CHAPTER 6 EXPONENTIALS AND LOGARITHMS

6.1 An Overview (page 234)

The laws of logarithms which are highlighted on pages 229 and 230 apply just as well to "natural logs." Thus $\ln yz = \ln y + \ln z$ and $b = e^{\ln b}$. Also important :

$$b^x = e^{x \ln b} \quad \text{and} \quad \ln x^a = a \ln x \quad \text{and} \quad \ln 1 = 0.$$

Problems 1 – 4 review the rules for logarithms. Don't use your calculator. Find the exponent or power.

1. $\log_7 \frac{1}{49}$ 2. $\log_{12} 72 + \log_{12} 2$ 3. $\log_{10} 6 \cdot \log_6 x$ 4. $\log_{0.5} 8$

   - To find $\log_7 \frac{1}{49}$, ask yourself "seven to *what power* is $\frac{1}{49}$?" Since $\frac{1}{49} = 7^{-2}$, the power is $\log_7 \frac{1}{49} = -2$.
   - $\log_{12} 72 + \log_{12} 2 = \log_{12}(72 \cdot 2) = \log_{12} 144$. Since the bases are the same (everything is base 12), the log of the product equals the sum of the logs. To find $\log_{12} 144$, ask $12^{\text{what power}} = 144$. The power is 2, so $\log_{12} 144 = 2$.
   - Follow the change of base formula $\log_a x = (\log_a b)(\log_b x)$. Here $a = 10$ and $b = 6$. The answer is $\log_{10} x$.
   - To find $\log_{0.5} 8$, ask $\frac{1}{2}^{\text{what power}} = 8$. Since $\frac{1}{2} = 2^{-1}$ and $8 = 2^3$, the power is -3. Therefore $\log_{0.5} 8 = -3$.

5. Solve $\log_x 10 = 2$. (This is Problem 6.1.6c) The unknown is the base x.

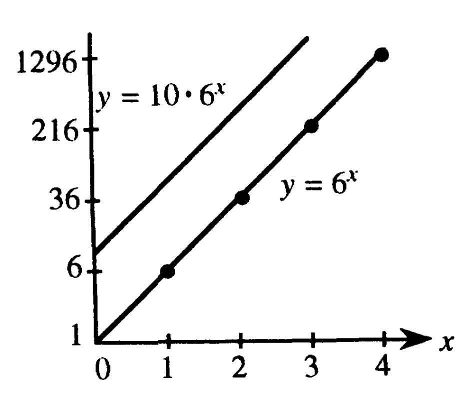
   - The statement $\log_x 10 = 2$ means exactly the same as $x^2 = 10$. Therefore $x = \sqrt{10}$. We can't choose $x = -\sqrt{10}$ since bases must be positive.

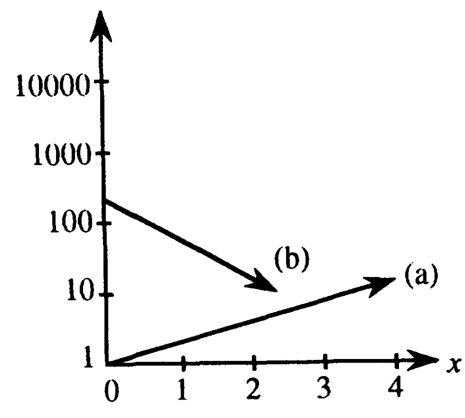
6. Draw the graphs for $y = 6^x$ and $y = 5 \cdot 6^x$ on semilog paper (preferably homemade).

   - The x axis is scaled normally. The y axis is scaled so that $6^0 = 1$, $6^1 = 6$, $6^2 = 36$, and $6^3 = 216$ are one unit apart. The axes cross at (0,1), not at (0,0) as on regular paper. Both graphs are straight lines. The line $y = 10 \cdot 6^x$ crosses the vertical axis when $x = 0$ and $y = 10$.

7. What are the equations of the functions represented in the right graph?

   - This is base 10 semilog paper, so both lines graph functions $y = A \cdot 10^{x \log b}$, where A is the intercept on the vertical axis and $\log b$ is the slope. One graph has $A = 1$ and the slope is $\frac{1}{4}$, so $y = 10^{x/4}$. The intercept on the second graph is 300 and the slope is $\frac{2}{3}$, so $y = 300 \cdot 10^{-2x/3}$.





Read-throughs and selected even-numbered solutions :

In $10^4 = 10{,}000$, the exponent 4 is the **logarithm** of 10,000. The base is $b = \mathbf{10}$. The logarithm of 10^m times 10^n is $\mathbf{m+n}$. The logarithm of $10^m/10^n$ is $\mathbf{m-n}$. The logarithm of $10{,}000^x$ is $\mathbf{4x}$. If $y = b^x$ then $x = \mathbf{\log_b y}$. Here x is any number, and y is always **positive**.

A base change gives $b = a^{\mathbf{\log_a b}}$ and $b^x = a^{\mathbf{x \log_a b}}$. Then 8^5 is 2^{15}. In other words $\log_2 y$ is $\mathbf{\log_2 8}$ times $\log_8 y$. When $y = 2$ it follows that $\log_2 8$ times $\log_8 2$ equals **1**.

On ordinary paper the graph of $y = \mathbf{mx+b}$ is a straight line. Its slope is $\mathbf{m}$. On semilog paper the graph of $y = \mathbf{Ab^x}$ is a straight line. Its slope is $\mathbf{\log b}$. On log-log paper the graph of $y = \mathbf{Ax^k}$ is a straight line. Its slope is $\mathbf{k}$.

The slope of $y = b^x$ is $dy/dx = \mathbf{cb^x}$, where c depends on b. The number c is the limit as $h \to 0$ of $\mathbf{\frac{b^h-1}{h}}$. Since $x = \log_b y$ is the inverse, $(dx/dy)(dy/dx) = \mathbf{1}$. Knowing $dy/dx = cb^x$ yields $dx/dy = \mathbf{1/cb^x}$. Substituting b^x for y, the slope of $\log_b y$ is $\mathbf{1/cy}$. With a change of letters, the slope of $\log_b x$ is $\mathbf{1/cx}$.

6 (a) 7 (b) 3 (c) $\sqrt{10}$ (d) $\frac{1}{4}$ (e) $\sqrt{8}$ (f) 5

12 $y = \log_{10} x$ is a straight line on "inverse" semilog paper: y axis normal, x axis scaled logarithmically (so $x = 1, 10, 100$ are equally spaced). Any equation $y = \log_b x + C$ will have a straight line graph.

14 $y = 10^{1-x}$ drops from 10 to 1 to .1 with slope -1 on semilog paper; $y = \frac{1}{2}\sqrt{10}^x$ increases with slope $\frac{1}{2}$ from $y = \frac{1}{2}$ at $x = 0$ to $y = 5$ at $x = 2$.

16 If 440/second is the frequency of middle A, then the next A is **880/second.** The 12 steps from A to A are approximately multiples of $2^{1/12}$. So 7 steps multiplies by $2^{7/12} \approx 1.5$ to give $(1.5)\ (440) = 660$. The seventh note from A is **E**.

22 The slope of $y = 10^x$ is $\frac{dy}{dx} = c10^x$ (later we find that $c = \ln 10$). At $x = 0$ and $x = 1$ the slope is c and $10c$. So the tangent lines are $y - 1 = c(x-0)$ and $y - 10 = 10c(x-1)$.

6.2 The Exponential e^x (page 241)

Problems 1 – 8 use the facts that $\frac{d}{dx}e^u = e^u\frac{du}{dx}$ and $\frac{d}{dx}\ln\ u = \frac{1}{u}\cdot\frac{du}{dx}$. If the base in the problem is not e, convert to base e. Use the change of base formulas $b^u = e^{(\ln b)u}$ and $\log_b u = \frac{\ln u}{\ln b}$. (And remember that $\ln b$ is just a constant.) In each problem, find dy/dx:

1. $y = \ln 3x$ • Take $u = 3x$ to get $\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} = \frac{1}{3x}\cdot 3 = \frac{1}{x}$. This is the same derivative as for $y = \ln x$. *Why*? The answer lies in the laws of logarithms: $\ln 3x = \ln 3 + \ln x$. Since $\ln 3$ is a constant, its derivative is zero. Because $\ln 3x$ and $\ln x$ differ only by a constant, they have the same derivative.

2. $y = \ln\cos 3x$. Assume $\cos 3x$ is positive so $\ln\cos 3x$ is defined.

   • Take $u = \cos 3x$. Then $\frac{du}{dx} = -3\sin 3x$. The answer is $\frac{dy}{dx} = \frac{1}{\cos 3x}(-3\sin 3x) = -3\tan 3x$. When we find a derivative we also find an integral: $-\int 3\tan 3x = \ln\cos 3x + C$.

3. $y = \ln(\ln x^2)$.

- Take $u = \ln x^2$. Then $\frac{du}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$. This means

$$\frac{dy}{dx} = (\frac{1}{\ln x^2})(\frac{2}{x}) = \frac{2}{x \ \ln x^2} = \frac{1}{x \ \ln x}.$$

Surprise to the author: This is also the derivative of $\ln(\ln x)$. Why does $\ln(\ln x^2)$ have the same derivative?

3. $y = \log_{10} \sqrt{x^2+5}$ • First change the base from 10 to e, by dividing by $\ln 10$. Now you are differentiating $\ln u$ instead of $\log u$: $y = \frac{1}{\ln 10} \ln \sqrt{x^2+5}$. For square roots, it is worthwhile to use the law that $\ln u^{1/2} = \frac{1}{2} \ln u$. Then $\ln \sqrt{x^2+5} = \frac{1}{2}\ln(x^2+5)$. [THIS IS NOT $\frac{1}{2}(\ln x^2 + \ln 5)$] This function is now

$$y = \frac{\ln(x^2+5)}{2 \ln 10} \quad \text{and} \quad \frac{dy}{dx} = (\frac{1}{2 \ \ln 10})(\frac{1}{x^2+5})(2x) = \frac{1}{\ln 10} \frac{x}{x^2+5}.$$

4. $y = \ln \frac{(x^4-8)^5}{(x^6+5x)\cos x}$ • Here again the laws of logarithms allow you to make things easier. Multiplication of numbers is addition of logs. Division is subtraction. Powers of u become multiples of $\ln u$: $y = 5\ln(x^4-8) - \ln(x^6+5x) - \ln\cos x$. Now dy/dx is long but easy:

$$\frac{dy}{dx} = \frac{5}{x^4-8}(4x^3) - \frac{6x^5+5}{x^6+5x} - \frac{(-\sin x)}{\cos x} = \frac{20x^3}{x^4-8} - \frac{6x^5+5}{x^6+5x} + \tan x.$$

5. $y = e^{\tan x}$ • dy/dx is $e^u du/dx = e^{\tan x}(\sec^2 x)$.

