CHAPTER 4 DERIVATIVES BY THE CHAIN RULE

4.1 The Chain Rule (page 158)

The function $\sin(3x+2)$ is "composed" out of two functions. The *inner* function is $u(x) = 3x+2$. The *outer* function is $\sin u$. I don't write $\sin x$ because that would throw me off. The derivative of $\sin(3x+2)$ is not $\cos x$ or even $\cos(3x+2)$. The chain rule produces the extra factor $\frac{du}{dx}$, which in this case is the number 3. *The derivative of* $\sin(3x+2)$ *is* $\cos(3x+2)$ *times* 3.

Notice again: Because the sine was evaluated at u (not at x), its derivative is also evaluated at u. We have $\cos(3x+2)$ not $\cos x$. The extra factor 3 comes because u changes as x changes:

$$(\textit{algebra}) \quad \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \quad \text{approaches} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad (\textit{calculus}).$$

These letters can and will change. Many many functions are chains of simpler functions.

1. Rewrite each function below as a composite function $y = f(u(x))$. Then find $\frac{dy}{dx} = f'(u)\frac{du}{dx}$ or $\frac{dy}{du}\frac{du}{dx}$.
   (a) $y = \tan(\sin x)$ (b) $y = \cos(3x^4)$ (c) $y = \frac{1}{(2x-5)^2}$

   - $y = \tan(\sin x)$ is the chain $y = \tan u$ with $u = \sin x$. The chain rule gives $\frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\cos x)$. Substituting back for u gives $\frac{dy}{dx} = \sec^2(\sin x)\cos x$.
   - $\cos(3x^4)$ separates into $\cos u$ with $u = 3x^4$. Then $\frac{dy}{du}\frac{du}{dx} = (-\sin u)(12x^3) = -12x^3\sin(3x^4)$.
   - $y = \frac{1}{(2x-5)^2}$ is $y = \frac{1}{u^2}$ with $u = 2x-5$. The chain rule gives $\frac{dy}{dx} = (-2u^{-3})(2) = -4(2x-5)^{-3}$. Another perfectly good "decomposition" is $y = \frac{1}{u}$, with $u = (2x-5)^2$. Then $\frac{dy}{du} = -\frac{1}{u^2}$ and $\frac{du}{dx} = 2(2x-5)(2)$ (really another chain rule). The answer is the same: $\frac{dy}{dx} = \frac{-1}{[(2x-5)^2]^2}\cdot 4(2x-5) = \frac{-4}{(2x-5)^3}$.

2. Write $y = \sin\sqrt{3x^2-5}$ and $y = \frac{1}{1-\frac{1}{x}}$ as triple chains $y = f(g(u(x)))$. Then find $\frac{dy}{dx} = f'(g(u))\cdot g'(u)\cdot\frac{du}{dx}$. You could write the chain as $y = f(w)$, $w = g(u)$, $u = u(x)$. Then you see the slope as a product of *three factors*, $\frac{dy}{dx} = (\frac{dy}{dw})(\frac{dw}{du})(\frac{du}{dx})$.

   - For $y(x) = \sin\sqrt{3x^2-5}$ the triple chain is $y = \sin w$, where $w = \sqrt{u}$ and $u = 3x^2-5$. The chain rule is $\frac{dy}{dx} = (\frac{dy}{dw})(\frac{dw}{du})(\frac{du}{dx}) = (\cos w)(\frac{1}{2\sqrt{u}})(6x)$. Substitute to get back to x:

$$\frac{dy}{dx} = \cos\sqrt{3x^2-5}\cdot\frac{1}{2\sqrt{(3x^2-5)}}\cdot 6x = \frac{6x\cos\sqrt{3x^2-5}}{2\sqrt{3x^2-5}}.$$

   - For $y(x) = \frac{1}{1-\frac{1}{x}}$ let $u = \frac{1}{x}$. Let $w = 1-u$. Then $y = \frac{1}{w}$. The derivative is

$$\frac{dy}{dx} = (\frac{dy}{dw})(\frac{dw}{du})(\frac{du}{dx}) = (-\frac{1}{w^2})(-1)(\frac{-1}{x^2}) = \frac{-1}{(1-u)^2x^2} = \frac{-1}{(1-\frac{1}{x})^2x^2} = \frac{-1}{(x-1)^2}.$$

With practice, you should get to the point where it is not necessary to write down u and w in full detail. Try this with exercises 1 – 22, doing as many as you need to get good at it. Problems 45 – 54 are excellent practice, too.

Questions 3 – 6 are based on the following table, which gives the values of functions f and f' and g and g' at a few points. You do not know what these functions are!

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	-1	0	undefined
$\frac{1}{3}$	$\frac{3}{4}$	$-\frac{9}{4}$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{4}{9}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
1	$\frac{1}{2}$	$-\frac{1}{4}$	1	$\frac{1}{2}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	$\frac{1}{3}$	$-\frac{1}{9}$	$\sqrt{2}$	$\frac{\sqrt{2}}{4}$
3	$\frac{1}{4}$	$-\frac{1}{16}$	$\sqrt{3}$	$\frac{\sqrt{3}}{6}$
4	$\frac{1}{5}$	$-\frac{1}{25}$	2	$\frac{1}{4}$
9	$\frac{1}{10}$	$-\frac{1}{100}$	3	$\frac{1}{6}$

3. Find: $f(g(4))$ and $f(g(1))$ and $f(g(0))$.
   - $g(4) = 2$ and $f(2) = \frac{1}{3}$ so $f(g(4)) = \frac{1}{3}$. Also $g(1) = 1$ so $f(g(1)) = f(1) = \frac{1}{2}$. Then $f(g(0)) = f(0) = 0$.
4. Find: $g(f(1))$ and $g(f(2))$ and $g(f(0))$.
   - Since $f(1) = \frac{1}{2}$, the chain $g(f(1))$ is $g(\frac{1}{2}) = \frac{\sqrt{2}}{2}$. Also $g(f(2)) = g(\frac{1}{3}) = \frac{\sqrt{3}}{3}$. Then $g(f(0)) = g(1) = 1$.

