathbf{\tan^{-1}x}$.

1 $\frac{2}{3}(2+x)^{3/2} + C$ **3** $(x+1)^{n+1}/(n+1) + C (n \ne -1)$ **5** $\frac{1}{12}(x^2+1)^6 + C$ **7** $-\frac{1}{4}\cos^4 x + C$
9 $-\frac{1}{8}\cos^4 2x + C$ **11** $\sin^{-1} t + C$ **13** $\frac{1}{3}(1+t^2)^{3/2} - (1+t^2)^{1/2} + C$ **15** $2\sqrt{x} + x + C$
17 $\sec x + C$ **19** $-\cos x + C$ **21** $\frac{1}{3}x^3 + \frac{2}{3}x^{3/2}$ **23** $-\frac{1}{3}(1-2x)^{3/2}$ **25** $y = \sqrt{2x}$
27 $\frac{1}{2}x^2$ **29** $a\sin x + b\cos x$ **31** $\frac{4}{15}x^{5/2}$ **33** F; F; F; F **35** $f(x-1); 2f(\frac{x}{2})$
37 $x - \tan^{-1} x$ **39** $\int \frac{1}{u} du$ **41** $4.9t^2 + C_1 t + C_2$ **43** $f(t+3); f(t) + 3t; 3f(t); \frac{1}{3}f(3t)$

2 $\frac{-2}{3}(3-x)^{3/2} + C$ **4** $\frac{1}{1-n}(x+1)^{1-n}$, for $n \ne 1$. **6** $\frac{-2}{9}(1-3x)^{3/2} + C$ **8** $\frac{-1}{2\sin^2 x} + C$ or $-\frac{1}{2}(\sin x)^{-2} + C$
10 $\cos^3 x \sin 2x$ equals $2\cos^4 x \sin x$ and its integral is $\frac{-2}{5}\cos^5 x + C$ **12** $\frac{-1}{3}(1-t^2)^{3/2} + C$
14 Write $u = 1 - t^2$ and $du = -2t\, dt$ to give $\int (1-u)\sqrt{u}\frac{du}{-2} = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C = -\frac{1}{3}(1-t^2)^{3/2} + \frac{1}{5}(1-t^2)^{5/2} + C$
16 The integral of $x^{1/2} + x^2$ is $\frac{2}{3}x^{3/2} + \frac{1}{3}x^3 + C$.
18 Set $u = \tan x$ and $du = \sec^2 x\, dx$. The integral of $u^2 du$ is $\frac{1}{3}\tan^3 x + C$.
20 Write $\sin^3 x$ as $(1 - \cos^2 x)\sin x$. The integrals of $-\cos^2 x \sin x$ and $\sin x$ give $\frac{1}{3}\cos^3 x - \cos x + C$.
22 Substitute $y = cx^n$ to find $ncx^{n-1} = (cx^n)^2$. Match exponents: $n - 1 = 2n$ or $n = -1$. Match coefficients: $nc = c^2$ or $c = n = -1$. Answer $\mathbf{y = -1/x}$.
24 $y = -\sqrt{1-2x} + C$ **26** $dy/dx = x/y$ gives $y\, dy = x\, dx$ or $y^2 = x^2 + C$ or $y = \sqrt{x^2 + C}$.

28 $y = \frac{1}{120}x^5 + C_1x^4 + C_2x^3 + C_3x^2 + C_4x + C_5$
30 $y = \frac{1}{9}x^3$ comes from $y^{-1/2}dy = x^{1/2}dx$ or $2y^{1/2} = \frac{2}{3}x^{3/2}(+C)$ **32** $\frac{dy}{dx} = x^{1/4}$ gives $y = \frac{4}{5}x^{5/4} + C$
34 (a) **False:** The derivative of $\frac{1}{2}f^2(x)$ is $f(x)\frac{df}{dx}$ (b) **True:** The chain rule gives $\frac{d}{dx}f(v(x)) = \frac{df}{dx}(v(x))$ times $\frac{dv}{dx}$ (c) **False:** These are inverse *operations* not inverse functions and (d) is **True.**
36 $\frac{1}{2}f(2x-1) + C; \frac{1}{2}f(x^2) + C$ **38** $\int(x^4 + 2x^2 + 1)dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C.$
40 Use $u = 1 + x^2$ and $du = 2x\,dx$ and $x^2 = u - 1$. Then $\int \frac{du}{u^2} - \int \frac{du}{u^3}$ is $-\frac{1}{u} + \frac{1}{2u^2} + C = \frac{-1}{1+x^2} + \frac{1}{2(1+x^2)^2} + C.$
42 $y = C_1x^3 + C_2x^2 + C_3x + C_4.$

5.5 The Definite Integral (page 205)

If $\int_a^x v(x)dx = f(x) + C$, the constant C equals $-\mathbf{f(a)}$. Then at $x = a$ the integral is **zero.** At $x = b$ the integral becomes $\mathbf{f(b) - f(a)}$. The notation $f(x)]_a^b$ means $\mathbf{f(b) - f(a)}$. Thus $\cos x]_0^\pi$ equals $\mathbf{-2}$. Also $[\cos x + 3]_0^\pi$ equals $\mathbf{-2}$, which shows why the antiderivative includes an arbitrary **constant.** Substituting $u = 2x - 1$ changes $\int_1^3 \sqrt{2x-1}\,dx$ into $\int_1^5 \frac{1}{2}\sqrt{u}\,du$ (with limits on u).