6. $y = \sin(e^{2x})$ • Set $u = e^{2x}$. Then $du/dx = 2e^{2x}$. Using the chain rule,

$$\frac{d}{dx}(\sin u) = (\cos u)(\frac{du}{dx}) = (\cos e^{2x})(2e^{2x}).$$

7. $y = 10^{x^2}$ • First change the base from 10 to e: $y = (e^{\ln 10})^{x^2} = e^{x^2 \ln 10}$. Let $u = x^2 \ln 10$. Then $\frac{du}{dx} = 2x \ln 10$. (Remember $\ln 10$ is a constant, you don't need the product rule.) We have

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^{x^2 \ln 10}(2x \ln 10).$$

8. $y = x^{-1/x}$ (This is Problem 6.2.18)

- First change to base e: $y = (e^{\ln x})^{-1/x}$. Since the exponent is $u = -\frac{1}{x}\ln x$, we need the product rule to get $du/dx = -\frac{1}{x}(\frac{1}{x}) + \frac{1}{x^2}\ln x$. Therefore

$$\frac{dy}{dx} = e^u \frac{du}{dx} = x^{-1/x} \cdot \frac{1}{x^2}(\ln x - 1).$$

Problems 9 - 14 use the definition $e = \lim_{h\to 0}(1+h)^{1/h}$. By substituting $h = \frac{1}{n}$ this becomes $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$. Evaluate these limits as $n \to \infty$:

9. $\lim(1+\frac{1}{n})^{6n}$ 10. $\lim(1+\frac{1}{6n})^{6n}$ 11. $\lim(1+\frac{1}{2n})^{3n}$ 12. $\lim(1+\frac{r}{n})^n$ (r is constant) 13. $\lim(\frac{n+8}{n})^n$

- Rewrite Problem 9 as $\lim((1+\frac{1}{n})^n)^6$. Since $(1+\frac{1}{n})^n$ goes to e the answer is $e^6 \approx 403$. The calculator shows $(1+\frac{1}{1000})^{6000} \approx 402$.
- Problem 10 is different because $6n$ is both inside and outside the parentheses. If you let $k = 6n$, and note $k \to \infty$ as $n \to \infty$, this becomes $\lim_{k\to\infty}(1+\frac{1}{k})^k = e$. The idea here is : If we have $\lim_{\Box\to\infty}(1+\frac{1}{\Box})^{\Box}$ and all the boxes *are the same*, the limit is e.

- Question 11 can be rewritten as $\lim(1+\frac{1}{2n})^{2n\cdot\frac{3}{2}} = e^{3/2}$. (The box is $\square = 2n$).
- In Question 12 write $n = mr$. Then $\frac{r}{n} = \frac{1}{m}$ and we have $\lim(1+\frac{1}{m})^{mr} = e^r$.
- In 13 write $(\frac{n+8}{n}) = 1 + \frac{8}{n}$. This is Problem 12 with $r = 8 : \lim_{n\to\infty}(1+\frac{8}{n})^n = e^8$.

14. (This is 6.2.21) Find the limit of $(\frac{11}{10})^{10}, (\frac{101}{100})^{100}, (\frac{1001}{1000})^{1000}, \cdots$. Then find the limit of $(\frac{10}{11})^{10}, (\frac{100}{101})^{100}, (\frac{1000}{1001})^{1000}, \cdots$ and the limit of $(\frac{10}{11})^{11}, (\frac{100}{101})^{101}, (\frac{1000}{1001})^{1001}, \cdots$.

    - The terms of the first sequence are $(\frac{n+1}{n})^n = (1+\frac{1}{n})^n$ where $n = 10, 100, 1000, \cdots$. The limit is e. The terms of the second sequence are the reciprocals of those of the first. So the second limit is $\frac{1}{e}$. The terms of the third are each $(\frac{n}{n+1})$ times those of the second. Since $\frac{n}{n+1} \to 1$ as $n \to \infty$, the third limit is again $\frac{1}{e}$.

    The third sequence can also be written $(\frac{n-1}{n})^n$ or $(1-\frac{1}{n})^n$. Its limit is e^{-1}. See Problem 12 with $r = -1$.

    Exercises 6.2.27 and 6.2.45 – 6.2.54 give plenty of practice in integrating exponential functions. Usually the trick is to locate $e^u du$. Problems 15 – 17 are three models.

15. (This is 6.2.32) Find an antiderivative for $v(x) = \frac{1}{e^x} + \frac{1}{x^e}$.

    - The first term is e^{-x}. Its antiderivative is $-e^{-x}$. The second term is just x^n with $n = -e$. Its antiderivative is $\frac{1}{1-e}x^{1-e}$. The answer is $f(x) = -e^{-x} + \frac{1}{1-e}x^{1-e} + C$.

16. Find an antiderivative for $v(x) = 3^{-2x}$. You may change to base e.

    - The change produces $e^{-2x\ln 3}$. The coefficient of x in the exponent is $-2\ln 3$. An antiderivative is $f(x) = \frac{-1}{2\ln 3}e^{-2x\ln 3}$ or $\frac{-1}{2\ln 3}3^{-2x}$.
      We need -2 and $\ln 3$ in the denominator, the same way that we needed $n+1$ when integrating x^n.

17. (This is 6.2.52) Find $\int_0^3 e^{(1+x^2)}x\,dx$. Set $u = 1+x^2$ and $du = 2x\,dx$. The integral is $\frac{1}{2}\int_{u(0)}^{u(3)} e^u du$. The new limits of integration are $u(0) = 1+0^2 = 1$ and $u(3) = 1+3^2 = 10$. Now $\frac{1}{2}\int_1^{10} e^u du = \frac{1}{2}e^u]_1^{10} = \frac{1}{2}(e^{10}-e)$. *This is not the same as* $\frac{1}{2}e^9$!

Read-throughs and selected even-numbered solutions :

The number e is approximately **2.78**. It is the limit of $(1+h)$ to the power **1/h**. This gives 1.01^{100} when $h = \mathbf{.01}$. An equivalent form is $e = \lim\,(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{n}})^{\mathbf{n}}$.

When the base is $b = e$, the constant c in Section 6.1 is **1**. Therefore the derivative of $y = e^x$ is $dy/dx = \mathbf{e^x}$. The derivative of $x = \log_e y$ is $dx/dy = \mathbf{1/y}$. The slopes at $x = 0$ and $y = 1$ are both **1**. The notation for $\log_e y$ is **ln y**, which is the **natural** logarithm of y.

The constant c in the slope of b^x is $c = \mathbf{\ln b}$. The function b^x can be rewritten as $\mathbf{e^{x\ln b}}$. Its derivative is $\mathbf{(\ln b)e^{x\ln b} = (\ln b)b^x}$. The derivative of $e^{u(x)}$ is $\mathbf{e^{u(x)}\frac{du}{dx}}$. The derivative of $e^{\sin x}$ is $\mathbf{e^{\sin x}\cos x}$. The derivative of e^{cx} brings down a factor **c**.

The integral of e^x is $\mathbf{e^x + C}$. The integral of e^{cx} is $\mathbf{\frac{1}{c}e^{cx} + C}$. The integral of $e^{u(x)}du/dx$ is $\mathbf{e^{u(x)} + C}$. In general the integral of $e^{u(x)}$ by itself is **impossible** to find.

18 $x^{-1/x} = e^{-(\ln x)/x}$ has derivative $(-\frac{1}{x^2} + \frac{\ln x}{x^2})e^{-(\ln x)/x} = (\frac{\ln x - 1}{x^2})x^{-1/x}$

20 $(1 + \frac{1}{n})^{2n} \to \mathbf{e^2} \approx 7.7$ and $(1 + \frac{1}{n})^{\sqrt{n}} \to \mathbf{1}$. Note that $(1 + \frac{1}{n})^{\sqrt{n}}$ is squeezed between 1 and $e^{1/\sqrt{n}}$ which approaches 1.

28 $(e^{3x})(e^{7x}) = e^{10x}$ which is the derivative of $\mathbf{\frac{1}{10}e^{10x}}$

42 $x^{1/x} = e^{(\ln x)/x}$ has slope $e^{(\ln x)/x}(\frac{1}{x^2} - \frac{\ln x}{x^2}) = \mathbf{x^{1/x}(\frac{1-\ln x}{x^2})}$. This slope is zero at $\mathbf{x = e}$, when $\ln x = 1$. The second derivative is *negative* so the maximum of $x^{1/x}$ is $e^{1/e}$. Check: $\frac{d}{dx}e^{(\ln x)/x}(\frac{1-\ln x}{x^2}) = e^{(\ln x)/x}[(\frac{1-\ln x}{x^2})^2 + \frac{(-2-1+2\ln x)}{x^3}] = -\frac{1}{e^3}e^{1/e}$ at $x = e$.

44 $x^e = e^x$ at $x = e$. This is the only point where $x^e e^{-x} = 1$ because the derivative is $x^e(-e^{-x}) + ex^{e-1}e^{-x} = (\frac{e}{x} - 1)x^e e^{-x}$. This derivative is positive for $x < e$ and negative for $x > e$. So the function $x^e e^{-x}$ increases to 1 at $x = e$ and then decreases: it never equals 1 again.

58 The asymptotes of $(1 + \frac{1}{x})^x = (\frac{x+1}{x})^x = (\frac{x}{x+1})^{-x}$ are $x = -1$ (from the last formula) and $y = e$ (from the first formula).

62 $\lim \frac{x^6}{e^x} = \lim \frac{6x^5}{e^x} = \lim \frac{30x^4}{e^x} = \lim \frac{120x^3}{e^x} = \lim \frac{360x^2}{e^x} = \lim \frac{720x}{e^x} = \lim \frac{720}{e^x} = 0$.

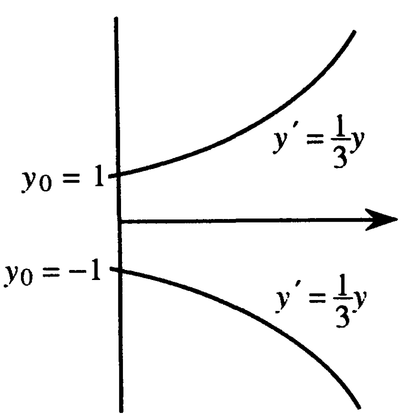
6.3 Growth and Decay in Science and Economics (page 250)

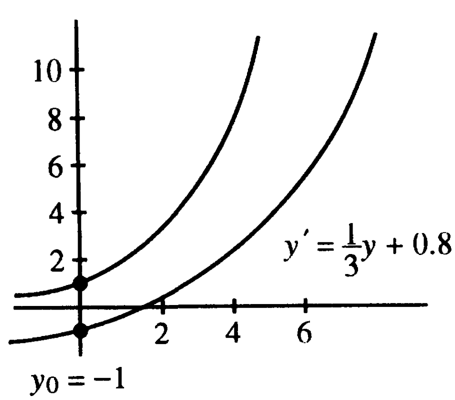
The applications in this section begin to suggest the power of the mathematics you are learning. Concentrate on understanding how to *use* $y = y_0e^{ct}$ and $y = y_0e^{ct} + \frac{s}{c}(e^{ct} - 1)$ as you work the examples.