     Note that $g(f(1))$ does not equal $f(g(1))$. Also $g(f(0)) \neq f(g(0))$. This is normal. Chains in a different order are different chains.
5. If $y = f(g(x))$ find $\frac{dy}{dx}$ at $x = 9$.
   - The chain rule says that $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$. At $x = 9$ we have $g(9) = 3$ and $g'(9) = \frac{1}{6}$. At $g = 3$ we have $f'(3) = -\frac{1}{16}$. Therefore at $x = 9$, $\frac{dy}{dx} = f'(g(9)) \cdot g'(9) = -\frac{1}{16} \cdot \frac{1}{6} = -\frac{1}{96}$.
6. If $y = g(f(x))$ find $\frac{dy}{dx}(1)$. Note that $f(1) = \frac{1}{2}$.
   - $g'(f(1)) \cdot f'(1) = g'(\frac{1}{2}) \cdot f'(1) = \frac{\sqrt{2}}{2}(-\frac{1}{4}) = \frac{-\sqrt{2}}{8}$.
7. If $y = f(f(x))$ find $\frac{dy}{dx}$ at $x = 2$. This chain repeats the same function ($f = g$). It is "iteration."
   - If you let $u = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ becomes $\frac{dy}{dx} = f'(u) \cdot f'(x)$. At $x = 2$ the table gives $u = \frac{1}{3}$. Then $\frac{dy}{dx} = f'(\frac{1}{3}) \cdot f'(2) = (-\frac{9}{4})(-\frac{1}{9}) = \frac{1}{4}$. Note that $(f'(2))^2 = (-\frac{1}{9})^2$. The derivative of $f(f(x))$ is *not* $(f'(x))^2$. And it is *not* the derivative of $(f(x))^2$.

Read-throughs and selected even-numbered solutions :

$z = f(g(x))$ comes from $z = f(y)$ and $y = \mathbf{g(x)}$. At $x = 2$ the chain $(x^2-1)^3$ equals $\mathbf{3^3 = 27}$. Its inside function is $y = \mathbf{x^2 - 1}$, its outside function is $z = \mathbf{y^3}$. Then dz/dx equals $\mathbf{3y^2 dy/dx}$. The first factor is evaluated at $y = \mathbf{x^2 - 1}$ (not at $y = x$). For $z = \sin(x^4 - 1)$ the derivative is $\mathbf{4x^3 \cos(x^4 - 1)}$. The triple chain $z = \mathbf{\cos(x+1)^2}$ has a shift and a **square** and a cosine. Then $dz/dx = \mathbf{2}\cos(\mathbf{x+1})(-\,\mathbf{\sin(x+1)})$.

The proof of the chain rule begins with $\Delta z/\Delta x = \mathbf{(\Delta z/\Delta y)\,(\Delta y/\Delta x)}$ and ends with $\mathbf{dz/dx = (dz/dy)\,(dy/dx)}$. Changing letters, $y = \cos u(x)$ has $dy/dx = \mathbf{-\sin u(x)\frac{du}{dx}}$. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx = \mathbf{nu^{n-1}\frac{du}{dx}}$. The slope of $5g(x)$ is $\mathbf{5g'(x)}$ and the slope of $g(5x)$ is $\mathbf{5g'(5x)}$. When f = cosine and g = sine and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are **1 and sin 1 and 0**.

18 $\frac{dz}{dx} = \frac{\cos(x+1)}{2\sqrt{\sin(x+1)}}$ **20** $\frac{dz}{dx} = \frac{\cos(\sqrt{x}+1)}{2\sqrt{x}}$ **22** $\frac{dz}{dx} = 4x(\sin x^2)(\cos x^2)$

28 $f(y) = y + 1; h(y) = \sqrt[3]{y}; k(y) \equiv 1$

38 For $g(g(x)) = x$ the graph of g should be **symmetric across the 45° line:** If the point (x,y) is on the graph so is (y,x). Examples: $g(x) = -\frac{1}{x}$ or $-x$ or $\sqrt[3]{1-x^3}$.

40 False (The chain rule produces -1 : so derivatives of even functions are odd functions)
False (The derivative of $f(x) = x$ is $f'(x) = 1$) **False** (The derivative of $f(1/x)$ is $f'(1/x)$ times $-1/x^2$)

True (The factor from the chain rule is 1) **False** (see equation (8)).

42 From $x = \frac{\pi}{4}$ go up to $y = \sin\frac{\pi}{4}$. Then go **across** to the parabola $z = y^2$. Read off $z = (\sin\frac{\pi}{4})^2$ on the horizontal z axis.

4.2 Implicit Differentiation and Related Rates (page 163)

Questions 1 – 5 are examples using *implicit differentiation* (**ID**).