The integral $\int_a^b v(x)dx$ can be defined for any **continuous** function $v(x)$, even if we can't find a simple **antiderivative.** First the meshpoints $x_1, x_2, \cdots$ divide $[a, b]$ into subintervals of length $\Delta x_k = \mathbf{x_k - x_{k-1}}$. The upper rectangle with base Δx_k has height $M_k =$ **maximum of v(x) in interval k.**
The upper sum S is equal to $\mathbf{\Delta x_1 M_1 + \Delta x_2 M_2} + \cdots$. The lower sum s is $\mathbf{\Delta x_1 m_1 + \Delta x_2 m_2} + \cdots$. The **area** is between s and S. As more meshpoints are added, S **decreases** and s **increases.** If S and s approach the same **limit**, that defines the integral. The intermediate sums S^*, named after **Riemann,** use rectangles of height $v(x_k^*)$. Here x_k^* is any point between $\mathbf{x_{k-1}}$ and $\mathbf{x_k}$, and $S^* = \sum \mathbf{\Delta x_k v(x_k^*)}$ approaches the area.

1 $C = -f(2)$ **3** $C = f(3)$ **5** $f(t)$ is wrong **7** $C = 0$ **9** $C = 0$
11 $u = x^2 + 1; \int_1^2 u^{10}\frac{du}{2} = \frac{u^{11}}{22}]_1^2 = \frac{2^{11}-1}{22}$ **13** $u = \tan x; \int_0^1 u\,du = \frac{1}{2}$
15 $u = \sec x; \int_1^{\sqrt{2}} u\,du = \frac{1}{2}$ (same as **13**) **17** $u = \frac{1}{x}, x = \frac{1}{u}, dx = \frac{-du}{u^2}; \int_1^{1/2} \frac{-du}{u}$
19 $S = \frac{1}{2}(\frac{1}{4}+1)^4 + \frac{1}{2}(1+1)^4; s = \frac{1}{2}(0) + \frac{1}{2}(\frac{1}{4}+1)^4$
21 $S = \frac{1}{2}[(\frac{1}{2})^3 + 1^3 + (\frac{3}{2})^3 + 2^3]; s = \frac{1}{2}[0^3 + (\frac{1}{2})^3 + 1^3 + (\frac{3}{2})^3]$
23 $S = \frac{1}{4}[(\frac{17}{16})^4 + (\frac{5}{4})^4 + (\frac{25}{16})^4 + 2^4]$ **25** Last rectangle minus first rectangle
27 $S = .07$ since 7 intervals have points where $W = 1$. The integral of $W(x)$ exists and equals zero.
29 M is increasing so Problem **25** gives $S - s = \Delta x(1 - 0)$; area from graph up to $y = 1$ is $\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot\frac{1}{2} + \cdots = \frac{1}{2}(1 + \frac{1}{4} + \frac{1}{16} + \cdots) = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$; area under graph is $\frac{1}{3}$.
31 $f(x) = 3 + \int_0^x v(x)dx; f(x) = \int_3^x v(x)dx$ **33** T;F;T;F;T;F;T

2 $C = -\mathbf{f(1)}$ so $\int_1^4 \frac{df}{dx}dx = f(4) - f(1)$.
4 $C = -f(\sin\frac{\pi}{2}) = -\mathbf{f(1)}$ so that $\int v(u)du = f(u) + C = f(\sin x)]_{\pi/2}^x$.
6 $C = \mathbf{0}$. No constant in the derivative! **8** $C = -f(0)$ so $\int_0^{x^2} v(t)dt = f(t)]_0^{x^2}$.
10 Set $x = 2t$ and $dx = 2dt$. Then $\int_{x=0}^2 v(x)dx = \int_{t=0}^1 v(2t)(2dt)$ so $C = \mathbf{2}$.

12 Choose $u = \sin x$. Then $u = 0$ at $x = 0$ and $u = 1$ at $x = \frac{\pi}{2}$. The integral is $\int_0^1 u^8 du = [\frac{1}{9}u^9]_0^1 = \mathbf{\frac{1}{9}}$.

14 $u = x^2$ has $du = 2x\,dx$; $u = 0$ at $x = 0$ and $u = 4$ at $x = 2$; then $\int_0^2 x^{2n} x\,dx = \int_0^4 u^n \frac{du}{2} = \frac{u^{n+1}}{2(n+1)}]_0^4 = \mathbf{\frac{4^{n+1}}{2(n+1)}}$.