In Problems 1 and 2, solve the differential equations starting from $y_0 = 1$ and $y_0 = -1$. Draw both solutions on the same graph.

1. $\frac{dy}{dt} = \frac{1}{3}y$ (pure exponential)   2. $\frac{dy}{dt} = \frac{1}{3}y + 0.8$ (exponential with source term)

- Problem 1 says that the rate of change is proportional to y. There are no other complicating terms. Use the exponential law y_0e^{ct} with $c = \frac{1}{3}$ and $y_0 = \pm 1$. The graphs of $y = \pm e^{t/3}$ are at left below.
- Problem 2 changes Problem 1 into $\frac{dy}{dt} = cy + s$. We have $c = \frac{1}{3}$ and $s = 0.8$. Its solution is $y = y_0e^{t/3} + \frac{0.8}{\frac{1}{3}}(e^{t/3} - 1) = y_0e^{t/3} + 2.4(e^{t/3} - 1)$. Study the graphs to see the effect of y_0 and s. With a graphing calculator you can carry these studies further. See what happens if s is very large, or if s is negative. Exercise 6.3.36 is also good for comparing the effects of various c's and s's.





3. (This is 6.3.5) Start from $y_0 = 10$. If $\frac{dy}{dt} = 4y$, at what time does y increase to 100?

   - The solution is $y = y_0 e^{ct} = 10e^{4t}$. Set $y = 100$ and solve for t:

$$100 = 10e^{4t} \text{ gives } 10 = e^{4t} \quad \text{and} \quad \ln 10 = 4t. \text{ Then } t = \frac{1}{4}\ln 10.$$

4. Problem 6.3.6 looks the same as the last question, but the right side is $4t$ instead of $4y$. Note that $\frac{dy}{dt} = 4t$ is *not* exponential growth. The slope $\frac{dy}{dt}$ is proportional to t and the solution is simply $y = 2t^2 + C$. Start at $y_0 = C = 10$. Setting $y = 100$ gives $100 = 2t^2 + 10$ and $t = \sqrt{45}$.

   Problems 5 – 10 involve $y = y_0 e^{ct}$.

5. Write the equation describing a bacterial colony growing exponentially. Start with 100 bacteria and end with 10^6 after 30 hours.

   - Right away we know $y_0 = 100$ and $y = 100e^{ct}$. We don't yet know c, but at $t = 30$ we have $y = 10^6 = 100e^{30c}$. Taking logarithms of $10^4 = e^{30c}$ gives $4\ln 10 = 30\,c$ or $\frac{4\ln 10}{30} = c$. The equation is $y = 100e^{(\frac{4\ln 10}{30}t)}$. More concisely, since $e^{\ln 10} = 10$ this is $y = 100 \cdot 10^{4t/30}$.

6. The number of cases of a disease increases by 2% a year. If there were 10,000 cases in 1992, how many will there be in 1995?

   - The direct approach is to multiply by 1.02 after every year. After three years $(1.02)^3 10,000 \approx 10,612$.
   - We can also use $y = 10^4 e^{ct}$. The 2% increase means $c = \ln(1.02)$. After three years (1992 to 1995) we set $t = 3$: $y = 10^4 e^{3(\ln 1.02)} = 10,612$.

     This is *not the same* as $y = 10^4 e^{.02t}$. That is *continuous* growth at 2%. It is continuous compounding, and $e^{.02} = 1.0202\cdots$ is a little different from 1.02.

7. How would Problem 6 change if the number of cases *decreases* by 2%?

   - A 2% decrease changes the multiplier to .98. Then $c = \ln(.98)$. In 3 years there would be 9,411 cases.

8. (This is 6.3.15) The population of Cairo grew exponentially from 5 million to 10 million in 20 years. Find the equation for Cairo's population. When was $y = 8$ million?

   - Starting from $y_0 = 5$ million $= 5 \cdot 10^6$ the population is $y = 5 \cdot 10^6 e^{ct}$. The doubling time $\frac{\ln 2}{c}$ is 20 years. We deduce that $c = \frac{\ln 2}{20} = .035$ and $y = 5 \cdot 10^6 e^{.035t}$. This reaches 8 million $= 8 \cdot 10^6$ when $\frac{8}{5} = e^{.035t}$. Then $t = \frac{\ln \frac{8}{5}}{.035} \approx 13.6$ years.

9. If $y = 4500$ at $t = 4$ and $y = 90$ at $t = 10$, what was y at $t = 0$? (We are assuming exponential decay.)

   - The first part says that $y = 4500e^{c(t-4)}$. The t in the basic formula is replaced by $(t - 4)$. [The "shifted" formula is $y = y_T e^{c(t-T)}$.] Note that $y = 4500$ when $t = 4$, as required. Since $y = 90$ when $t = 10$, we have $90 = 4500e^{6c}$ and $e^{6c} = \frac{90}{4500} = .02$. This means $6c = \ln .02$ and $c = \frac{1}{6}\ln .02$. Finally, set $t = 0$ to get the amount at that time: $y = 4500e^{(\frac{1}{6}\ln .02)(0-4)} \approx 61074$.

10. (Problem 6.3.13) How old is a skull containing $\frac{1}{5}$ as much radiocarbon as a modern skull?

    - Information about radioactive dating is on pages 243-245. Since the half-life of carbon 14 is 5568 years, the amount left at time t is $y_0 e^{ct}$ with exponent $c = \frac{\ln 1/2}{5568} = \frac{-\ln 2}{5568}$. We do not know the initial amount y_0. But we can use $y_0 = 1$ (100% at the start) and $y = \frac{1}{5} = 0.2$ at the unknown age t. Then

$$0.2 = e^{\frac{-\ln 2}{5568}t} \text{ yields } t = \frac{(\ln 0.2)5568}{-\ln 2} = 31,425 \text{ years.}$$

11. (Problem 6.3.37) What value $y =$ constant solves $\frac{dy}{dt} = 4 - y$? Show that $y(t) = Ae^{-t} + 4$ is also a solution. Find $y(1)$ and y_∞ if $y_0 = 3$.

    - If y is constant, then $\frac{dy}{dt} = 0$. Therefore $y - 4 = 0$. The steady state y_∞ is the constant $y = 4$.
    - A non-constant solution is $y(t) = Ae^{-t} + 4$. Check: $\frac{dy}{dt} = -Ae^{-t}$ equals $4 - y = 4 - (Ae^{-t} + 4)$.
    - If we know $y(0) = A + 4 = 3$, then $A = -1$. In this case $y(t) = -1e^{-t} + 4$ gives $y(1) = 4 - \frac{1}{e}$.
    - To find y_∞, let $t \to \infty$. Then $y = -e^{-t} + 4$ goes to $y_\infty = 4$, the expected steady state.

12. (Problem 6.3.46) (a) To have \$50,000 for college tuition in 20 years, what gift y_0 should a grandparent make now? Assume $c = 10\%$. (b) What continuous deposit should a parent make during 20 years to save \$50,000? (c) If the parent saves $s = \$1000$ per year, when does the account reach \$50,000?

    - Part (a) is a question about the present value y_0, if the gift is worth \$50,000 in 20 years. The formula $y = y_0e^{ct}$ turns into $y_0 = ye^{-ct} = (50,000)e^{-0.1(20)} = \$6767$.
    - Part (b) is different because there is a continuous deposit instead of one lump sum. In the formula $y_0e^{ct} + \frac{s}{c}(e^{ct} - 1)$ we know $y_0 = 0$ and $c = 10\% = 0.1$. We want to choose s so that $y = 50,000$ when $t = 20$. Therefore $50,000 = \frac{s}{0.1}(e^{(0.1)20} - 1)$. This gives $s = 782.59$. The parents should continuously deposit \$782.59 per year for 20 years.
    - Part (c) asks how long it would take to accumulate \$50,000 if the deposit is $s = \$1000$ per year.

    $$50,000 = \frac{1000}{0.1}(e^{0.1t} - 1) \text{ leads to } 5 = e^{0.1t} - 1 \text{ and } t = \frac{\ln 6}{0.1} = 17.9 \text{ years.}$$

    This method takes 17.9 years to accumulate the tuition. The smaller deposit $s = \$782.59$ took 20 years.

13. (Problem 6.3.50) For how long can you withdraw \$500/year after depositing \$5000 at a continuous rate of 8%? At time t you run dry: and $y(t) = 0$.

    - This situation uses both terms of the formula $y = y_0e^{ct} + \frac{s}{c}(e^{ct} - 1)$. There is an initial value $y_0 = 5000$ and a *sink* (negative source) of $s = -500$/year. With $c = .08$ we find the time t when $y = 0$:

    $$\text{Multiply } 0 = 5000e^{.08t} - \frac{500}{.08}(e^{.08t} - 1) \text{ by } .08 \text{ to get } 0 = 400e^{.08t} - 500(e^{.08t} - 1).$$

    Then $e^{.08t} = \frac{500}{100} = 5$ and $t = \frac{\ln 5}{.08} = 20.1$. You have 20 years of income.

14. Your Thanksgiving turkey is at 40°F when it goes into a 350° oven at 10 o'clock. At noon the meat thermometer reads 110°. When will the turkey be done (195°)?

    - Newton's law of cooling applies even though the turkey is warming. Its temperature is approaching $y_\infty = 350°$ from $y_0 = 40°$. Using method 3 (page 250) we have $(y - 350) = (40 - 350)e^{ct}$. The value of c varies from turkey to turkey. To find c for your particular turkey, substitute $y = 110$ when $t = 2$:

    $$110 - 350 = -310e^{2c} \Rightarrow \frac{-240}{-310} = e^{2c} \Rightarrow \ln\frac{24}{31} = 2c \Rightarrow c = \frac{1}{2}\ln\frac{24}{31} = -.128.$$

    The equation for y is $350 - 310e^{-.128t}$. The turkey is done when $y = 195$:

$$195 = 350 - 310e^{-.128t} \text{ or } \frac{195-350}{-310} = e^{-.128t} \text{ or } -.128\,t = \ln\frac{-155}{-310} = \ln\frac{1}{2}.$$

This gives $t = 5.4$ hours. You can start making gravy at 3 : 24.

Read-throughs and selected even-numbered solutions :

If $y' = cy$ then $y(t) = \mathbf{y_0 e^{ct}}$. If $dy/dt = 7y$ and $y_0 = 4$ then $y(t) = \mathbf{4e^{7t}}$. This solution reaches 8 at $t = \mathbf{\frac{\ln 2}{7}}$. If the doubling time is T then $c = \mathbf{\frac{\ln 2}{T}}$. If $y' = 3y$ and $y(1) = 9$ then y_0 was $\mathbf{9e^{-3}}$. When c is negative, the solution approaches **zero** as $t \to \infty$.

