CHAPTER 14

Multiple Integrals

14.1 Double Integrals

This chapter shows how to integrate functions of two or more variables. First, a double integral is defined as the limit of sums. Second, we find a fast way to compute it. ***The key idea is to replace a double integral by two ordinary "single" integrals***.

The double integral $\iint f(x,y)dy\,dx$ starts with $\int f(x,y)dy$. For each fixed x we integrate with respect to y. The answer depends on x. Now integrate again, this time with respect to x. The limits of integration need care and attention! Frequently those limits on y and x are the hardest part.

Why bother with sums and limits in the first place? Two reasons. There has to be a definition and a computation to fall back on, when the single integrals are difficult or impossible. And also—this we emphasize—*multiple integrals represent more than area and volume*. Those words and the pictures that go with them are the easiest to understand. You can almost see the volume as a "sum of slices" or a "double sum of thin sticks." The true applications are mostly to other things, but the central idea is always the same: ***Add up small pieces and take limits***.

We begin with the area of R and the volume of V, by double integrals.

A LIMIT OF SUMS

The graph of $z = f(x,y)$ is a curved surface above the xy plane. At the point (x,y) in the plane, the height of the surface is z. (The surface is *above* the xy plane only when z is positive. Volumes below the plane come with minus signs, like areas below the x axis.) We begin by choosing a positive function—for example $z = 1 + x^2 + y^2$.

The base of our solid is a region R in the xy plane. That region will be chopped into small rectangles (sides Δx and Δy). When R itself is the rectangle $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$, the small pieces fit perfectly. For a triangle or a circle, the rectangles miss part of R. But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

Question What is the volume above R and below the graph of $z = f(x,y)$?
Answer It is a double integral—***the integral of*** $f(x,y)$ ***over*** R. To reach it we begin with a sum, as suggested by Figure 14.1.

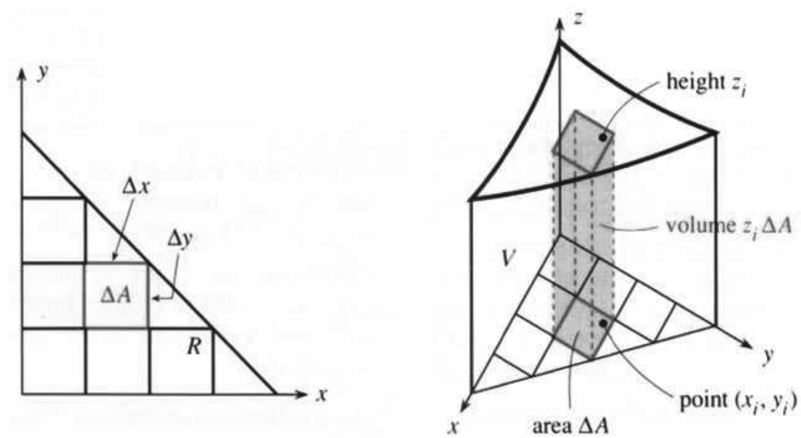

Fig. 14.1 Base R cut into small pieces ΔA. Solid V cut into thin sticks $\Delta V = z\Delta A$.

For single integrals, the interval $[a,b]$ is divided into short pieces of length Δx. For double integrals, R is divided into small rectangles of area $\Delta A = (\Delta x)(\Delta y)$. Above the ith rectangle is a "thin stick" with small volume. That volume is the base area ΔA times the height above it—except that this height $z = f(x,y)$ varies from point to point. Therefore we select a point (x_i, y_i) in the ith rectangle, and compute the volume from the height above that point:

$$\textit{volume of one stick} = f(x_i, y_i)\Delta A \quad \textit{volume of all sticks} = \sum f(x_i, y_i)\Delta A.$$

This is the crucial step for any integral—to see it as a sum of small pieces.

Now take limits: $\Delta x \to 0$ and $\Delta y \to 0$. The height $z = f(x,y)$ is nearly constant over each rectangle. (We assume that f is a continuous function.) The sum approaches a limit, which depends only on the base R and the surface above it. The limit is the volume of the solid, and it is the ***double integral*** of $f(x,y)$ over R:

$$\iint_R f(x,y)\,dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum f(x_i, y_i)\Delta A. \tag{1}$$

To repeat: The limit is the same for all choices of the rectangles and the points (x_i, y_i). The rectangles will not fit exactly into R, if that base area is curved. The heights are not exact, if the surface $z = f(x,y)$ is also curved. But the errors on the sides and top, where the pieces don't fit and the heights are wrong, approach zero. Those errors are the volume of the "icing" around the solid, which gets thinner as $\Delta x \to 0$ and $\Delta y \to 0$. A careful proof takes more space than we are willing to give. But the properties of the integral need and deserve attention:

1. Linearity: $\iint (f+g)\,dA = \iint f\,dA + \iint g\,dA$
2. Constant comes outside: $\iint c f(x,y)\,dA = c\iint f(x,y)\,dA$
3. R splits into S and T (not overlapping): $\iint\limits_R f\,dA = \iint\limits_S f\,dA + \iint\limits_T f\,dA$.