16 Choose $u = x^2$ with $du = 2x\,dx$ and $u = 0$ at $x = 0$ and $u = 1$ at $x = 1$. Then $\int_0^1 \frac{du}{2\sqrt{1-u}} = -\sqrt{1-u}]_0^1 = \mathbf{+1}$. (Could also choose $u = 1 - x^2$.)

18 With $u = 1 - x$ and $du = -dx$ the limits are $u = 1$ at $x = 0$ and $u = 0$ at $x = 1$. The integral $\int_0^1 x^3(1-x)^3 dx$ becomes $\int_1^0 (1-u)^3 u^3(-du)$. Reverse limits by Property 3 on the next page: $\int_0^1 (1-u)^3 u^3 du$ which is the **same as the original** (no progress). Compute by writing out $x^3(1 - 3x + 3x^2 - x^3)$ and integrating each term: $[\frac{1}{4}x^4 - \frac{3}{5}x^5 + \frac{3}{6}x^6 - \frac{1}{7}x^7]_0^1 = \frac{1}{4} - \frac{3}{5} + \frac{3}{6} - \frac{1}{7}$.

20 $\sin 2\pi x$ has maximum $M_1 = 1$ and minimum $m_1 = 0$ in the interval to $x = \frac{1}{2}$; then $M_2 = 0$ and $m_2 = -1$ in the interval to $x = 1$. Thus $S = \mathbf{\frac{1}{2}(1)}$ and $s = \mathbf{\frac{1}{2}(-1)}$.

22 Maximum of x in the four intervals is: $M_k = -\frac{1}{2}, 0, \frac{1}{2}, 1$. Minimum is $m_k = -1, -\frac{1}{2}, 0, \frac{1}{2}$. Then $S = \frac{1}{2}(-\frac{1}{2} + 0 + \frac{1}{2} + 1) = \mathbf{\frac{1}{2}}$ and $s = \frac{1}{2}(-1 - \frac{1}{2} + 0 + \frac{1}{2}) = \mathbf{-\frac{1}{2}}$.

24 The exact area is $\int_0^2 x^3 dx = \frac{x^4}{4}]_0^2 = 4$. Then $S - 4$ is less than $S - s = 2^3\Delta x$. So $S < 4.001$ if $2^3\Delta x < .001$ or $\Delta x < \mathbf{\frac{1}{8}(.001) = .000125}$.

26 All midpoints of the intervals with $\Delta x = \frac{1}{n}$ are fractions. So $\mathbf{V(x^*) = 1}$ **at these midpoints** $\mathbf{x^*}$. The upper Riemann sum S^* is the sum of Δx's times $1 =$ length of interval of integration. This stays the same as $n \to \infty$ but other choices of x^* give $S^* = 0$: not Riemann integrable.

28 (**Correction:** Change v to M.) The graph of $M(x)$ is above *horizontal* rectangles of total area $(\frac{1}{2})(\frac{1}{2}) + (\frac{1}{4})(\frac{1}{4}) + \cdots = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \mathbf{\frac{1}{3}}$. With $\Delta x = \frac{1}{3}$ the M's are $0, \frac{1}{2}, 1$ with $S = \frac{1}{3}(0 + \frac{1}{2} + 1) = \frac{1}{2}$. The m's are $0, 0, \frac{1}{2}$ with $s = \frac{1}{3}(0 + 0 + \frac{1}{2}) = \frac{1}{6}$.

30 Check $f(1) = \int_1^1 v(x)dx = 0$. Check $\frac{d}{dx}\int_x^1 v(x)dx = \frac{d}{dx}(-\int_1^x v(x)dx) = -v(x)$. Then $f(x)$ is correct.

5.6 Properties of the Integral and Average Value (page 212)

The integrals $\int_0^b v(x)dx$ and $\int_b^5 v(x)dx$ add to $\mathbf{\int_0^5 v(x)dx}$. The integral $\int_3^1 v(x)\,dx$ equals $\mathbf{-\int_1^3 v(x)dx}$. The reason is **that the steps Δx are negative.** If $v(x) \le x$ then $\int_0^1 v(x)\,dx \le \mathbf{\frac{1}{2}}$. The average value of $v(x)$ on the interval $1 \le x \le 9$ is defined by $\mathbf{\frac{1}{8}\int_1^9 v(x)\,dx}$. It is equal to $v(c)$ at a point $x = c$ which is **between 1 and 9.** The rectangle across the interval with height $v(c)$ has the same area as **the region under $v(x)$.** The average value of $v(x) = x + 1$ on the interval $1 \le x \le 9$ is **6.**

If x is chosen from 1,3,5,7 with equal probabilities $\frac{1}{4}$, its expected value (average) is **4.** The expected value of x^2 is **21.** If x is chosen from $1, 2, \cdots, 8$ with probabilities $\frac{1}{8}$, its expected value is **4.5.** If x is chosen from $1 \le x \le 9$, the chance of hitting an integer is **zero.** The chance of falling between x and $x + dx$ is $p(x)dx = \mathbf{\frac{1}{8}dx}$. The expected value $E(x)$ is the integral $\mathbf{\int_1^9 \frac{x}{8}\,dx}$. It equals **5.**