The constant solution to $dy/dt = y + 6$ is $y = \mathbf{-6}$. The general solution is $y = Ae^t - 6$. If $y_0 = 4$ then $A = \mathbf{10}$. The solution of $dy/dt = cy + s$ starting from y_0 is $y = Ae^{ct} + B = \mathbf{(y_0 + \frac{s}{c})e^{ct} - \frac{s}{c}}$. The output from the source is $\mathbf{\frac{s}{c}(e^{ct} - 1)}$. An input at time T grows by the factor $\mathbf{e^{c(t-T)}}$ at time t.

At $c = 10\%$, the interest in time dt is $dy = \mathbf{.01\,y\,dt}$. This equation yields $y(t) = \mathbf{y_0 e^{.01t}}$. With a source term instead of y_0, a continuous deposit of $s = 4000$/year yields $y = \mathbf{40,000(e-1)}$ after ten years. The deposit required to produce 10,000 in 10 years is $s = \mathbf{yc/(e^{ct}-1) = 1000/(e-1)}$. An income of 4000/year forever (!) comes from $y_0 = \mathbf{40,000}$. The deposit to give 4000/year for 20 years is $y_0 = \mathbf{40,000(1-e^{-2})}$. The payment rate s to clear a loan of 10,000 in 10 years is $\mathbf{1000e/(e-1)}$ per year.

The solution to $y' = -3y + s$ approaches $y_\infty = \mathbf{s/3}$.

12 To multiply again by 10 takes ten more hours, a total of **20 hours**. If $e^{10c} = 10$ (and $e^{20c} = 100$) then $10c = \ln 10$ and $\mathbf{c = \frac{\ln 10}{10}} \approx \mathbf{.23}$.

16 $8e^{.01t} = 6e^{.014t}$ gives $\frac{8}{6} = e^{.004t}$ and $t = \frac{1}{.004}\ln\frac{8}{6} = 250\ln\frac{4}{3} = \mathbf{72}$ **years.**

24 Go from 4 mg back down to 1 mg in T hours. Then $e^{-.01T} = \frac{1}{4}$ and $-.01T = \ln\frac{1}{4}$ and $T = \frac{\ln\frac{1}{4}}{-.01} = 139$ hours (not so realistic).

28 Given $mv = mv - v\Delta m + m\Delta v - (\Delta m)\Delta v + \Delta m(v-7)$; cancel terms to leave $m\Delta v - (\Delta m)\Delta v = 7\Delta m$; divide by Δm and approach the limit $\mathbf{m\frac{dv}{dm} = 7}$. Then $v = 7\ln m + C$. At $t = 0$ this is $20 = 7\ln 4 + C$ so that $v = 7\ln m + 20 - 7\ln 4 = \mathbf{7\ln\frac{m}{4} + 20}$.

36 (a) $\frac{dy}{dt} = 3y + 6$ gives $y \to \infty$ (b) $\frac{dy}{dt} = -3y + 6$ gives $y \to 2$ (c) $\frac{dy}{dt} = -3y - 6$ gives $y \to -2$ (d) $\frac{dy}{dt} = 3y - 6$ gives $y \to -\infty$.

42 \$1000 changes by (\$1000) $(-.04dt)$, a decrease of $40dt$ dollars in time dt. The printing rate should be $s = 40$.

48 The deposit of $4dT$ grows with factor c from time T to time t, and reaches $e^{c(t-T)}4dT$. With $t = 2$ add deposits from $T = 0$ to $T = 1$: $\int_0^1 e^{c(2-T)}4dT = [\frac{4e^{c(2-T)}}{-c}]_0^1 = \mathbf{\frac{4e^c - 4e^{2c}}{-c}}$.

58 If $\frac{dy}{dt} = -y + 7$ then $\frac{dy}{dt}$ is zero at $y_\infty = \mathbf{7}$ (this is $-\frac{s}{c} = \frac{7}{1}$). The derivative of $y - y_\infty$ is $\frac{dy}{dt}$, so the derivative of $y - 7$ is $-(y-7)$. The decay rate is $c = -1$, and $\mathbf{y - 7 = e^{-t}(y_0 - 7)}$.

60 All solutions to $\frac{dy}{dt} = c(y - 12)$ converge to $\mathbf{y = 12}$ provided c is **negative.**

66 (a) The white coffee cools to $y_\infty + (y_0 - y_\infty)e^{ct} = \mathbf{20 + 40e^{ct}}$. (b) The black coffee cools to $\mathbf{20 + 50e^{ct}}$. The milk warms to $\mathbf{20 - 10e^{ct}}$. The mixture $\frac{5(\text{black coffee})+1(\text{milk})}{6}$ has $20 + \frac{250-10}{6}e^{ct} = \mathbf{20 + 40e^{ct}}$. So it doesn't matter when you add the milk!

6.4 Logarithms (page 258)

This short section is packed with important information and techniques – how to differentiate and integrate logarithms, logarithms as areas, approximation of logarithms, and logarithmic differentiation (LD). The examples cover each of these topics:

Derivatives The rule for $y = \ln u$ is $\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$. With a different base b, the rule for $y = \log_b u = \frac{\ln u}{\ln b}$ is $\frac{dy}{dx} = \frac{1}{u \ln b}\frac{du}{dx}$. Find $\frac{dy}{dx}$ in Problems 1 – 4.

1. $y = \ln(5 - x)$. • $u = 5 - x$ so $\frac{dy}{dx} = (\frac{1}{5-x})(-1) = \frac{1}{x-5}$.
2. $y = \log_{10}(\sin x)$. • Change to base e with $y = \frac{\ln(\sin x)}{\ln 10}$. Now $\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{\sin x} \cdot \cos x$.
3. $y = (\ln x)^3$. • This is $y = u^3$, so $\frac{dy}{dx} = 3u^2\frac{du}{dx} = 3(\ln x)^2\frac{1}{x}$.
4. $y = \tan x \ln \sin x$. • The product rule gives

$$\frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x} \cdot \cos x + \sec^2 x(\ln \sin x) = 1 + \sec^2 x(\ln \sin x).$$

5. (This is 6.4.53) Find $\lim_{x\to 0} \frac{\log_b(1+x)}{x}$.

   • This limit takes the form $\frac{0}{0}$, so turn to l'Hôpital's rule (Section 3.8). The derivative of $\log_b(1 + x)$ is $(\frac{1}{\ln b})(\frac{1}{1+x})$. The derivative of x is 1. The ratio is $\frac{1}{(\ln b)(1+x)}$ which approaches $\frac{1}{\ln b}$.

Logarithms as areas

6. (This is 6.4.56) Estimate the area under $y = \frac{1}{x}$ for $4 \le x \le 8$ by four trapezoids. What is the exact area?

   • Each trapezoid has base $\Delta x = 1$, so four trapezoids take us from $x = 4$ to $x = 8$. With $y = \frac{1}{x}$ the sides of the trapezoids are the heights $y_0, y_1, y_2, y_3, y_4 = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$. The total trapezoidal area is

$$\Delta x(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \frac{1}{2}y_4) = 1(\frac{1}{8} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{16}) = 0.6970.$$

   To get the exact area we integrate $\int_4^8 \frac{1}{x}dx = \ln 8 - \ln 4 = \ln\frac{8}{4} = \ln 2 \approx 0.6931$.
   It is interesting to compare with the trapezoidal area from $x = 1$ to $x = 2$. *The exact area* $\int_1^2 \frac{1}{x}dx$ *is still* $\ln 2$. Now $\Delta x = \frac{1}{4}$ and the heights are $\frac{1}{1}, \frac{1}{\frac{5}{4}}, \frac{1}{\frac{6}{4}}, \frac{1}{\frac{7}{4}}, \frac{1}{2}$. The total trapezoidal area comes from the same rule:

$$\frac{1}{4}(\frac{1}{2}\cdot\frac{1}{1} + \frac{4}{5} + \frac{4}{6} + \frac{4}{7} + \frac{1}{2}\cdot\frac{1}{2}) = (\frac{1}{8} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{16}) = 0.6970 \text{ as before.}$$

The sum is not changed! This is another way to see why $\ln 8 - \ln 4$ is equal to $\ln 2 - \ln 1$. The area stays the same when we integrate $\frac{1}{x}$ from any a to $2a$.

Questions 7 and 8 are about approximations going as far as the x^3 term:

$$\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \quad \text{and} \quad e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

7. Approximate $\ln(.98)$ by choosing $x = -.02$. Then $1 + x = .98$.

- $\ln(1-.02) \approx (-.02) - \frac{(-.02)^2}{2} + \frac{(-.02)^3}{3} = -.0202026667$.
  The calculator gives $\ln .98 = -.0202027073$. Somebody is wrong by $4 \cdot 10^{-8}$.

8. Find a quadratic approximation (this means x^2 terms) near $x = 0$ for $y = 2^x$.

   - 2^x is the same as $e^{x \ln 2}$. Put $x \ln 2$ into the series. The approximation is $1 + x\ln 2 + \frac{(\ln 2)^2}{2}x^2$.

Integration The basic rule is $\int \frac{du}{u} = \ln|u| + C$. Why not just $\ln u + C$? Go back to the definition of $\ln u$ = area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = u$. Here u must be positive since we cannot cross $x = 0$, where $\frac{1}{x}$ blows up. However if u stays negative, there is something we can do. Write $\int \frac{du}{u} = \int \frac{-du}{-u}$. The denominator $-u$ is positive and the numerator is its derivative! In that case, $\int \frac{-du}{-u} = \ln(-u) + C$. The expression $\int \frac{du}{u} = \ln|u| + C$ covers both cases. When you *know* u is positive, as in $\ln(x^2+1)$, leave off the absolute value sign.

For definite integrals, the limits of integration should tell you whether u is negative or positive. Here are two examples with $u = \sin x$:

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}dx = \ln(\sin x)]_{\pi/4}^{\pi/2} \qquad \int_{\pi/2}^{3\pi/4} \frac{\cos x}{\sin x}dx = \ln|\sin x|]_{\pi/2}^{3\pi/4}.$$

The integral $\int_0^{\pi} \frac{\cos x}{\sin x}dx$ is illegal. It starts and ends with $u = \sin x = 0$

9. Integrate $\int \frac{x\,dx}{1-x^2}$.

   - Let $1 - x^2$ equal u. Then $du = -2x\,dx$. The integral becomes $-\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C$. This is $-\frac{1}{2}\ln|1-x^2| + C$. Avoid $x = \pm 1$ where $u = 0$.

10. Integrate $\int \frac{x\,dx}{1-x}$. This is *not* $\int \frac{du}{u}$. But we can write $\frac{x}{1-x}$ as $-1 + \frac{1}{1-x}$:

    - $\int \frac{x\,dx}{1-x} = \int(-1 + \frac{1}{1-x})dx = -x - \ln|1-x| + C$.

11. (This is 6.4.18) Integrate $\int_2^e \frac{dx}{x(\ln x)^2}$.