In **1** the volume under $f+g$ has two parts. The "thin sticks" of height $f+g$ split into thin sticks under f and under g. In **2** the whole volume is stretched upward by c. In **3** the volumes are side by side. As with single integrals, these properties help in computations.

By writing dA, we allow shapes other than rectangles. Polar coordinates have an extra factor r in $dA = r\,dr\,d\theta$. By writing $dx\,dy$, we choose rectangular coordinates and prepare for the splitting that comes now.

SPLITTING A DOUBLE INTEGRAL INTO TWO SINGLE INTEGRALS

The double integral $\iint f(x,y)dy\,dx$ will now be reduced to single integrals in y and then x. (Or vice versa. Our first integral could equally well be $\int f(x,y)dx$.) Chapter 8 described the same idea for solids of revolution. First came the area of a slice, which is a single integral. Then came a second integral to add up the slices. For solids formed by revolving a curve, all slices are circular disks—now we expect other shapes.

Figure 14.2 shows a slice of area $A(x)$. It cuts through the solid at a fixed value of x. The cut starts at $y = c$ on one side of R, and ends at $y = d$ on the other side. This particular example goes from $y = 0$ to $y = 2$ (R is a rectangle). The area of a slice is the y integral of $f(x,y)$. *Remember that x is fixed and y goes from c to d*:

$$A(x) = \textbf{\textit{area of slice}} = \int_c^d f(x,y)dy \quad \text{(the answer is a function of } x\text{)}.$$

EXAMPLE 1 $\quad A = \displaystyle\int_{y=0}^{2} (1+x^2+y^2)dy = \left[y + x^2y + \frac{y^3}{3}\right]_{y=0}^{y=2} = 2 + 2x^2 + \frac{8}{3}.$

This is the reverse of a partial derivative! The integral of x^2dy, with x constant, is x^2y. This "partial integral" is actually called an ***inner integral***. After substituting the limits $y = 2$ and $y = 0$ and subtracting, we have the area $A(x) = 2 + 2x^2 + \frac{8}{3}$. Now the ***outer integral*** adds slices to find the volume $\int A(x)\,dx$. The answer is a *number*:

$$\text{volume} = \int_{x=0}^{1}\left(2 + 2x^2 + \frac{8}{3}\right)dx = \left[2x + \frac{2x^3}{3} + \frac{8}{3}x\right]_0^1 = 2 + \frac{2}{3} + \frac{8}{3} = \frac{16}{3}.$$

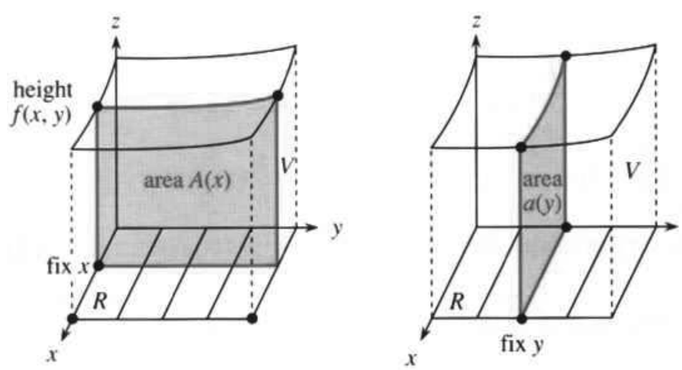

Fig. 14.2 A slice of V at a fixed x has area $A(x) = \int f(x,y)dy$.

To complete this example, check the volume when the x integral comes first:

$$\text{inner integral} = \int_{x=0}^{1}(1+x^2+y^2)dx = \left[x + \frac{1}{3}x^3 + y^2x\right]_{x=0}^{x=1} = \frac{4}{3} + y^2$$

$$\text{outer integral} = \int_{y=0}^{2}\left(\frac{4}{3} + y^2\right)dy = \left[\frac{4}{3}y + \frac{1}{3}y^3\right]_{y=0}^{y=2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}.$$

The fact that double integrals can be split into single integrals is *Fubini's Theorem*.