1. Find $\frac{dy}{dx}$ from the equation $x^2 + xy = 2$. Take the x derivative of all terms.

   - The derivative of x^2 is $2x$. The derivative of xy (a product) is $x\,\frac{dy}{dx} + y$. The derivative of 2 is 0. Thus $2x + x\,\frac{dy}{dx} + y = 0$, and $\frac{dy}{dx} = -\frac{y+2x}{x}$.

     In this example the original equation can be solved for $y = \frac{1}{x}(2 - x^2)$. Ordinary *explicit* differentiation yields $\frac{dy}{dx} = \frac{-2}{x^2} - 1$. This must agree with our answer from **ID**.

2. Find $\frac{dy}{dx}$ from $(x+y)^3 = x^4 + y^4$. This time we cannot solve for y.

   - The chain rule tells us that the x-derivative of $(x+y)^3$ is $3(x+y)^2(1 + \frac{dy}{dx})$. Therefore **ID** gives $3(x+y)^2(1 + \frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$. Now algebra separates out $\frac{dy}{dx} = \frac{3(x+y)^2 - 4y^3}{4x^3 - 3(x+y)^2}$.

3. Use **ID** to find $\frac{dy}{dx}$ for $y = x\sqrt{1-x}$.

   - Implicit differentiation (**ID** for short) is not necessary, but you might appreciate how it makes the problem easier. Square both sides to eliminate the square root: $y^2 = x^2(1-x) = x^2 - x^3$, so that

$$2y\frac{dy}{dx} = 2x - 3x^2 \quad \text{and} \quad \frac{dy}{dx} = \frac{2x - 3x^2}{2y} = \frac{2x - 3x^2}{2x\sqrt{1-x}} = \frac{2 - 3x}{2\sqrt{1-x}}.$$

4. Find $\frac{d^2y}{dx^2}$ when $xy + y^2 = 1$. Apply **ID** twice to this equation.

   - First derivative: $x\,\frac{dy}{dx} + y + 2y\,\frac{dy}{dx} = 0$. Rewrite this as $\frac{dy}{dx} = \frac{-y}{x+2y}$. Now take the derivative again. The second form needs the quotient rule , so I prefer to use **ID** on the first derivative equation:

$$x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + 2y\,\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^2 = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -2\frac{\frac{dy}{dx} + (\frac{dy}{dx})^2}{x + 2y}.$$

   Now substitute $\frac{-y}{x+2y}$ for $\frac{dy}{dx}$ and simplify the answer to $\frac{d^2y}{dx^2} = \frac{2}{(x+2y)^3}$.

5. Find the equation of the tangent line to the ellipse $x^2 + xy + y^2 = 1$ through the point (1,0).

   - The line has equation $y = m(x-1)$ where m is the slope at (1,0). To find that slope, apply **ID** to the equation of the ellipse: $2x + x\,\frac{dy}{dx} + y + 2y\,\frac{dy}{dx} = 0$. Do not bother to solve this for $\frac{dy}{dx}$. Just plug in $x = 1$ and $y = 0$ to obtain $2 + \frac{dy}{dx} = 0$. Then $m = \frac{dy}{dx} = -2$ and the tangent equation is $y = -2(x-1)$.

Questions 6–8 are problems about ***related rates***. The slope of one function is known, we want the slope of a *related* function. Of course slope = *rate* = derivative. You must find the relation between functions.

6. Two cars leave point A at the same time $t = 0$. One travels north at 65 miles/hour, the other travels east at 55 miles/hour. How fast is the distance D between the cars changing at $t = 2$?

   - The distance satisfies $D^2 = x^2 + y^2$. This is the relation between our functions! Find the rate of change (take the derivative): $2D\frac{dD}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}$. We need to know $\frac{dD}{dt}$ at $t = 2$. We already know $\frac{dx}{dt} = 55$ and $\frac{dy}{dt} = 65$. At $t = 2$ the cars have traveled for two hours: $x = 2(55) = 110$, $y = 2(65) = 130$ and $D = \sqrt{110^2 + 130^2} \approx 170.3$.

     Substituting these values gives $2(170.3)\,\frac{dD}{dt} = 2(110)(55) + 2(130)(65)$, so $\frac{dD}{dt} \approx 85$ miles/hour.

7. Sand pours out from a conical funnel at the rate of 5 cubic inches per second. The funnel is 6″ wide at the top and 6″ high. At what rate is the sand height falling when the remaining sand is 1″ high?

   - Ask yourself what rate(s) you know and what rate you want to know. In this case you know $\frac{dV}{dt} = -5$ (V is the volume of the sand). You want to know $\frac{dh}{dt}$ when $h = 1$ (h is the height of the sand). Can you get an equation *relating* V and h? This is usually the crux of the problem.

     The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. If we could eliminate r, then V would be related to h. Look at the figure. By similar triangles $\frac{r}{h} = \frac{3}{6}$, so $r = \frac{1}{2}h$. This means that $V = \frac{1}{3}\pi(\frac{h}{2})^2 h = \frac{1}{12}\pi h^3$.

     Now take the t derivative: $\frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\frac{dh}{dt}$. *After* the derivative has been taken, substitute what is known at $h = 1$: $-5 = \frac{1}{12}\pi(3)\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{-20}{\pi}$ in/sec ≈ -6.4 in/sec.

8. (This is Problem 4.2.21) The bottom of a 10-foot ladder moves away from the wall at 2 ft/sec. How fast is the top going down the wall when the top is (a) 6 feet high? (b) 5 feet high? (c) zero feet high?