    - A sneaky one, not $\frac{du}{u}$. Set $u = \ln x$ and $du = \frac{dx}{x}$:

$$\int_{\ln 2}^{1} \frac{du}{u^2} = -\frac{1}{u}]_{\ln 2}^{1} = -1 + \frac{1}{\ln 2}.$$

Logarithmic differentiation (**LD**) greatly simplifies derivatives of powers and products, and quotients. To find the derivative of $x^{1/x}$, **LD** is the best way to go. (Exponential differentiation in Problem 6.5.70 amounts to the same thing.) The secret is in decomposing the original expression. Here are examples:

12. $y = \frac{(x^2+7)^3}{\sqrt{x^3-9}}(4x^8)$ leads to $\ln y = 3\ln(x^2+7) + \ln 4 + 8\ln x - \frac{1}{2}\ln(x^3-9)$.

    Multiplication has become addition. Division has become subtraction. The powers 3, 8, $\frac{1}{2}$ now multiply. This is as far as logarithms can go. *Do not try to separate* $\ln x^2$ *and* $\ln 7$. Take the derivative of $\ln y$:

    - $\frac{1}{y}\frac{dy}{dx} = 3\frac{2x}{x^2+7} + 0 + \frac{8}{x} - \frac{3x^2}{2(x^3-9)}$.
      If you substitute back for y then $\frac{dy}{dx} = \frac{(x^2+7)^3 \cdot 4x^8}{\sqrt{x^3-9}}[\frac{6x}{x^2+7} + \frac{8}{x} - \frac{3x^2}{2(x^3-9)}]$.

13. $y = (\sin x)^{x^2}$ has a function $\sin x$ raised to a functional power x^2. **LD** is necessary.

    - First take logarithms: $\ln y = x^2 \ln \sin x$. Now take the derivative of both sides. Notice especially the left side: $\frac{1}{y}\frac{dy}{dx} = x^2\frac{\cos x}{\sin x} + 2x\ln\sin x$. Multiply by y to find $\frac{dy}{dx}$.

14. Find the tangent line $y^2(2-x) = x^3$ at the point (1,1). **ID** and **LD** are useful but not necessary.

- We need to know the slope dy/dx at (1,1). Taking logarithms gives

$$\ln y^2 + \ln(2-x) = \ln x^3 \text{ or } 2\ln y + \ln(2-x) = 3\ln x.$$

Now take the x derivative of both sides: $\frac{2}{y}\frac{dy}{dx} + \frac{-1}{2-x} = \frac{3}{x}$. Plug in $x = 1, y = 1$ to get $2\frac{dy}{dx} + \frac{-1}{1} = 3$ or $\frac{dy}{dx} = 2$. The tangent line through (1,1) with slope 2 is $y - 1 = 2(x-1)$.

Read-throughs and selected even-numbered solutions :

The natural logarithm of x is $\int_1^x \frac{\mathbf{dt}}{\mathbf{t}}$ (or $\int_{\mathbf{1}}^{\mathbf{x}} \frac{\mathbf{dx}}{\mathbf{x}}$). This definition leads to $\ln xy = \mathbf{\ln x + \ln y}$ and $\ln x^n = \mathbf{n \ln x}$. Then e is the number whose logarithm (area under $1/x$ curve) is **1**. Similarly e^x is now defined as the number whose natural logarithm is **x**. As $x \to \infty, \ln x$ approaches **infinity**. But the ratio $(\ln x)/\sqrt{x}$ approaches **zero**. The domain and range of $\ln x$ are $\mathbf{0 < x < \infty, -\infty < \ln x < \infty}$.

The derivative of $\ln x$ is $\mathbf{\frac{1}{x}}$. The derivative of $\ln(1+x)$ is $\mathbf{\frac{1}{1+x}}$. The tangent approximation to $\ln(1+x)$ at $x = 0$ is **x**. The quadratic approximation is $\mathbf{x - \frac{1}{2}x^2}$. The quadratic approximation to e^x is $\mathbf{1 + x + \frac{1}{2}x^2}$.

The derivative of $\ln u(x)$ by the chain rule is $\mathbf{\frac{1}{u(x)}\frac{du}{dx}}$. Thus $(\ln\cos x)' = -\mathbf{\frac{\sin x}{\cos x}} = \mathbf{-\tan x}$. An antiderivative of $\tan x$ is $\mathbf{-\ln\cos x}$. The product $p = xe^{5x}$ has $\ln p = \mathbf{5x + \ln x}$. The derivative of this equation is $\mathbf{p'/p = 5 + \frac{1}{x}}$. Multiplying by p gives $p' = \mathbf{xe^{5x}(5 + \frac{1}{x}) = 5xe^{5x} + e^{5x}}$, which is **LD** or logarithmic differentiation.

The integral of $u'(x)/u(x)$ is $\mathbf{\ln u(x)}$. The integral of $2x/(x^2+4)$ is $\mathbf{\ln(x^2+4)}$. The integral of $1/cx$ is $\mathbf{\frac{\ln x}{c}}$. The integral of $1/(ct+s)$ is $\mathbf{\frac{\ln(ct+s)}{c}}$. The integral of $1/\cos x$, after a trick, is $\mathbf{\ln(\sec x + \tan x)}$. We should write $\ln|x|$ for the antiderivative of $1/x$, since this allows $\mathbf{x < 0}$. Similarly $\int du/u$ should be written $\mathbf{\ln|u|}$.

4 $\frac{x(\frac{1}{x}) - (\ln x)}{x^2} = \mathbf{\frac{1 - \ln x}{x^2}}$ **6** Use $(\log_e 10)(\log_{10} x) = \log_e x$. Then $\frac{d}{dx}(\log_{10} x) = \frac{1}{\log_e 10} \cdot \frac{1}{x} = \mathbf{\frac{1}{x \ln 10}}$.
16 $y = \frac{x^3}{x^2+1}$ equals $x - \frac{x}{x^2+1}$. Its integral is $[\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)]_0^2 = \mathbf{2 - \frac{1}{2}\ln 5}$.
20 $\int \frac{\sin x}{\cos x}dx = \int \frac{-du}{u} = -\ln u = -\ln(\cos x)]_0^{\pi/4} = -\ln\frac{1}{\sqrt{2}} + 0 = \mathbf{\frac{1}{2}\ln 2}$.
24 Set $u = \ln\ln x$. By the chain rule $\frac{du}{dx} = \frac{1}{\ln x}\frac{1}{x}$. Our integral is $\int \frac{du}{u} = \ln u = \mathbf{\ln(\ln(\ln x))} + C$.
28 $\ln y = \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(x^2-1)$. Then $\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1} + \frac{x}{x^2-1} = \frac{2x^3}{x^4-1}$. Then $\frac{dy}{dx} = \frac{2x^3 y}{x^4-1} = \frac{2x^3}{\sqrt{x^4-1}}$.
36 $\ln y = -\ln x$ so $\frac{1}{y}\frac{dy}{dx} = \frac{-1}{x}$ and $\frac{dy}{dx} = -\frac{e^{-\ln x}}{x}$. Alternatively we have $y = \frac{1}{x}$ and $\frac{dy}{dx} = \mathbf{-\frac{1}{x^2}}$.
40 $\frac{d}{dx}\ln x = \mathbf{\frac{1}{x}}$. Alternatively use $\frac{1}{x^2}\frac{d}{dx}(x^2) - \frac{1}{x}\frac{d}{dx}(x) = \frac{1}{x}$.
54 Use l'Hôpital's Rule: $\lim\limits_{x\to 0}\frac{b^x \ln b}{1} = \mathbf{\ln b}$. We have redone the derivative of b^x at $x = 0$.
62 $\frac{1}{x}\ln\frac{1}{x} = -\frac{\ln x}{x} \to 0$ as $x \to \infty$. This means $\mathbf{y \ln y \to 0}$ as $y = \frac{1}{x} \to 0$. (Emphasize: The factor $y \to 0$ is "stronger" than the factor $\ln y \to -\infty$.)
70 LD: $\ln p = x\ln x$ so $\frac{1}{p}\frac{dp}{dx} = 1 + \ln x$ and $\frac{dp}{dx} = p(1+\ln x) = x^x(1+\ln x)$. Now find the same answer by **ED:** $\frac{d}{dx}(e^{x\ln x}) = e^{x\ln x}\frac{d}{dx}(x\ln x) = x^x(1+\ln x)$.

6.5 Separable Equations Including the Logistic Equation (page 266)

Separation of variables works so well (when it works) that there is a big temptation to use it often and wildly. I asked my class to integrate the function $y(x) = \frac{d}{dx}(e^{1+x^2})$ from $x = 0$ to $x = 3$. The point of this question is that you don't have to take the derivative of e^{1+x^2}. When you integrate, that brings back the original function. So the answer is

$$\int_0^3 y(x)dx = [e^{1+x^2}]_0^3 = e^{10} - e.$$

One mistake was to write that answer as e^9. The separation of variables mistake was in $y\ dy$:

from $y = \frac{d}{dx}(e^{1+x^2})$ the class wrote $\int y\ dy = \int \frac{d}{dx}(e^{1+x^2})dx$.

You can't multiply one side by dy and the other side by dx. This mistake leads to $\frac{1}{2}y^2$ which shouldn't appear. Separation of variables starts from $\frac{dy}{dx} = u(y)v(x)$ and does *the same thing to both sides.* Divide by $u(y)$, multiply by dx, and integrate. Then $\int dy/u(y) = \int v(x)dx$. Now a y-integral equals an x-integral.

Solve the differential equations in Problems 1 and 2 by separating variables.

1. $\frac{dy}{dx} = \sqrt{xy}$ with $y_0 = 4$ (which means $y(0) = 4$.)

   - First, move dx to the right side and $\sqrt{y}$ to the left: $\frac{dy}{y^{1/2}} = x^{1/2}dx$. Second, integrate both sides: $2y^{1/2} = \frac{2}{3}x^{3/2} + C$. (This constant C combines the constants for each integral.) Third, solve for $y = (\frac{1}{3}x^{3/2} + C)^2$. Here $C/2$ became C. Half a constant is another constant. This is the general solution. Fourth, use the starting value $y_0 = 4$ to find C:

$$4 = (\frac{1}{3}(0)^{3/2} + C)^2 \text{ yields } C = \pm 2. \text{ Then } y = (\frac{1}{3}x^{3/2} \pm 2)^2.$$

2. Solve $(x-3)t\ dt + (t^2+1)dx = 0$ with $x = 5$ when $t = 0$.

   - Divide both sides by $(x-3)(t^2+1)$ to separate t from x :

$$\frac{t\ dt}{t^2+1} + \frac{dx}{x-3} = 0 \text{ or } -\int \frac{t\ dt}{t^2+1} = \int \frac{dx}{x-3}.$$

   Integrating gives $-\frac{1}{2}\ln(t^2+1) = \ln(x-3) + C$ or $(t^2+1)^{-1/2} = e^C(x-3)$. Since $x = 5$ when $t = 0$ we have $1 = 2e^C$. Put $e^C = \frac{1}{2}$ into the solution to find $x - 3 = 2(t^2+1)^{-1/2}$ or $x = 3 + 2(t^2+1)^{-1/2}$.