14A if $f(x,y)$ is continuous on the rectangle R, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy. \quad (2)$$

The inner integrals are the cross-sectional areas $A(x)$ and $a(y)$ of the slices. The outer integrals add up the volumes $A(x)dx$ and $a(y)dy$. Notice the reversing of limits.

Normally the brackets in (2) are omitted. When the y integral is first, dy is written inside dx. ***The limits on y are inside too***. I strongly recommend that you compute the inner integral on one line and the outer integral on a *separate line*.

EXAMPLE 2 Find the volume below the plane $z = x - 2y$ and above the base triangle R.

The triangle R has sides on the x and y axes and the line $x + y = 1$. The strips in the y direction have varying lengths. (So do the strips in the x direction.) This is the main point of the example—the base is not a rectangle. The upper limit on the inner integral changes as x changes. ***The top of the triangle is at $y = 1 - x$***.

Figure 14.3 shows the strips. The region should always be drawn (except for rectangles). Without a figure the limits are hard to find. A sketch of R makes it easy:

y goes from $c = 0$ to $d = 1 - x$. Then x goes from $a = 0$ to $b = 1$.

The inner integral has *variable limits* and the outer integral has *constant limits*:

$$\text{inner: } \int_{y=0}^{y=1-x} (x-2y)\,dy = \Big[xy - y^2\Big]_{y=0}^{y=1-x} = x(1-x) - (1-x)^2 = -1 + 3x - 2x^2$$

$$\text{outer: } \int_{x=0}^{1} (-1+3x-2x^2)\,dx = \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3\right]_0^1 = -1 + \frac{3}{2} - \frac{2}{3} = -\frac{1}{6}.$$

The volume is negative. Most of the solid is below the xy plane. To check the answer $-\frac{1}{6}$, do the x integral first: *x goes from 0 to $1-y$. Then y goes from 0 to 1.*

$$\text{inner: } \int_{x=0}^{1-y} (x-2y)\,dx = \left[\frac{1}{2}x^2 - 2xy\right]_0^{1-y} = \frac{1}{2}(1-y)^2 - 2(1-y)y = \frac{1}{2} - 3y + \frac{5}{2}y^2$$

$$\text{outer: } \int_{y=0}^{1} \left(\frac{1}{2} - 3y + \frac{5}{2}y^2\right) dy = \left[\frac{1}{2}y - \frac{3}{2}y^2 + \frac{5}{6}y^3\right]_0^1 = \frac{1}{2} - \frac{3}{1} + \frac{5}{6} = -\frac{1}{6}.$$

Same answer, very probably right. The next example computes $\iint 1\,dx\,dy =$ area of R.

EXAMPLE 3 The area of R is $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} dy\,dx$ and also $\displaystyle\int_{y=0}^{1}\int_{x=0}^{1-y} dx\,dy$.

The first has vertical strips. The inner integral equals $1 - x$. Then the outer integral (of $1 - x$) has limits 0 and 1, and the area is $\frac{1}{2}$. It is like an indefinite integral inside a definite integral.

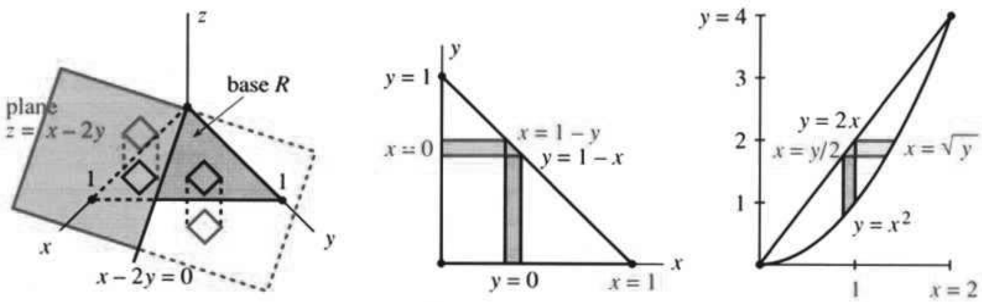

Fig. 14.3 Thin sticks above and below (Example 2). Reversed order (Examples 3 and 4).

EXAMPLE 4 Reverse the order of integration in $\displaystyle\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3 dy\,dx$.