   - We are given $\frac{dx}{dt} = 2$. We want to know dy/dt. *The equation relating x and y is* $x^2 + y^2 = 100$. This gives $2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$. Substitute $\frac{dx}{dt} = 2$ to find $\frac{dy}{dt} = -\frac{2x}{y}$.

   (a) If $y = 6$, then $x = 8$ (use $x^2 + y^2 = 100$) and $\frac{dy}{dt} = -\frac{8}{3}$ ft/sec.

   (b) If $y = 5$, then $x = 5\sqrt{3}$ (use $x^2 + y^2 = 100$) and $\frac{dy}{dt} = -2\sqrt{3}$ ft/sec.

   (c) If $y = 0$, then we are dividing by zero: $\frac{dy}{dx} = -\frac{2x}{0}$. Is the speed infinite? How is this possible?

Read-throughs and selected even-numbered solutions :

For $x^3 + y^3 = 2$ the derivative dy/dx comes from **implicit** differentiation. We don't have to solve for **y**. Term by term the derivative is $3x^2 + \mathbf{3y^2\frac{dy}{dx}} = 0$. Solving for dy/dx gives $\mathbf{-x^2/y^2}$. At $x = y = 1$ this slope is $\mathbf{-1}$. The equation of the tangent line is $y - 1 = \mathbf{-1(x-1)}$.

A second example is $y^2 = x$. The x derivative of this equation is $\mathbf{2y\frac{dy}{dx}} = 1$. Therefore $dy/dx = \mathbf{1/2y}$. Replacing y by $\sqrt{x}$ this is $dy/dx = \mathbf{1/2\sqrt{x}}$.

In related rates, we are given dg/dt and we want df/dt. We need a relation between f and **g**. If $f = g^2$, then $(df/dt) = \mathbf{2g}(dg/dt)$. If $f^2 + g^2 = 1$, then $df/dt = \mathbf{-\frac{g}{f}\frac{dg}{dt}}$. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt = \mathbf{3s^2(2)} = \mathbf{6s^2}$. To find a number (8 is wrong), you also need to know **s**.

6 $f'(x) + F'(y)\frac{dy}{dx} = y + x\frac{dy}{dx}$ so $\frac{dy}{dx} = \frac{\mathbf{y - f'(x)}}{\mathbf{F'(y) - x}}$

12 $2(x-2) + 2y\frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = \mathbf{1}$ at (1,1); $2x + 2(y-2)\frac{dy}{dx} = 0$ also gives $\frac{dy}{dx} = \mathbf{1}$.

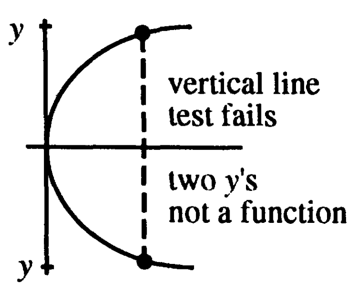
20 x is a constant (fixed at 7) and therefore **a change Δ x is not allowed**

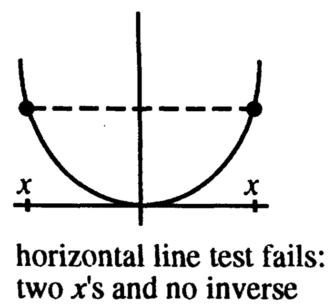
24 Distance to you is $\sqrt{x^2+8^2}$, rate of change is $\frac{x}{\sqrt{x^2+8^2}}\frac{dx}{dt}$ with $\frac{dx}{dt} = 560$. (a) Distance = 16 and $x = 8\sqrt{3}$ and rate is $\frac{8\sqrt{3}}{16}(560) = \mathbf{280\sqrt{3}}$; (b) $x = 8$ and rate is $\frac{8}{\sqrt{8^2+8^2}}(560) = \mathbf{280\sqrt{2}}$; (c) $x = 0$ and rate = 0.

28 Volume = $\frac{4}{3}\pi r^3$ has $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. If this equals twice the surface area $4\pi r^2$ (with minus for evaporation) than $\frac{dr}{dt} = -\mathbf{2}$.

4.3 Inverse Functions and Their Derivatives (page 170)

The vertical line test and the horizontal line test are good for visualizing the meaning of "function" and "invertible." If a vertical line hits the graph twice, we have two y's for the same x. *Not a function.* If a horizontal line hits the graph twice, we have two x's for the same y. *Not invertible.* This means that the inverse is not a function.





These tests tell you that the sideways parabola $x = y^2$ does not give y as a function of x. (Vertical lines intersect the graph twice. There are two square roots $y = \sqrt{x}$ and $y = -\sqrt{x}$.) Similarly the function $y = x^2$ has no inverse. This is an ordinary parabola – horizontal lines cross it twice. If $y = 4$ then $x = f^{-1}(4)$ has two answers $x = 2$ and $x = -2$. In questions 1 – 2 find the inverse function $x = f^{-1}(y)$.

1. $y = x^2 + 2$. This function fails the horizontal line test. It has no inverse. Its graph is a parabola opening upward, which is crossed twice by some horizontal lines (and not crossed at all by other lines).

   Here's another way to see why there is no inverse: $x^2 = y - 2$ leads to $x = \pm\sqrt{y-2}$. Then $x+ = \sqrt{y-2}$ represents the right half of the parabola, and $x = -\sqrt{y-2}$ is the left half. We can get an inverse by reducing the domain of $y = x^2 + 2$ to $x \geq 0$. With this restriction, $x = f^{-1}(y) = \sqrt{y-2}$. The positive square root is the inverse. The domain of $f(x)$ matches the range of $f^{-1}(y)$.