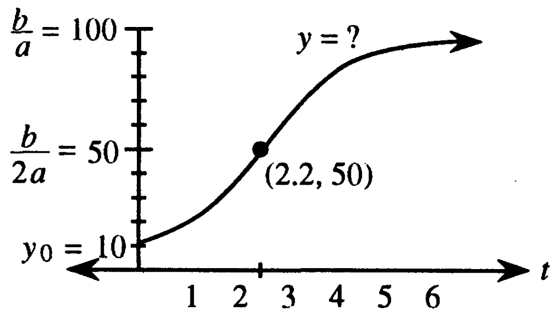
   Problems 3 – 5 deal with the logistic equation $y' = cy - by^2$.

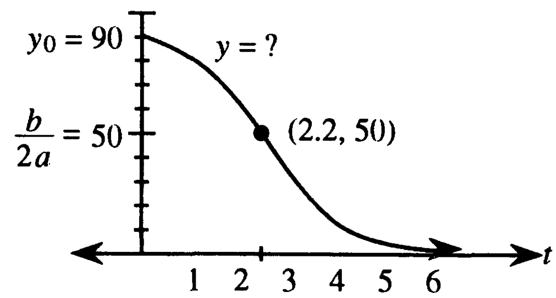
3. (This is 6.5.15.) Solve $\frac{dz}{dt} = -z + 1$ with $z_0 = 2$. Turned upside down, what is $y = \frac{1}{z}$? Graph y and z.

   - Separation of variables gives $\frac{dz}{-z+1} = dt$ or $-\ln|-z+1| = t + C$. Put in $z = 2$ when $t = 0$ to find $C = 0$. Also notice that $-2+1$ is *negative*. The absolute value is reversing the sign. So we have

$$-\ln(z-1) = t \quad \text{or} \quad z - 1 = e^{-t} \quad \text{or} \quad z = e^{-t} + 1.$$

   Now $y = \frac{1}{z} = \frac{1}{1+e^{-t}}$. According to Problem 6.3.15, this y solves the logistic equation $y' = y - y^2$.





4. Each graph above is an S-curve that solves a logistic equation $y' = \pm y \pm by^2$ with $c = 1$ or $c = -1$. Each has an inflection point at (2.2,50). Find the differential equations and the solutions.

- The first graph shows $y_0 = 10$. The inflection point is at height $\frac{c}{2b} = 50$. Then $c = 1$ and $b = \frac{c}{100} = .01$. The limiting value $y_\infty = \frac{c}{b}$ is twice as high at $y_\infty = 100$. The differential equation is $dy/dt = y - .01y^2$. The solution is given by equation (12) on page 263:

$$y = \frac{c}{b + de^{-ct}} \text{ where } d = \frac{c - by_0}{y_0} = \frac{1 - (.01)(10)}{10} = .09. \text{ Then } y = \frac{1}{.01 - .09e^{-t}}.$$

The second graph must solve the differential equation $\frac{dy}{dt} = -y + by^2$. Its slope is just the opposite of the first. Again we have $\frac{c}{2b} = 50$ and $b = 0.01$. Substitute $c = -1$ and $y_0 = 90$:

$$d = \frac{c - by_0}{y_0} = \frac{-1 + .01(90)}{90} = \frac{-1}{900} \text{ and } y(t) = \frac{-1}{-.01 - \frac{e^t}{900}} = \frac{900}{9 + e^t}.$$

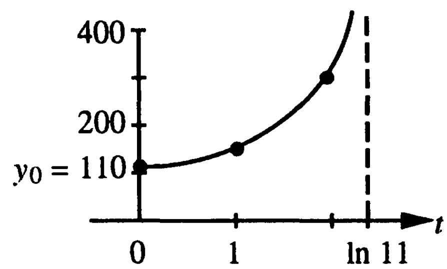
This is a case where *death wins.* Since $y_0 < \frac{c}{b} = 100$ the population dies out before the cooperation term $+by^2$ is strong enough to save it. See Example 6 on page 264 of the text.

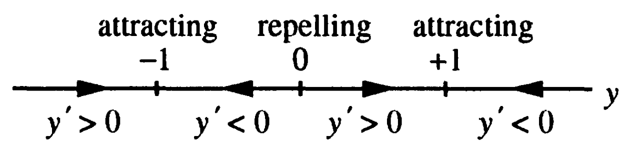
5. Change y_0 in Problem 4 to 110. Then $y_0 > \frac{c}{b} = 100$. Find the solution $y(t)$ and graph it.

- As in 4(b) the equation is $\frac{dy}{dt} = -y + .01y^2$. Since y_0 is now 110, the solution has

$$d = \frac{-1 + .01(110)}{110} = \frac{1}{1100} \text{ and } y(t) = \frac{-1}{-.01 + \frac{e^t}{1100}} = \frac{1100}{11 - e^t}.$$

The graph is sketched below. After a sluggish start, the population blows up at $t = \ln 11$.





6. Draw a y–line for $y' = y - y^3$. Which steady states are approached from which initial values y_0?

   - Factor $y - y^3$ to get $y' = y(1-y)(1+y)$. A steady state has $y' = 0$. This occurs at $y = 0$, 1, and -1. Plot those points on the straight line. They are not all attracting.

     Now consider the sign of $y(1-y)(1+y) = y'$. If y is below $-1, y'$ is positive. (Two factors y and $1+y$ are negative but their product is positive.) If y is between -1 and 0, y' is negative and y decreases. If y is between 0 and 1, all factors are positive and so is y'. Finally, if $y > 1$ then y' is negative.

The signs of y' are $+-+-$. The curved line $f(y)$ is sketched to show those signs. A positive y' means an increasing y. So the solution moves toward -1 and also toward $+1$. It moves away from $y = 0$, because y is increasing on the right of zero and decreasing on the left of zero.

The arrows in the y–line point to the left when y' is negative. The sketch shows that $y = -1$ and $y = +1$ are *stable steady states*. They are attracting, while $y = 0$ is an unstable (or *repelling*) stationary point. The solution approaches -1 from $y_0 < 0$, and it approaches $+1$ from $y_0 > 0$.

Read-throughs and selected even-numbered solutions :

The equations $dy/dt = cy$ and $dy/dt = cy + s$ and $dy/dt = u(y)v(t)$ are called **separable** because we can separate y from t. Integration of $\int dy/y = \int c\,dt$ gives $\mathbf{\ln y = ct + constant}$. Integration of $\int dy/(y + s/c) = \int c\,dt$ gives $\mathbf{\ln(y + \frac{s}{c}) = ct + C}$. The equation $dy/dx = -x/y$ leads to $\int \mathbf{y\,dy} = -\int \mathbf{x\,dx}$. Then $y^2 + x^2 =$ **constant** and the solution stays on a circle.

The logistic equation is $dy/dt = \mathbf{cy - by^2}$. The new term $-by^2$ represents **competition** when cy represents growth. Separation gives $\int dy/(cy - by^2) = \int dt$, and the y-integral is $1/c$ times $\ln \frac{\mathbf{y}}{\mathbf{c-by}}$. Substituting y_0 at $t = 0$ and taking exponentials produces $y/(c - by) = e^{ct}\mathbf{y_0}/(\mathbf{c - by_0})$. As $t \to \infty$,
y approaches $\frac{\mathbf{c}}{\mathbf{b}}$. That is the steady state where $cy - by^2 = \mathbf{0}$. The graph of y looks like an **S**, because it has an inflection point at $\frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{c}}{\mathbf{b}}$.

In biology and chemistry, concentrations y and z react at a rate proportional to y times $\mathbf{z}$. This is the Law of **Mass Action.** In a model equation $dy/dt = c(y)y$, the rate c depends on $\mathbf{y}$. The MM equation is

$dy/dt = -\mathbf{cy}/(\mathbf{y}+\mathbf{K})$. Separating variables yields $\int \frac{\mathbf{y}+\mathbf{K}}{\mathbf{y}}dy = \int -\mathbf{c}\,\mathbf{dt} = -ct + C$.

6 $\frac{dy}{\tan y} = \cos x dx$ gives $\ln(\sin y) = \sin x + C$. Then $C = \ln(\sin 1)$ at $x = 0$. After taking exponentials $\mathbf{\sin y = (\sin 1)e^{\sin x}}$. No solution after $\sin y$ reaches 1 (at the point where $(\sin 1)e^{\sin x} = 1$).

8 $e^y dy = e^t dt$ so $e^y = e^t + C$. Then $C = e^e - 1$ at $t = 0$. After taking logarithms $y = \mathbf{\ln(e^t + e^e - 1)}$.

10 $\frac{d(\ln y)}{d(\ln x)} = \frac{dy/y}{dx/x} = n$. Therefore $\ln y = n \ln x + C$. Therefore $y = (x^n)(e^C) =$ **constant times** $\mathbf{x^n}$.

16 Equation (14) is $z = \frac{1}{c}(b + \frac{c-by_0}{y_0}e^{-ct})$. Turned upside down this is $y = \frac{c}{b+de^{-ct}}$ with $d = \frac{c-by_0}{y_0}$.

20 $y' = y + y^2$ has $c = 1$ and $b = -1$ with $y_0 = 1$. Then $y(t) = \mathbf{\frac{1}{-1+2e^{-t}}}$ by formula (12). The denominator is zero and y blows up when $2e^{-t} = 1$ or $\mathbf{t = \ln 2}$.

26 At the middle of the S-curve $y = \frac{c}{2b}$ and $\frac{dy}{dt} = c(\frac{c}{2b}) - b(\frac{c}{2b})^2 = \frac{c^2}{4b}$. If b and c are multiplied by 10 then so is this slope $\frac{c^2}{4b}$, which becomes **steeper**.

28 If $\frac{cy}{y+K} = d$ then $cy = dy + dK$ and $\mathbf{y} = \mathbf{\frac{dK}{c-d}}$. At this steady state the maintenance dose replaces the aspirin being eliminated.

30 The rate $R = \frac{cy}{y+K}$ is a **decreasing** function of K because $\frac{dR}{dK} = \frac{-cy}{(y+K)^2}$.

34 $\frac{d[A]}{dt} = -r[A][B] = -r[A](b_0 - \frac{n}{m}(a_0 - [A]))$. The changes $a_0 - [A]$ and $b_0 - [B]$ are in the proportion m to n; we solved for $[B]$.