1 $\bar{v} = \frac{1}{2}\int_{-1}^1 x^4 dx = \frac{1}{5}$ equals c^4 at $c = \pm(\frac{1}{5})^{1/4}$ **3** $\bar{v} = \frac{1}{\pi}\int_0^\pi \cos^2 x\,dx = \frac{1}{2}$ equals $\cos^2 c$ at $c = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

5 $\bar{v} = \int_1^2 \frac{dx}{x^2} = \frac{1}{2}$ equals $\frac{1}{c^2}$ at $c = \sqrt{2}$ **7** $\int_3^5 v(x)dx$ **9** False, take $v(x) < 0$

11 True; $\frac{1}{3}\int_0^1 v(x)dx + \frac{2}{3}\cdot\frac{1}{2}\int_1^3 v(x)dx = \frac{1}{3}\int_0^3 v(x)dx$ **13** False; when $v(x) = x^2$ the function $x^2 - \frac{1}{3}$ is even

15 False; take $v(x) = 1$; factor $\frac{1}{2}$ is missing **17** $\bar{v} = \frac{1}{b-a}\int_a^b v(x)dx$ **19** 0 and $\frac{2}{\pi}$

21 $v(x) = Cx^2; v(x) = C$. This is "constant elasticity" in economics (Section 2.2) **23** $\overline{V} \to 0; \overline{V} \to 1$
25 $\frac{1}{2}\int_0^2(a-x)dx = a+1$ if $a > 2$; $\frac{1}{2}\int_0^2|a-x|dx = \frac{1}{2}$ area $= \frac{a^2}{2} - a + 1$ if $a < 2$; distance = absolute value
27 Small interval where $y = \sin\theta$ has probability $\frac{d\theta}{\pi}$; the average y is $\int_0^\pi \frac{\sin\theta\, d\theta}{\pi} = \frac{2}{\pi}$
29 Area under $\cos\theta$ is 1. Rectangle $0 \le \theta \le \frac{\pi}{2}, 0 \le y \le 1$ has area $\frac{\pi}{2}$. Chance of falling across a crack is $\frac{1}{\pi/2} = \frac{2}{\pi}$.
31 $\frac{1}{6^3}, \frac{3}{6^3}, \cdots, \frac{1}{6^3}$; 10.5 **33** $\frac{1}{t}\int_0^t 220\cos\frac{2\pi t}{60}dt = \frac{1}{t}\cdot 220 \cdot \frac{60}{2\pi}\sin\frac{2\pi t}{60} = V_{\text{ave}}$
35 Any $v(x) = v_{\text{even}}(x) + v_{\text{odd}}(x)$; $(x+1)^3 = (3x^2+1) + (x^3+3x)$; $\frac{1}{x+1} = \frac{1}{1-x^2} - \frac{x}{1-x^2}$
37 16 per class; $\frac{6}{64}$; $E(x) = \frac{1800}{64} = \frac{225}{8}$ **39** F; F; T; T
41 $f(x) = \left\{\begin{array}{ll} \frac{1}{2}(x-2)^2 & x \ge 2 \\ -\frac{1}{2}(x-2)^2 & x \le 2 \end{array}\right\} + C$; $f(5) - f(0) = \frac{9}{2} + \frac{4}{2} = \frac{13}{2}$