2. $y = f(x) = \frac{x}{x-1}$. (This is Problem 4.3.4) Find x as a function of y.

   - Write $y = \frac{x}{x-1}$ as $y(x-1) = x$ or $yx - y = x$. ***We always have to solve for** x.* We have $yx - x = y$ or $x(y-1) = y$ or $x = \frac{y}{y-1}$. Therefore $f^{-1}(y) = \frac{y}{y-1}$.

   Note that f and f^{-1} are the same! If you graph $y = f(x)$ and the line $y = x$ you will see that $f(x)$ is symmetric about the 45° line. In this unusual case, $x = f(y)$ when $y = f(x)$.

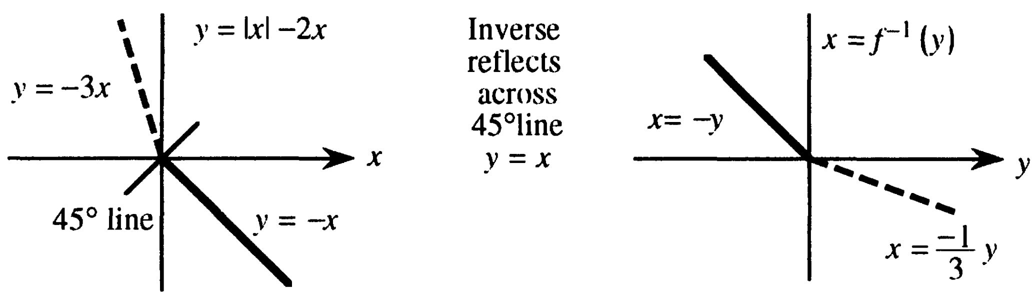
You might wonder at the statement that $f(x) = \frac{x}{x-1}$ is the same as $g(y) = \frac{y}{y-1}$. The definition of a function does not depend on the particular choice of letters. The functions $h(r) = \frac{r}{r-1}$ and $F(t) = \frac{t}{t-1}$ and $G(z) = \frac{z}{z-1}$ are also the same. To graph them, you would put r, t, or z on the horizontal axis–they are the input (domain) variables. Then $h(r)$, $F(t)$, $G(z)$ would be on the vertical axis as output variables.

The function $y = f(x) = 3x$ and its inverse $x = f^{-1}(y) = \frac{1}{3}y$ (***absolutely not*** $\frac{1}{3y}$) are graphed on page 167. For $f(x) = 3x$, the domain variable x is on the horizontal axis. For $f^{-1}(y) = \frac{1}{3}y$, the *domain variable for* f^{-1} is y.

This can be confusing since we are so accustomed to seeing x along the horizontal axis. The advantage of $f^{-1}(x) = \frac{1}{3}x$ is that it allows you to keep x on the horizontal and to stick with x for domain (input). The advantage of $f^{-1}(y) = \frac{1}{3}y$ is that it emphasizes: f takes x to y and f^{-1} takes y back to x.

3. (This is 4.3.34) Graph $y = |x| - 2x$ and its inverse on separate graphs.

   - $y = |x| - 2x$ should be analyzed in two parts: positive x and negative x. When $x \geq 0$ we have $|x| = x$. The function is $y = x - 2x = -x$. When x is negative we have $|x| = -x$. Then $y = -x - 2x = -3x$. Then $y = -x$ on the right of the y axis and $y = -3x$ on the left. Inverses $x = -y$ and $x = -\frac{y}{3}$. The second graph shows the inverse function.



4. Find $\frac{dx}{dy}$ when $y = x^2 + x$. Compare implicit differentiation with $\frac{1}{dy/dx}$.

   - The x derivative of $y = x^2 + x$ is $\frac{dy}{dx} = 2x + 1$. Therefore $\frac{dx}{dy} = \frac{1}{2x+1}$.
   - The y derivative of $y = x^2 + x$ is $1 = 2x\,\frac{dx}{dy} + \frac{dx}{dy} = (2x+1)\frac{dx}{dy}$. This also gives $\frac{dx}{dy} = \frac{1}{2x+1}$.
   - It might be desirable to know $\frac{dx}{dy}$ as a function of y, not x. In that case solve the quadratic equation $x^2 + x - y = 0$ to get $x = \frac{-1 \pm \sqrt{1+4y}}{2}$. Substitute this into $\frac{dx}{dy} = \frac{1}{2x+1} = \frac{\pm 1}{\sqrt{1+4y}}$.
   - Now we know $x = \frac{-1 \pm \sqrt{1+4y}}{2}$ (this is the inverse function). So we can directly compute $\frac{dx}{dy} = \pm\frac{1}{2} \cdot \frac{1}{2}(1+4y)^{-1/2} \cdot 4 = \frac{\pm 1}{\sqrt{1+4y}}$. Same answer four ways!

5. Find $\frac{dx}{dy}$ at $x = \pi$ for $y = \cos x + x^2$.

   $\frac{dy}{dx} = -\sin x + 2x$. Substitute $x = \pi$ to find $\frac{dy}{dx} = -\sin \pi + 2\pi = 2\pi$. Therefore $\frac{dx}{dy} = \frac{1}{2\pi}$.