6.6 Powers Instead of Exponentials (page 276)

1. Write down a power series for $y(x)$ whose derivative is $\frac{1}{2}y(x)$. Assume that $y(0) = 1$.

- *First method*: Look for $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$, and choose the a's so that $y' = \frac{1}{2}y$. Start with $a_0 = 1$ so that $y(0) = 1$. Then take the derivative of each term:

$$y' = 0 + a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1} + \cdots$$

Matching this series with $\frac{1}{2}y$ gives $a_1 = \frac{1}{2}a_0$ and $2a_2 = \frac{1}{2}a_1$. Therefore $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$. Similarly $3a_3$ matches $\frac{1}{2}a_2$ and na_n matches $\frac{1}{2}a_{n-1}$. The pattern continues with $a_3 = \frac{1}{3}\cdot\frac{1}{2}\cdot a_2$ and $a_4 = \frac{1}{4}\cdot\frac{1}{2}\cdot a_3$. The typical term is $a_n = \frac{1}{n!2^n}$:

$$\text{The series is } y(x) = 1 + \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 2^2} + \frac{x^3}{3!2^3} + \cdots + \frac{x^n}{n!2^n} + \cdots.$$

- *Second method*: We already know the solution to $y' = \frac{1}{2}y$. It is $y_0e^{\frac{1}{2}x}$. Starting from $y_0 = 1$, the solution is $y = e^{\frac{1}{2}x}$. We also know the exponential series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$. So just substitute the new exponent $\frac{1}{2}x$ in place of x:

$$y = e^{\frac{1}{2}x} = 1 + \frac{1}{2}x + \frac{1}{2!}(\frac{x}{2})^2 + \frac{1}{3!}(\frac{x}{2})^3 + \cdots + \frac{1}{n!}(\frac{x}{2})^n + \cdots = \text{ same answer.}$$

2. (This is 6.6.19) Solve the difference equation $y(t+1) = 3y(t) + 1$ with $y_0 = 0$.

- Follow equations 8 and 9 on page 271. In this problem $a = 3$ and $s = 1$. Each step multiplies the previous y by 3 and adds 1. From $y_0 = 0$ we have $y_1 = 1$ and $y_2 = 4$. Then $y_3 = 13$ and $y_4 = 40$. The solution is

$$y(t) = 3^t \cdot 0 + 1\frac{(3^t - 1)}{3 - 1} \quad \text{or} \quad y(t) = \frac{3^t - 1}{2}.$$

3. If prices rose $\frac{3}{10}\%$ in the last month, what is the equivalent annual rate of inflation?

   - *The answer is not* 12 times $\frac{3}{10} = 3.6\%$. The monthly increases are ***compounded***. A \$1 price at the beginning of the year would be $(1 + .003)^{12} \approx 1.0366$ at the end of the year. The annual rate of inflation is .0366 or 3.66%.

4. If inflation stays at 4% a year, find the present value that yields a dollar after 10 years.

   - Use equation 2 on page 273 with $n = 1$ and $y = 1$. The rate is .04 instead of .05, for 10 years instead of 20. We get $y_0 = (1 + \frac{.04}{1})^{-10}1 = 0.6755$. In a decade a dollar will be worth what 67.55 cents is worth today.

5. Write the difference equation and find the steady state for this situation: Every week 80% of the cereal is sold and 400 more boxes are delivered to the supermarket.

   - If $C(t)$ represents the number of cereal boxes after t weeks, the problem states that $C(t + 1) = 0.2C(t) + 400$. The reason for 0.2 is that 80% are sold and 20% are left. The difference equation has $a = 0.2$ and $s = 400$. Since $|a| < 1$, a steady state is approached: $C_\infty = \frac{s}{1-a} = \frac{400}{.8} = 500$. At that steady state, 80% of 500 boxes are sold (that means 400) and they are replaced by 400 new boxes.

Read-throughs and selected even-numbered solutions :

The infinite series for e^x is $\mathbf{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots}$. Its derivative is $\mathbf{e^x}$. The denominator $n!$ is called "**n factorial**" and is equal to $\mathbf{n(n-1)\cdots(1)}$. At $x = 1$ the series for e is $\mathbf{1 + 1 + \frac{1}{2} + \frac{1}{6}} + \cdots$.

To match the original definition of e, multiply out $(1 + 1/n)^n = \mathbf{1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n})^2}$ (first three terms). As $n \to \infty$ those terms approach $\mathbf{1 + 1 + \frac{1}{2}}$ in agreement with e. The first three terms of $(1 + x/n)^n$ are $\mathbf{1 + n(\frac{x}{n}) + \frac{n(n-1)}{2}(\frac{x}{n})^2}$. As $n \to \infty$ they approach $\mathbf{1 + x + \frac{1}{2}x^2}$ in agreement with e^x. Thus $(1 + x/n)^n$ approaches $\mathbf{e^x}$. A quicker method computes $\ln(1 + x/n)^n \approx \mathbf{x}$ (first term only) and takes the exponential.

Compound interest (n times in one year at annual rate x) multiplies by $\mathbf{(1 + \frac{x}{n})^n}$. As $n \to \infty$, continuous compounding multiplies by $\mathbf{e^x}$. At $x = 10\%$ with continuous compounding, \$1 grows to $\mathbf{e^{.1}} \approx$ **\$1.105** in a year.

The difference equation $y(t+1) = ay(t)$ yields $y(t) = \mathbf{a^t}$ times y_0. The equation $y(t+1) = ay(t) + s$ is solved by $y = a^t y_0 + s[1 + a + \cdots + a^{t-1}]$. The sum in brackets is $\mathbf{\frac{1-a^t}{1-a}}$ or $\mathbf{\frac{a^t-1}{a-1}}$. When $a = 1.08$ and $y_0 = 0$, annual deposits of $s = 1$ produce $y = \mathbf{\frac{1.08^t - 1}{.08}}$ after t years. If $a = \frac{1}{2}$ and $y_0 = 0$, annual deposits of $s = 6$ leave $\mathbf{12(1 - \frac{1}{2^t})}$ after t years, approaching $y_\infty = \mathbf{12}$. The steady equation $y_\infty = ay_\infty + s$ gives $y_\infty = \mathbf{s/(1-a)}$.

When i = interest rate per period, the value of $y_0 = \$1$ after N periods is $y(N) = \mathbf{(1 + i)^N}$. The deposit to produce $y(N) = 1$ is $y_0 = \mathbf{(1 + i)^{-N}}$. The value of $s = \$1$ deposited after each period grows to $y(N) =$

$\frac{1}{i}((1+i)^N - 1)$. The deposit to reach $y(N) = 1$ is $s = \frac{1}{i}(1 - (1+i)^{-N})$.

Euler's method replaces $y' = cy$ by $\Delta y = cy\Delta t$. Each step multiplies y by $1 + c\Delta t$. Therefore y at $t = 1$ is $(1 + c\Delta t)^{1/\Delta t} y_0$, which converges to $y_0 e^c$ as $\Delta t \to 0$. The error is proportional to Δt, which is too large for scientific computing.

4 A larger series is $1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3$. This is greater than $1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots = e$.
8 The exact sum is $e^{-1} \approx .37$ (Problem 6). After five terms $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = .375$.
14 $y(0) = 0, y(1) = 1, y(2) = 3, y(3) = 7$ (and $y(n) = 2^n - 1$). **24** Ask for $\frac{1}{2}y(0) - 6 = y(0)$. Then $y(0) = -12$.
30 The equation $-dP(t+1) + b = cP(t)$ becomes $-2P(t+1) + 8 = P(t)$ or $P(t+1) = -\frac{1}{2}P(t) + 4$. Starting from $P(0) = 0$ the solution is $P(t) = 4[\frac{(-\frac{1}{2})^t - 1}{-\frac{1}{2} - 1}] = \frac{8}{3}(1 - (-\frac{1}{2})^t) \to \frac{8}{3}$.
38 Solve $\$1000 = \$8000\ [\frac{.1}{1-(1.1)^{-n}}]$ for n. Then $1 - (1.1)^{-n} = .8$ or $(1.1)^{-n} = .2$. Thus $1.1^n = 5$ and $n = \frac{\ln 5}{\ln 1.1} \approx 17$ **years**.
40 The interest is $(.05)1000 = \$50$ in the first month. You pay \$60. So your debt is now $\$1000 - \$10 = \$990$. Suppose you owe $y(t)$ after month t, so $y(0) = \$1000$. The next month's interest is $.05y(t)$. You pay \$60. So $y(t+1) = 1.05y(t) - 60$. After 12 months $y(12) = (1.05)^{12}1000 - 60[\frac{(1.05)^{12}-1}{1.05-1}]$. This is also $\frac{60}{.05} + (1000 - \frac{60}{.05})(1.05)^{12} \approx \$841$.
44 Use the loan formula with $.09/n$ not $.09n$: payments $s = 80{,}000\frac{.09/12}{[1-(1+\frac{.09}{12})^{-360}]} \approx \$643.70$. Then 360 payments equal \$231,732.

6.7 Hyperbolic Functions (page 280)

1. Given $\sinh x = \frac{5}{12}$, find the values of $\cosh x, \tanh x, \coth x, \operatorname{sech} x$ and $\operatorname{csch} x$.

   • Use the identities on page 278. The one to remember is similar to $\cos^2 x + \sin^2 x = 1$:

   $$\cosh^2 x - \sinh^2 x = 1 \text{ gives } \cosh^2 x = 1 + \frac{25}{144} = \frac{169}{144} \text{ and } \cosh x = \frac{13}{12}.$$

   Note that $\cosh x$ is always positive. Then $\tanh x = \frac{\sinh x}{\cosh x}$ is $\frac{5/12}{13/12} = \frac{5}{13}$. The others are upside down:

   $$\coth x = \frac{1}{\tanh x} = \frac{13}{5} \text{ and } \operatorname{sech} x = \frac{1}{\cosh x} = \frac{12}{13} \text{ and } \operatorname{csch} x = \frac{1}{\sinh x} = \frac{12}{5}.$$

2. Find $\cosh(2\ln 10)$. Substitute $x = 2\ln 10 = \ln 100$ into the definition of $\cosh x$:

   • $\cosh(2\ln 10) = \frac{e^{\ln 100} + e^{-\ln 100}}{2} = \frac{100 + \frac{1}{100}}{2} = \frac{100.01}{2} = 50.005$.

3. Find $\frac{dy}{dx}$ when $y = \sinh(4x^3)$. Use the chain rule with $u = 4x^3$ and $\frac{du}{dx} = 12x^2$

   • The derivative of $\sinh u(x)$ is $(\cosh u)\frac{du}{dx} = 12x^2 \cosh(4x^3)$.

4. Find $\frac{dy}{dx}$ when $y = \ln \tanh 2x$. • Let $y = \ln u$, where $u = \tanh 2x$. Then

   $$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{2\operatorname{sech}^2 2x}{\tanh 2x} = \frac{2}{\sinh 2x \cosh 2x}.$$

5. Find $\frac{dy}{dx}$ when $y = \text{sech}^{-1}6x$. • See equation (3) on page 279. If $u = 6x$ then

$$\frac{dy}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx} = \frac{-6}{6x\sqrt{1-36x^2}} = \frac{-1}{x\sqrt{1-36x^2}}.$$

6. Find $\int \frac{dx}{\sqrt{x^2+9}}$. • Except for the 9, this looks like $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x + C$ on page 279. Factoring out $\sqrt{9}$ leaves $\sqrt{x^2+9} = \sqrt{9}\sqrt{\frac{x^2}{9}+1}$. So the problem has $u = \frac{x}{3}$ and $du = \frac{1}{3}dx$:

$$\int \frac{dx}{3\sqrt{(\frac{x}{3})^2+1}} = \int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1}u + C = \sinh^{-1}(\frac{x}{3} + C).$$

7. Find $\int \cosh^2 x \sinh\ x\ dx$ (This is 6.7.53.) Remember that $u = \cosh\ x$ has $\frac{du}{dx} = +\sinh\ x$:

   • The problem is really $\int u^2 du$ with $u = \cosh\ x$. The answer is $\frac{1}{3}u^3 + C = \frac{1}{3}\cosh^3 x + C$.