2 $v_{\text{ave}} = \frac{1}{2}\int_{-1}^1 x^5 dx = 0$ which equals c^5 at $c = 0$.
4 $v_{\text{ave}} = \frac{1}{4}\int_0^4 \sqrt{x}\, dx = \frac{1}{6}4^{3/2}$ which equals $\sqrt{c}$ at $c = \frac{1}{36}4^3 = \frac{16}{9}$.
6 $v_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sin x)^9 dx = 0$ (odd function over symmetric interval $-\pi$ to π). This equals $(\sin c)^9$ at $c = -\pi$ and 0 and π.
8 $2\int_1^5 x\, dx = x^2]_1^5 = 24$. Remember to reverse sign in the integral from 5 to 1.
10 **False.** The interval keeps length 3 but if $v(x) = x$ the integral changes.
12 **False:** This is the average value of $\frac{df}{dx}$.
14 **False:** $-1 \le \sin x \le 1$ but the derivatives do *not* satisfy $0 \le \cos x \le 0$.
16 (a) **False:** strictly speaking the antiderivatives of x^2 are $\frac{1}{3}x^3 + C$; this is odd only when $C = 0$
(b) **False:** $(x)^2$ is even.
18 The average of $\frac{df}{dx}$ is $\frac{f(6)-f(2)}{6-2} = -1$.
20 **Property 6** proves both (a) and (b) because $v(x) \le |v(x)|$ and also $-v(x) \le |v(x)|$. So their integrals maintain these inequalities.
22 If v is increasing then $v(t) \le v(x)$ when $t \le x$. Apply Property 6: $\int_0^x v(t)dt \le \int_0^x v(x)dt$. Note $v(x)$ is constant in the last integral, which is $t\, v(x)]_0^x = xv(x)$.
24 Suppose $v_n < \epsilon$ for n larger than N. (N is now *fixed*.) Then the average $\frac{v_1+\cdots+v_n}{n}$ is less than $\frac{v_1+\cdots+v_N+(n-N)\epsilon}{n}$. As $n \to \infty$ this approaches $\frac{n\epsilon}{n} = \epsilon$. So the average goes below any ϵ and must approach zero.
28 $Y_{\text{ave}} = \frac{1}{2}\int_{-1}^1 \sqrt{1-x^2}dx = \frac{1}{2}$ (area) $= \frac{1}{4}\pi$. A uniform distribution of Q along the base is different from a uniform distribution of P along the semicircle.
30 This needle falls across a crack when $y < x\cos\theta$ (change the 1's in the Buffon needle figure to x's).
Following Problem 29, the shaded region lies under $y = x\cos\theta$ and under $y = 1$.
Keeping $x < 1$ (*shorter needles only*) the area is $\int_0^{\pi/2} x\cos\theta\, d\theta = x\sin\theta]_0^{\pi/2} = x$.
This fraction $\frac{x}{\pi/2} = \frac{2x}{\pi}$ of the total area is the probability of falling across a crack.
32 The square has area 1. The area under $y = \sqrt{x}$ is $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$.
34 When x is replaced by $-x$, the function $\frac{1}{2}(v(x) + v(-x))$ is unchanged (even). The function $\frac{1}{2}(v(x) - v(-x))$ becomes $\frac{1}{2}(v(-x) - v(x))$ so signs are reversed (odd function).
36 $f'(-x) = \lim \frac{f(-x+h)-f(-x)}{h} =$ (when f is even) $\lim \frac{f(x-h)-f(x)}{h} = -f'(x)$. Thus f' is odd.
38 Average size is $\frac{G}{n}$. The chance of an individual belonging to group 1 is $\frac{x_1}{G}$. The expected size is sum of size times probability: $E(x) = \sum \frac{x_i^2}{G}$. This exceeds $\frac{G}{n}$ by the Schwarz inequality:
$(1x_1 + \cdots + 1x_n)^2 \le (1^2 + \cdots + 1^2)(x_1^2 + \cdots + x_n^2)$ is the same as $G^2 \le n\sum x_i^2$.
40 This formula for $f(x)$ jumps from 9 to -9. The correct formula (with continuous f) is x^2 then $18 - x^2$. Then $f(4) - f(0) = 2$, which is $\int_0^4 v(x)dx$.
42 The integral of $v(x) - v_{\text{ave}}$ is zero (equal positive and negative areas): $\int_a^b v_{\text{ave}}dx = (b-a)v_{\text{ave}} = \int_a^b v(x)dx$.

5.7 The Fundamental Theorem and Its Applications (page 219)

The area $f(x) = \int_a^x v(t)\,dt$ is a function of **x**. By Part **1** of the Fundamental Theorem, its derivative is $\mathbf{v(x)}$. In the proof, a small change Δx produces the area of a thin **rectangle.** This area Δf is approximately $\mathbf{\Delta x}$ times $\mathbf{v(x)}$. So the derivative of $\int_a^x t^2\,dt$ is $\mathbf{x^2}$.

The integral $\int_x^b t^2\,dt$ has derivative $\mathbf{-x^2}$. The minus sign is because **x is the lower limit.** When both limits $a(x)$ and $b(x)$ depend on x, the formula for df/dx becomes $\mathbf{v(b(x))\frac{db}{dx}}$ minus $\mathbf{v(a(x))\frac{da}{dx}}$. In the example $\int_2^{3x} t\,dt$, the derivative is **9x**.

By Part **2** of the Fundamental Theorem, the integral of df/dx is $\mathbf{f(x) + C}$. In the special case when $df/dx = 0$, this says that **the integral is constant.** From this special case we conclude: If $dA/dx = dB/dx$ then $A(x) = \mathbf{B(x) + C}$. If an antiderivative of $1/x$ is $\ln x$ (whatever that is), then automatically $\int_a^b dx/x = \mathbf{\ln b - \ln a}$.

The square $0 \le x \le s, 0 \le y \le s$ has area $A = \mathbf{s^2}$. If s is increased by Δs, the extra area has the shape of **an L**. That area ΔA is approximately $\mathbf{2s\,\Delta s}$. So $dA/ds = \mathbf{2s}$.