Read-throughs and selected even-numbered solutions :

The functions $g(x) = x - 4$ and $f(y) = y + 4$ are **inverse** functions, because $f(g(x)) = \mathbf{x}$. Also $g(f(y)) = \mathbf{y}$. The notation is $f = g^{-1}$ and $g = \mathbf{f^{-1}}$. The composition **of f and** $\mathbf{f^{-1}}$ is the identity function. By definition

$x = g^{-1}(y)$ if and only if $y = \mathbf{g(x)}$. When y is in the range of g, it is in the **domain** of g^{-1}. Similarly x is in the **domain** of g when it is in the **range** of g^{-1}. If g has an inverse then $g(x_1) \neq g(x_2)$ at any two points. The function g must be steadily **increasing** or steadily **decreasing**.

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)(\mathbf{dg/dx}) = \mathbf{1}$. The slope of g^{-1} times the slope of g equals **1**. More directly $dx/dy = 1/(\mathbf{dy/dx})$. For $y = 2x+1$ and $x = \frac{1}{2}(y-1)$, the slopes are $dy/dx = \mathbf{2}$ and $dx/dy = \mathbf{\frac{1}{2}}$. For $y = x^2$ and $x = \sqrt{\mathbf{y}}$, the slopes are $dy/dx = \mathbf{2x}$ and $dx/dy = \mathbf{1/2\sqrt{y}}$. Substituting x^2 for y gives $dx/dy = \mathbf{1/2x}$. Then $(dx/dy)(dy/dx) = \mathbf{1}$.

The graph of $y = g(x)$ is also the graph of $x = \mathbf{g^{-1}(y)}$, but with x across and y up. For an ordinary graph of g^{-1}, take the reflection in the line $y = x$. If (3,8) is on the graph of g, then its mirror image **(8,3)** is on the graph of g^{-1}. Those particular points satisfy $8 = 2^3$ and $3 = \log_{\mathbf{2}} \mathbf{8}$.

The inverse of the chain $z = h(g(x))$ is the chain $x = \mathbf{g^{-1}(h^{-1}(z))}$. If $g(x) = 3x$ and $h(y) = y^3$ then $z = \mathbf{(3x)^3} = \mathbf{27x^3}$. Its inverse is $x = \mathbf{\frac{1}{3}z^{1/3}}$, which is the composition of $\mathbf{g^{-1}(y)} = \mathbf{\frac{1}{3}y}$ and $\mathbf{h^{-1}(z)} = \mathbf{z^{1/3}}$.

4 $x = \frac{y}{y-1}$ (f^{-1} matches f)

14 f^{-1} does not exist because $f(3)$ is the same as $f(5)$.

16 No two x's give the same y. **22** $\frac{dy}{dx} = -\frac{1}{(x-1)^2}$; $\frac{dx}{dy} = -\mathbf{\frac{1}{y^2}} = -(x-1)^2$.

44 **First proof** Suppose $y = f(x)$. We are given that $y > x$. This is the same as $y > f^{-1}(y)$.

Second proof The graph of $f(x)$ is above the 45° line, because $f(x) > x$. The mirror image is below the 45° line so $f^{-1}(y) < y$.

48 $g(x) = x+6, f(y) = y^3, g^{-1}(y) = y-6, f^{-1}(z) = \sqrt[3]{z}; \mathbf{x} = \sqrt[3]{z} - \mathbf{6}$

4.4 Inverses of Trigonometric Functions (page 175)

The table on page 175 summarizes what you need to know – the six inverse trig functions, their domains, and their derivatives. The table gives you $\frac{dx}{dy}$ since the inverse functions have input y and output x. The input y is a *number* and the output x is an *angle*. Watch the restrictions on y and x (to permit an inverse).

1. Compute (a) $\sin^{-1}(\sin\frac{\pi}{4})$ (b) $\cos^{-1}(\sin\frac{\pi}{3})$ (c) $\sin^{-1}(\sin\pi)$ (d) $\tan^{-1}(\cos 0)$ (e) $\cos^{-1}(\cos(-\frac{\pi}{2}))$

- (a) $\sin\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$ and $\sin^{-1}\frac{\sqrt{2}}{2}$ brings us back to $\frac{\pi}{4}$.
- (b) $\sin\frac{\pi}{3} = \frac{1}{2}$ and then $\cos^{-1}(\frac{1}{2}) = +\frac{2\pi}{3}$. Note that $\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{2}$. The angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are complementary (they add to 90° or $\frac{\pi}{2}$). Always $\sin^{-1}y + \cos^{-1}y = \frac{\pi}{2}$.
- (c) $\sin^{-1}(\sin\pi)$ *is not* π! Certainly $\sin\pi = 0$. But $\sin^{-1}(0) = 0$. The $\sin^{-1}$ function or arcsin function only yields angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- (d) $\tan^{-1}(\cos 0) = \tan^{-1} 1 = \frac{\pi}{4}$
- (e) $\cos^{-1}(\cos(-\frac{\pi}{2}))$ looks like $-\frac{\pi}{2}$. But $\cos(-\frac{\pi}{2}) = 0$ and then $\cos^{-1}(0) = \frac{\pi}{2}$.

2. Find $\frac{dx}{dy}$ if $x = \sin^{-1} 3y$. What are the restrictions on y?

   We know that $x = \sin^{-1} u$ yields $\frac{dx}{du} = \frac{1}{\sqrt{1-u^2}}$. Set $u = 3y$ and use the chain rule: $\frac{dx}{du}\frac{du}{dy} = \frac{3}{\sqrt{1-u^2}} = \frac{3}{\sqrt{1-9y^2}}$. The restriction $|u| \leq 1$ on sines means that $|3y| \leq 1$ and $|y| \leq \frac{1}{3}$.