8. Find $\int \frac{\sinh\ x}{1+\cosh\ x}dx$. (This is 6.7.29.) The top is the derivative of the bottom!

   • $\int \frac{du}{u} = \ln|u| + C = \ln(1+\cosh\ x) + C$.

   The absolute value sign is dropped because $1 + \cosh\ x$ is always positive.

9. (This is Problem 6.7.54) A falling body with friction equal to velocity squared obeys $\frac{dv}{dt} = g - v^2$.
   (a) Show that $v(t) = \sqrt{g}\tanh\sqrt{g}t$ satisfies the equation. (b) Derive this yourself by integrating $\frac{dv}{g-v^2} = dt$.
   (c) Integrate $v(t)$ to find the distance $f(t)$.

   • (a) The derivative of $\tanh\ x$ is $\text{sech}^2\ x$. The derivative of $v(t) = \sqrt{g}\ \tanh\sqrt{g}\ t$ has $u = \sqrt{g}\ t$. The chain rule gives $\frac{dv}{dt} = \sqrt{g}(\text{sech}^2 u)\frac{du}{dt} = g\ \text{sech}^2\sqrt{g}t$. Now use the identity $\text{sech}^2 u = 1 - \tanh^2 u$:

   $$\frac{dv}{dt} = g(1 - \tanh^2\sqrt{g}t) = g - v^2.$$

   • (b) The differential equation is $\frac{dv}{dt} = g - v^2$. Separate variables to find $\frac{dv}{g-v^2} = dt$:

   $$\int \frac{dv}{g-v^2} = \int \frac{dv}{g[1-(\frac{v}{\sqrt{g}})^2]} = \frac{1}{\sqrt{g}}\tanh^{-1}\frac{v}{\sqrt{g}} \text{ by equation (2), on page 279.}$$

   The integral of dt is $t + C$. Assuming the body falls from rest ($v = 0$ at $t = 0$), we have $C = 0$. Then $t = \frac{1}{\sqrt{g}}\tanh^{-1}\frac{v}{\sqrt{g}}$ turns into $v = \sqrt{g}\ \tanh\sqrt{g}t$.

   • (c) $\int v\ dt = \int \sqrt{g}\tanh\sqrt{g}t\ dt = \ln\cosh\sqrt{g}t + C$.

Read-throughs and selected even-numbered solutions :

Cosh $x = \mathbf{\frac{1}{2}}(\mathbf{e^x + e^{-x}})$ and $\sinh x = \mathbf{\frac{1}{2}}(\mathbf{e^x - e^{-x}})$ and $\cosh^2 x - \sinh^2 x = \mathbf{1}$. Their derivatives are **sinh x** and **cosh x** and **zero**. The point $(x, y) = (\cosh t, \sinh t)$ travels on the hyperbola $\mathbf{x^2 - y^2 = 1}$. A cable hangs in the shape of a catenary $y = \mathbf{a\ cosh\frac{x}{a}}$.

The inverse functions $\sinh^{-1} x$ and $\tanh^{-1} x$ are equal to $\ln[x+\sqrt{x^2+1}]$ and $\frac{1}{2}\ln\frac{\mathbf{1+x}}{\mathbf{1-x}}$. Their derivatives are $\mathbf{1/\sqrt{x^2+1}}$ and $\frac{1}{1-x^2}$. So we have two ways to write the anti**derivative**. The parallel to $\cosh x + \sinh x = e^x$ is Euler's formula $\mathbf{\cos x + i \sin x = e^{ix}}$. The formula $\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$ involves **imaginary** exponents. The parallel formula for $\sin x$ is $\mathbf{\frac{1}{2i}(e^{ix}-e^{-ix})}$.

12 $\sinh(\ln x) = \frac{1}{2}(e^{\ln x} - e^{-\ln x}) = \frac{1}{2}(x - \frac{1}{x})$ with derivative $\mathbf{\frac{1}{2}(1+\frac{1}{x^2})}$.

16 $\frac{1+\tanh x}{1-\tanh x} = e^{2x}$ by the equation following (4). Its derivative is $\mathbf{2e^{2x}}$. More directly the quotient rule gives $\frac{(1-\tanh x)\text{sech}^2 x + (1+\tanh x)\text{sech}^2 x}{(1-\tanh x)^2} = \frac{2\,\text{sech}^2 x}{(1-\tanh x)^2} = \frac{2}{(\cosh x - \sinh x)^2} = \frac{2}{e^{-2x}} = \mathbf{2e^{2x}}$.

18 $\frac{d}{dx}\ln u = \frac{du/dx}{u} = \frac{\text{sech}\,x\tanh x - \text{sech}^2 x}{\text{sech}\,x + \tanh x}$. Because of the minus sign we do not get $\text{sech}\,x$. The integral of $\text{sech}\,x$ is $\sin^{-1}(\tanh x) + C$.

30 $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}dx = \mathbf{\ln(\sinh x)} + C$. **32** $\sinh x + \cosh x = e^x$ and $\int e^{nx}dx = \mathbf{\frac{1}{n}e^{nx}} + C$.

36 $y = \text{sech} x$ looks like a bell-shaped curve with $y_{\max} = 1$ at $x = 0$. The x axis is the asymptote. But note that y decays like $2e^{-x}$ and not like e^{-x^2}.

40 $\frac{1}{2}\ln(\frac{1+x}{1-x})$ approaches $+\infty$ as $x \to 1$ and $-\infty$ as $x \to -1$. The function is *odd* (so is the tanh function). The graph is an **S** curve rotated by 90°.

44 The x derivative of $x = \sinh y$ is $1 = \cosh y\frac{dy}{dx}$. Then $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \mathbf{\frac{1}{\sqrt{1+x^2}}}$ = slope of $\sinh^{-1} x$.

50 Not hyperbolic! Just $\int(x^2+1)^{-1/2}x\,dx = \mathbf{(x^2+1)^{1/2}} + C$.

58 $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x}+e^x) = \cosh x$. Then $\cos i = \cosh 1 = \mathbf{\frac{e+e^{-1}}{2}}$ (real!).

6 Chapter Review Problems

Graph Problems (Sketch the graphs and locate maxima, minima, and inflection points)

G1 $y = x \ln x$

G2 $y = e^{-x^2}$

G3 $y = e^{-x^3}$

G4 $y = x^2 - 72\ln x$

G5 $y = x^6 e^{-x}$

G6 $y = e^{\ln x}$ (watch the domain)

G7 Sketch ln 3 as an area under a curve. Approximate the area using four trapezoids.

G8 Sketch $y = \ln x$ and $y = \ln\frac{1}{x}$. Also sketch $y = e^x$ and $y = e^{-1/x}$.

G9 Sketch $y = 2 + e^x$ and $y = e^{x+2}$ and $y = 2e^x$ on the same axes.

Review Problems

R1 Give an example of a linear differential equation and a nonlinear differential equation. If possible find their solutions starting from $y(0) = A$.

R2 Give examples of differential equations that can and cannot be solved by separation of variables.

R3 In exponential growth, the rate of change of y is directly proportional to ______. In exponential decay, dy/dt is proportional to ______. The difference is that ______.

R4 What is a steady state? Give an example for $\frac{dy}{dt} = y + 3$.

R5 Show from the definition that $d(\cosh x) = \sinh x\, dx$ and $d(\operatorname{sech} x) = \operatorname{sech} x \tanh x\, dx$.

R6 A particle moves along the curve $y = \cosh x$ with $dx/dt = 2$. Find dy/dt when $x = 1$.

R7 A chemical is decomposing with a half-life of 3 hours. Starting with 120 grams how much remains after 3 hours and how much after 9 hours?

R8 A radioactive substance decays with a half-life of 10 hours. Starting with 100 grams, show that the average during the first 10 hours is $100/\ln 2$ grams.

R9 How much money must be deposited now at 6% interest (compounded continuously) to build a nest egg of \$40,000 in 15 years?

R10 Show that a continuous deposit of \$1645 per year at 6% interest yields more than \$40,000 after 15 years.

Drill Problems (Find dy/dx in **D1** to **D 12**.)

D1 $y = e^{\cos x}$

D2 $e^y + e^{-y} = 2x$

D3 $\sin x = e^y$

D4 $y = \pi^x + \pi^{-x}$

D5 $y = \frac{e^x}{x}$

D6 $y = \sec e^x$

D7 $y = \ln \frac{x-2}{x+2}$

D8 $y^2 = \ln(x^2 + y^2)$

D9 $y = \frac{\sqrt{x^2+5}(2x-3)^2}{\sqrt[3]{x^4(x+1)}}$ (use **LD**)

D10 $y = x^{\cos x}$

D11 $y = \ln(\tanh x^2)$

D12 $y = \cosh x \sinh x$

Find the integral in **D13** to **D20**.

D13 $\int 5^x \, dx$

D14 $\int x \, e^{x^2+1} \, dx$

D15 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

D16 $\int \frac{e^x}{5+e^x} \, dx$

D17 $\int \frac{\cos x}{4+\sin x} \, dx$

D18 $\int \sinh x \cosh x \, dx$

D19 $\int \tanh^2 x \, \mathrm{sech}^2 x \, dx$

D20 $\int \frac{dx}{x \ln \frac{1}{x}}$

Solve the differential equations **D21** *to* **D26**

D21 $y' = -4y$ with $y(0) = 2$

D22 $\frac{dy}{dt} = 2 - 3y$ with $y_0 = 1$

D23 $\frac{dy}{dt} = t^2\sqrt{y}$ with $y_0 = 9$

D24 $\frac{dy}{dt} = 2ty^2$ with $y_0 = 1$

D25 $\frac{dy}{dx} = e^{xy}$ with $y_0 = 10$

D26 $\frac{dy}{dt} = y - 2y^2$ with $y_0 = 100$

Solutions $y = 2e^{-4t}$ $\quad y = \frac{1}{3}e^{-3t} + \frac{2}{3}$ $\quad y = (\frac{t^3}{6} + 3)^2$ $\quad y = \frac{-1}{t^2-1}$ $\quad y = -\ln|e^{-10} - e^x|$ $\quad y = \frac{1}{2-1.99e^{-t}}$

D27 If a population grows continuously at 2% a year, what is its percentage growth after 20 years?