1 $\cos^2 x$ **3** 0 **5** $(x^2)^3(2x) = 2x^7$ **7** $v(x+1) - v(x)$ **9** $\frac{\sin^2 x}{x} - \frac{1}{x^2}\int_0^x \sin^2 t\,dt$
11 $\int_0^x v(u)du$ **13** 0 **15** 0 **17** $u(x)v(x)$ **19** $\sin^{-1}(\sin x)\cos x = x\cos x$
21 F; F; F; T **23** Taking derivatives $v(x) = (x\cos x)' = \cos x - x\sin x$
25 Taking derivatives $-v(-x)(-1) = v(x)$ so v is even **27** F; T; T; F
29 $\int_1^x v(x)dx = \int_0^x v(x)dx - \int_0^1 v(x)dx = \frac{1}{x+2} - \frac{1}{1+2}$
31 $V = s^3; A = 3s^2$; half of hollow cube; $\Delta V \approx 3s^2 dS; 3s^2$ (which is A)
33 $dH/dr = 2\pi^2 r^3$ **35** Wedge has length $r \approx$ height of triangle; $\int_0^{\pi/2} \frac{1}{2}r^2 d\theta = \frac{\pi r^2}{4}$
37 $r = \frac{1}{\cos\theta}; \frac{d\theta}{2\cos^2\theta}; \int_0^{\pi/4} \frac{d\theta}{2\cos^2\theta} = \frac{\tan\theta}{2}\Big]_0^{\pi/4} = \frac{1}{2}$
39 $x = y^2; \int_0^2 y^2 dy = \frac{y^3}{3}\Big]_0^2 = \frac{8}{3}$; vertical strips have length $2 - \sqrt{x}$
41 Length $\sqrt{2}a$; width $\frac{da}{\sqrt{2}}; \int_0^1 a\,da = \frac{1}{2}$ **43** The differences of the sums $f_j = v_1 + v_2 + \cdots + v_j$ are $f_j - f_{j-1} = v_j$
45 No, $\int_0^x a(t)dt = \frac{df}{dx}(x) - \frac{df}{dx}(0)$ and $\int_0^1(\int_0^x a(t)dt)dx = f(1) - f(0) - \frac{df}{dx}(0)$

2 $\frac{d}{dx}\int_x^1 \cos 3t\,dt = -\cos \mathbf{3x}$. **4** $\frac{d}{dx}\int_0^2 x^n dt = \frac{d}{dx}2x^n = \mathbf{2nx^{n-1}}$.
6 $\frac{d}{dx}\int_{-x}^{x/2} v(u)du = \frac{1}{2}v(\frac{x}{2}) - (-1)v(-x) = \mathbf{\frac{1}{2}v(\frac{x}{2}) + v(-x)}$
8 $\frac{d}{dx}(\frac{1}{x}\int_0^x v(t)dt)$ by the product rule is $\mathbf{\frac{1}{x}v(x) - \frac{1}{x^2}\int_0^x v(t)dt}$ which is $\frac{1}{x^2}\int_0^x (v(x) - v(t))dt$.
10 $\frac{d}{dx}(\frac{1}{2}\int_x^{x+2} x^3 dx) = \mathbf{\frac{1}{2}(x+2)^3 - \frac{1}{2}x^3}$ **12** $\frac{d}{dx}\int_0^x (\frac{df}{dx})^2 dx = \mathbf{(\frac{df}{dx})^2(x)}$ **14** $\frac{d}{dx}\int_0^x v(-t)dt = \mathbf{v(-x)}$
16 $\frac{d}{dx}\int_{-x}^x \sin t\,dt = \sin x - (-1)\sin(-x) = \mathbf{0}$. (The integral is zero because $\sin t$ is odd)
18 $\frac{d}{dx}\int_{a(x)}^{b(x)} 5dt = 5\frac{db}{dx} - 5\frac{da}{dx}$. **20** $\frac{d}{dx}(\int_0^{f(x)} \frac{df}{dt}dt) = \frac{d}{dx}f(f(x)) = \mathbf{f'(f(x))f'(x)}$.
22 $F(\pi + \Delta x) - F(\pi)$ is the strip of width $2\Delta x$ beyond $x = 2\pi$ on the sine graph minus the strip of width Δx beyond $x = \pi$ (compare Figure 5.15b). $F(\Delta x) - F(0)$ is the strip from Δx to $2\Delta x$.
24 If $\frac{df}{dx} = 2x$ then the derivative of $f(x) - x^2$ is zero. So $f(x) - x^2$ is a constant C (this was the point of equation (7)).
26 $\int_{2x}^{3x} \frac{dt}{t} = \int_{u=2}^{3} \frac{x\,du}{xu} = \mathbf{\int_2^3 \frac{du}{u}}$ (which is a number – not dependent on x). **28** $\int_1^x v(x)dx = x^n\Big]_1^x = \mathbf{x^n - 1}$.
30 When the side s is increased, only *two* strips are added to the square (on the right side and top). So $dA = 2s\,ds$

which agrees with $A = s^2$.

32 The 4-dimensional cube has volume $H = \mathbf{s^4}$. The face with $x = s$ is a **3-dimensional cube**. Its volume is $V = s^3$. Four faces have volume $\mathbf{4s^3}$. Increase by Δs gives $\Delta H = (\mathbf{s} + \mathbf{\Delta s})^4 - s^4$. So $dH/ds = \mathbf{4s^3}$.