3. Find $\frac{dz}{dx}$ when $z = \cos^{-1}(\frac{1}{x})$. What are the restrictions on x?

   $\cos^{-1}$ accepts inputs between -1 and 1, inclusive. For this reason $|\frac{1}{x}| \leq 1$ and $|x| \geq 1$. To find the derivative, use the chain rule with $z = \cos^{-1} u$ and $u = \frac{1}{x}$:

   $$\frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{-1}{x^2} = \frac{1}{x\sqrt{x^2 - x^2u^2}} = \frac{1}{x\sqrt{x^2-1}}.$$

4. Find $\frac{dy}{dx}$ when $y = \sec^{-1}\sqrt{x^2+1}$. (This is Problem 4.4.23)

   - The derivative of $y = \sec^{-1} u$ is $\frac{1}{|u|\sqrt{u^2-1}}$. In this problem $u = \sqrt{x^2+1}$. Then

     $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{x}{\sqrt{x^2+1}} = (\text{substitute for } u) = \frac{x}{(x^2+1)|x|} = \pm\frac{1}{x^2+1}.$$

     Here is another way to do this problem. Since $y = \sec^{-1}\sqrt{x^2+1}$, we have $\sec y = \sqrt{x^2+1}$ and $\sec^2 y = x^2+1$. *This is a trig identity provided* $x = \pm\tan y$. Then $y = \pm\tan^{-1} x$ and $\frac{dy}{dx} = \pm\frac{1}{x^2+1}$.

5. Find $\frac{dy}{dx}$ if $y = \tan^{-1}\frac{2}{x} - \cot^{-1}\frac{x}{2}$. Explain zero.

   - The derivative of $\tan^{-1}\frac{2}{x}$ is $\frac{1}{1+(\frac{2}{x})^2} \cdot \frac{-2}{x^2} = \frac{-2}{x^2+4}$. The derivative of $\cot^{-1}\frac{x}{2}$ is $-\frac{1}{1+(\frac{x}{2})^2} \cdot \frac{1}{2} = -\frac{2}{x^2+4}$. By subtraction $\frac{dy}{dx} = 0$. *Why do* $\tan^{-1}\frac{2}{x}$ *and* $\cot^{-1}\frac{x}{2}$ *have the same derivative? Are they equal?* Think about domain and range before you answer that one.

The relation $x = \sin^{-1} y$ means that $\mathbf{y}$ is the sine of $\mathbf{x}$. Thus x is the angle whose sine is $\mathbf{y}$. The number y lies between $\mathbf{-1}$ and $\mathbf{1}$. The angle x lies between $-\pi/\mathbf{2}$ and $\pi/\mathbf{2}$. (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle $\sin^{-1} y$ is $\sqrt{\mathbf{1-y^2}}$. The derivative of $x = \sin^{-1} y$ is $dx/dy = \mathbf{1}/\sqrt{\mathbf{1-y^2}}$.

The relation $x = \cos^{-1} y$ means that y equals $\mathbf{\cos x}$. Again the number y lies between $\mathbf{-1}$ and $\mathbf{1}$. This time the angle x lies between $\mathbf{0}$ and π (so that each y comes from only one angle x). The sum $\sin^{-1} y + \cos^{-1} y = \pi/\mathbf{2}$. (The angles are called **complementary**, and they add to a **right** angle.) Therefore the derivative of $x = \cos^{-1} y$ is $dx/dy = \mathbf{-1}/\sqrt{\mathbf{1-y^2}}$, the same as for $\sin^{-1} y$ except for a **minus** sign.

The relation $x = \tan^{-1} y$ means that $y = \mathbf{\tan x}$. The number y lies between $-\infty$ and ∞. The angle x lies between $-\pi/\mathbf{2}$ and $\pi/\mathbf{2}$. The derivative is $dx/dy = \mathbf{1/(1+y^2)}$. Since $\tan^{-1} y + \cot^{-1} y = \pi/\mathbf{2}$, the derivative of $\cot^{-1} y$ is the same except for a **minus** sign.

The relation $x = \sec^{-1} y$ means that $\mathbf{y} = \mathbf{\sec x}$. The number y *never* lies between $\mathbf{-1}$ and $\mathbf{1}$. The angle x lies between $\mathbf{0}$ and π, but never at $x = \pi/\mathbf{2}$. The derivative of $x = \sec^{-1} y$ is $dx/dy = \mathbf{1}/\mid \mathbf{y} \mid \sqrt{\mathbf{y^2-1}}$.

10 The sides of the triangle are $y, \sqrt{1-y^2}$, and 1. The tangent is $\frac{\mathbf{y}}{\sqrt{\mathbf{1-y^2}}}$.

14 $\frac{d(\sin^{-1} y)}{dy}|_{x=0} = \mathbf{1}$; $\frac{d(\cos^{-1} y)}{dy}|_{x=0} = -\infty$; $\frac{d(\tan^{-1} y)}{dy}|_{x=0} = \mathbf{1}$; $\frac{d(\sin^{-1} y)}{dy}|_{x=1} = \frac{\mathbf{1}}{\mathbf{\cos 1}}$; $\frac{d(\cos^{-1} y)}{dy}|_{x=1}$ $= -\frac{\mathbf{1}}{\mathbf{\sin 1}}$; $\frac{d(\tan^{-1} y)}{dy}|_{x=1} = \frac{\mathbf{1}}{\mathbf{\sec^2 1}}$.