34 $\int x\,dy = \int_0^1 \sqrt{y}\,dy = \frac{2}{3}y^{3/2}]_0^1 = \frac{2}{3}$.

36 A is the area under $y = \sqrt{r^2 - x^2}$ (quarter of a circle). Then $\int_{x=0}^{r} \sqrt{r^2 - x^2}dx = \int_{\theta=0}^{\pi/2}(r\cos\theta)(r\cos\theta\,d\theta) = \frac{\pi}{4}\mathbf{r^2}$ because the average value of $\cos^2\theta$ is $\frac{1}{2}$. (Its integral is $\frac{1}{2}(\theta + \sin\theta\cos\theta)]_0^{\pi/2} = \frac{\pi}{4}$.)

38 The triangle ends at the line $x + y = 1$ or $r\cos\theta + r\sin\theta = 1$. The area is $\frac{1}{2}$, by geometry. So the area integral $\int_{\theta=0}^{\pi/2} \frac{1}{2}r^2 d\theta = \frac{1}{2}$: Substitute $r = \frac{1}{\cos\theta + \sin\theta}$.

40 Rings have area $2\pi r\,dr$, and $\int_2^3 2\pi r\,dr = \pi r^2]_2^3 = 5\pi$. Strips are difficult because they go in and out of the ring (see Figure 14.5b on page 528).

42 The strip around the ellipse does not have constant width dr. The width is dr in the x direction and 2 dr in the y direction.

44 The sum to $j = n$ of the differences $f_j - f_{j-1}$ is $f_n + C$ (and the constant is $C = -f_0$). This sum telescopes: $(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2)\cdots$

46 At $t = 1$ the area is under the parabola $y = -x^2 + 1$. The line along the base has length $\frac{dA}{dt}$, because an increase Δt raises the mountain by Δt and adds a strip along the base. These strips have increasing length so $\frac{d}{dt}(\frac{dA}{dt}) > 0$.

5.8 Numerical Integration (page 226)

To integrate $y(x)$, divide $[a, b]$ into n pieces of length $\Delta x = (\mathbf{b} - \mathbf{a})/\mathbf{n}$. R_n and L_n place a **rectangle** over each piece, using the height at the right or **left** endpoint: $R_n = \Delta x(y_1 + \cdots + y_n)$ and $L_n = \mathbf{\Delta x(y_0 + \cdots + y_{n-1})}$. These are **first**-order methods, because they are incorrect for $y = \mathbf{x}$. The total error on $[0, 1]$ is approximately $\mathbf{\frac{\Delta x}{2}(y(1) - y(0))}$. For $y = \cos\pi x$ this leading term is $\mathbf{-\Delta x}$. For $y = \cos 2\pi x$ the error is very small because $[0, 1]$ is a complete **period**.

A much better method is $T_n = \frac{1}{2}R_n + \mathbf{\frac{1}{2}L_n} = \Delta x[\frac{1}{2}y_0 + \mathbf{1}y_1 + \cdots + \mathbf{\frac{1}{2}}y_n]$. This **trapezoidal** rule is **second**-order because the error for $y = x$ is **zero**. The error for $y = x^2$ from a to b is $\mathbf{\frac{1}{6}(\Delta x)^2(b - a)}$. The **midpoint** rule is twice as accurate, using $M_n = \Delta x[\mathbf{y_{\frac{1}{2}}} + \cdots + \mathbf{y_{n-\frac{1}{2}}}]$.

Simpson's method is $S_n = \frac{2}{3}M_n + \mathbf{\frac{1}{3}T_n}$. It is **fourth**-order, because the powers $\mathbf{1, x, x^2, x^3}$ are integrated correctly. The coefficients of $y_0, y_{1/2}, y_1$ are $\mathbf{\frac{1}{6}, \frac{4}{6}, \frac{1}{6}}$ times Δx. Over three intervals the weights are $\Delta x/6$ times $1 - \mathbf{4} - \mathbf{2} - \mathbf{4} - \mathbf{2} - \mathbf{4} - \mathbf{1}$. Gauss uses **two** points in each interval, separated by $\Delta x/\sqrt{3}$. For a method of order p the error is nearly proportional to $\mathbf{(\Delta x)^p}$.

1 $\frac{1}{2}\Delta x(v_0 - v_n)$ **3** $1, .5625, .3025; 0, .0625, .2025$ **5** $L_8 \approx .1427, T_8 \approx .2052, S_8 \approx .2000$

7 $p = 2$: for $y = x^2, \frac{1}{4}\cdot 0^2 + \frac{1}{2}\cdot(\frac{1}{2})^2 + \frac{1}{4}\cdot 1^2 \neq \frac{1}{3}$ **9** For $y = x^2$, error $\frac{1}{6}(\Delta x)^2$ from $\frac{1}{2} - \frac{1}{3}, y_1' = 2\Delta x$

13 8 intervals give $\frac{(\Delta x)^2}{12}[-\frac{1}{b^2} + \frac{1}{a^2}] = \frac{1}{1024} < .001$ **15** $f''(c)$ is $y'(c)$ **17** $\infty; .683, .749, .772 \to \frac{\pi}{4}$

19 $A + B + C = 1, \frac{1}{2}B + C = \frac{1}{2}, \frac{1}{4}B + C = \frac{1}{3}$; Simpson

21 $y = 1$ and x on $[0,1]$: $L_n = 1$ and $\frac{1}{2} - \frac{1}{2n}, R_n = 1$ and $\frac{1}{2} + \frac{1}{2n}$, so only $\frac{1}{2}L_n + \frac{1}{2}R_n$ gives 1 and $\frac{1}{2}$

23 $T_{10} \approx 500{,}000{,}000; T_{100} \approx 50{,}000{,}000; 25{,}000\pi$

25 $a = 4, b = 2, c = 1; \int_0^1 (4x^2 + 2x + 1)dx = \frac{10}{3}$; Simpson fits parabola **27** $c = \frac{1}{4320}$

2 The trapezoidal error has a factor $(\Delta x)^2$. It is reduced by 4 when Δx is cut in half. The error in Simpson's rule is proportional to $(\Delta x)^4$ and is reduced by 16.

4 Computing L_n and R_n requires n evaluations each. $T_n = \frac{1}{2}y_0 + y_1 + \cdots$ requires $n + 1$: more efficient.

8 The trapezoidal rule for $\int_0^{2\pi} \frac{dx}{3+\sin x} = \frac{\pi}{\sqrt{2}} = 2.221441$ gives $\frac{2\pi}{3} \approx 2.09$ (two intervals), $\frac{7\pi}{9} \approx 2.221$ (three intervals), $\frac{17\pi}{24} \approx 2.225$ (four intervals is worse??), and 7 digits for T_5. Curious that $M_n = T_n$ for odd n.

10 The midpoint rule is exact for 1 and x. For $y = x^2$ the integral from 0 to Δx is $\frac{1}{3}(\Delta x)^3$ and the rule gives $(\Delta x)(\frac{\Delta x}{2})^2$. This error $\frac{1}{4}(\Delta x)^3 - \frac{1}{3}(\Delta x)^3 = -\frac{1}{12}(\Delta x)^3$ does equal $-\frac{(\Delta x)^2}{24}(y'(\Delta x) - y'(0))$.

12 The first and third integrals give accurate answers more easily.

14 Correct answer $\frac{2}{3}$. $T_1 = .5, T_{10} \approx .66051, T_{100} \approx .66646$. $M_1 \approx .707, M_{10} \approx .66838, M_{100} \approx .66673$. What is the rate of decrease of the error?

16 $\int_{-1}^{1} \frac{dx}{2+\cos 6\pi x} = \frac{2}{\sqrt{3}}$ is approximated by $T_2 = 1(\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}) = \frac{2}{3}$ and $S_2 = \frac{1}{6}(\frac{1}{3} + 4 \cdot \frac{1}{1} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{1} + \frac{1}{3}) = \frac{14}{9}$ and $G_1 = \frac{1}{2+\cos(-6\pi/\sqrt{3})} + \frac{1}{2+\cos(6\pi/\sqrt{3})} = .776$ (large error) and $G_2 = \frac{1}{2+\cos(6\pi\frac{1+1/\sqrt{3}}{2})} + \frac{1}{2+\cos(6\pi\frac{1-1\sqrt{3}}{2})} \approx 1.5$.

18 The trapezoidal rule $T_4 = \frac{\pi}{8}(\frac{1}{2} + \cos^2\frac{\pi}{8} + \cos^2\frac{\pi}{4} + \cos^2\frac{3\pi}{8} + 0)$ gives the correct answer $\frac{\pi}{4}$.

20 $\frac{1}{90}(7y_0 + 32y_{1/4} + 12y_{1/2} + 32y_{3/4} + 7y_1)$ is correct over an interval for $y = 1, x, x^2, x^3, x^4$. Those five requirements give the five coefficients.

22 Any of these stopping points should give the integral as $0.886227 \cdots$ Extra correct digits depend on the computer design.

24 Directly $T_4 \approx 5.4248$. Separately on the intervals $[0, \pi]$ and $[\pi, 4]$, a single trapezoidal step T_1 is exact because $|x - \pi|$ is linear. Integral $= \frac{\pi^2}{2} + (8 - 4\pi + \frac{\pi^2}{2})$.

26 Simpson's rule gives $\frac{1}{6}(0^4 + 4(\frac{1}{2})^4 + 1^4) = \frac{5}{24}$. The difference from $\int_0^1 x^4 dx = \frac{1}{5}$ is $\frac{1}{120}$. Then $y'''(1) = 24$ and $y'''(0) = 0$ and $\frac{1}{120} = c(24)$ gives $c = \frac{1}{2880}$.

28 $y(a) = y(b)$.

Resource: Calculus
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