16 $\cos^{-1}(\sin x)$ is the complementary angle $\frac{\pi}{2} - x$. The tangent of that angle is $\frac{\mathbf{\cos x}}{\mathbf{\sin x}} = \mathbf{\cot x}$.

34 The requirement is $u' = \frac{1}{1+t^2}$. To satisfy this requirement take $u = \mathbf{\tan^{-1} t}$.

36 $u = \tan^{-1} y$ has $\frac{du}{dy} = \frac{1}{1+y^2}$ and $\frac{d^2u}{dy^2} = \frac{\mathbf{-2y}}{\mathbf{(1+y^2)^2}}$.

42 By the product rule $\frac{dz}{dx} = (\cos x)(\sin^{-1} x) + (\sin x)\frac{1}{\sqrt{1-x^2}}$. Note that $z \neq x$ and $\frac{dz}{dx} \neq 1$.

48 $u(x) = \mathbf{\frac{1}{2}\tan^{-1} 2x}$ (need $\frac{1}{2}$ to cancel 2 from the chain rule).

50 $u(x) = \frac{x-1}{x+1}$ has $\frac{du}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. Then $\frac{d}{dx}\tan^{-1} u(x) = \frac{1}{1+u^2}\frac{du}{dx} = \frac{1}{1+(\frac{x-1}{x+1})^2}\frac{2}{(x+1)^2} =$ $\frac{2}{(x+1)^2+(x-1)^2} = \frac{\mathbf{1}}{\mathbf{x^2+1}}$. This is also the derivative of $\mathbf{\tan^{-1} x}$! So $\tan^{-1} u(x) - \tan^{-1} x$ is a **constant**.

4 Chapter Review Problems

Review Problems

R1 Give the domain and range of the six inverse trigonometric functions.

R2 Is the derivative of $u(v(x))$ ever equal to the derivative of $u(x)v(x)$?

R3 Find y' and the second derivative y'' by implicit differentiation when $y^2 = x^2 + xy$.

R4 Show that $y = x + 1$ is the tangent line to the graph of $y = x + \cos xy$ through the point (0,1).

R5 If the graph of $y = f(x)$ passes through the point (a, b) with slope m, then the graph of $y = f^{-1}(x)$ passes through the point ____ with slope ____ .

R6 Where does the graph of $y = \cos x$ intersect the graph of $y = \cos^{-1} x$? Give an equation for x and show that $x = .7391$ in Section 3.6 is a solution.

R7 Show that the curves $xy = 4$ and $x^2 - y^2 = 15$ intersect at right angles.

R8 "The curve $y^2 + x^2 + 1 = 0$ has $2y\frac{dy}{dx} + 2x = 0$ so its slope is $-x/y$." What is the problem with that statement?

R9 Gas is escaping from a spherical balloon at 2 cubic feet/minute. How fast is the surface area shrinking when the area is 576π square feet?

R10 A 50 foot rope goes up over a pulley 18 feet high and diagonally down to a truck. The truck drives away at 9 ft/sec. How fast is the other end of the rope rising from the ground?

R11 Two concentric circles are expanding, the outer radius at 2 cm/sec and the inner radius at 5 cm/sec. When the radii are 10 cm and 3 cm, how fast is the area between them increasing (or decreasing)?

R12 A swimming pool is 25 feet wide and 100 feet long. The bottom slopes steadily down from a depth of 3 feet to 10 feet. The pool is being filled at 100 cubic feet/minute. How fast is the water level rising when it is 6 feet deep at the deep end?

R13 A five-foot woman walks at night toward a 12-foot street lamp. Her speed is 4 ft/sec. Show that her shadow is shortening by $\frac{20}{7}$ ft/sec when she is 3 feet from the lamp.

R14 A 40 inch string goes around an 8 by 12 rectangle – but we are changing its shape (same string). If the 8 inch sides are being lengthened by 1 inch/second, how fast are the 12 inch sides being shortened? Show that the area is increasing at 4 square inches per second. (For some reason it will take *two* seconds before the area increases from 96 to 100.)

R15 The volume of a sphere (when we know the radius) is $V(r) = 4\pi r^3/3$. The radius of a sphere (when we know the volume) is $r(V) = (3V/4\pi)^{1/3}$. This is the inverse! The surface area of a sphere is $A(r) = 4\pi r^2$. The radius (when we know the area) is $r(A) =$ ____ . The chain $r(A(r))$ equals ____ .

R16 The surface area of a sphere (when we know the volume) is $A(V) = 4\pi(3V/4\pi)^{2/3}$. The volume (when we know the area) is $V(A) =$ ____ .

Drill Problems (Find dy/dx in Problems **D1** to **D6**).

D1 $y = t^3 - t^2 + 2$ with $t = \sqrt{x}$

D2 $y = \sin^3(2x - \pi)$

D3 $y = \tan^{-1}(4x^2 + 7x)$

D4 $y = \csc\sqrt{x}$

D5 $y = \sin(\sin^{-1} x)$ for $|x| \le 1$

D6 $y = \sin u \cos u$ with $u = \cos^{-1} x$

In **D7** to **D10** find y' by implicit differentiation.

D7 $x^2 - 2xy + y^2 = 4$

D8 $y = \sin(xy) + x$

D9 $9x^2 + 16y^2 = 144$

D10 $9y - 6x + y^4 = 0$

D11 The area of a circle is $A(r) = \pi r^2$. Find the radius r when you know the area A. (This is the inverse function $r(A)$!). The derivative of $A = \pi r^2$ is $dA/dr = 2\pi r$. Find dr/dA.