Solution Draw a figure! The inner integral goes from the parabola $y = x^2$ up to the straight line $y = 2x$. This gives vertical strips. The strips sit side by side between $x = 0$ and $x = 2$. They stop where $2x$ equals x^2, and the line meets the parabola.

The problem is to put the x integral first. It goes along horizontal strips. On each line $y =$ constant, we need the *entry value* of x and the *exit value* of x. From the figure, x goes from $\frac{1}{2}y$ to $\sqrt{y}$. Those are the inner limits. Pay attention also to the outer limits, because they now apply to y. The region starts at $y = 0$ and ends at $y = 4$. *No change in the integrand* x^3—that is the height of the solid:

$$\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3 dy\,dx \quad \textit{is reversed to} \quad \int_{y=0}^{4}\int_{x=\frac{1}{2}y}^{\sqrt{y}} x^3 dx\,dy. \tag{3}$$

EXAMPLE 5 Find the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.

Solution The solid is a tetrahedron (four sides). It goes from $z = 0$ (the xy plane) up to the plane $2x + y + z = 4$. On that plane $z = 4 - 2x - y$. This is the height function $f(x, y)$ to be integrated.

Figure 14.4 shows the base R. To find its sides, set $z = 0$. The sides of R are the lines $x = 0$ and $y = 0$ and $2x + y = 4$. Taking vertical strips, dy is inner:

$$\text{inner:} \quad \int_{y=0}^{4-2x}(4-2x-y)dy = \left[(4-2x)y - \frac{1}{2}y^2\right]_0^{4-2x} = \frac{1}{2}(4-2x)^2$$

$$\text{outer:} \quad \int_{x=0}^{2}\frac{1}{2}(4-2x)^2 dx = \left[-\frac{(4-2x)^3}{2\cdot 3\cdot 2}\right]_0^2 = \frac{4^3}{2\cdot 3\cdot 2} = \frac{16}{3}.$$

Question What is the meaning of the inner integral $\frac{1}{2}(4-2x)^2$ $\left(\text{and also}\frac{16}{3}\right)$?

Answer The first is $A(x)$, the area of the slice. $\frac{16}{3}$ is the solid volume.

Question What if the inner integral $\int f(x, y)dy$ has limits that depend on y?
Answer It can't. Those limits must be wrong. Find them again.

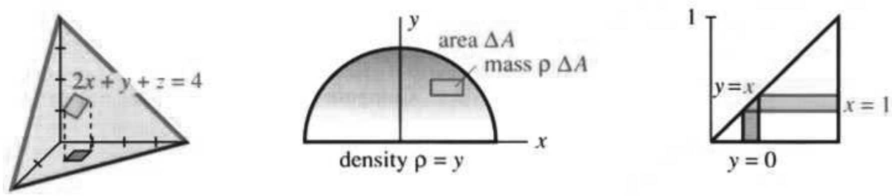

Fig. 14.4 Tetrahedron in Example 5, semicircle in Example 6, triangle in Example 7.

EXAMPLE 6 Find the mass in a semicircle $0 \leqslant y \leqslant \sqrt{1-x^2}$ if the density is $\rho = y$.

This is a new application of double integrals. The total mass is a sum of small masses (ρ times ΔA) in rectangles of area ΔA. The rectangles don't fit perfectly inside the semicircle R, and the density is not constant in each rectangle—but those problems disappear in the limit. We are left with a double integral:

$$\textbf{\textit{total mass}}\ M = \iint_R \rho\, dA = \iint_R \rho(x,y)\, dx\, dy. \tag{4}$$

Set $\rho = y$. Figure 14.4 shows the limits on x and y (try both $dy\,dx$ and $dx\,dy$):

$$\text{mass } M = \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} y\, dy\, dx \quad \text{and also} \quad M = \int_{y=0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx\, dy.$$

The first inner integral is $\frac{1}{2}y^2$. Substituting the limits gives $\frac{1}{2}(1-x^2)$. The outer integral of $\frac{1}{2}(1-x^2)$ yields the total mass $M = \frac{2}{3}$.

The second inner integral is xy. Substituting the limits on x gives _____ . Then the outer integral is $-\frac{2}{3}(1-y^2)^{3/2}$. Substituting $y = 1$ and $y = 0$ yields $M =$ _____ .

Remark This same calculation also produces the *moment* around the x axis, when the density is $\rho = 1$. The factor y is the distance to the x axis. ***The moment is*** $M_x = \iint y\, dA = \frac{2}{3}$. Dividing by the area of the semicircle (which is $\pi/2$) locates the centroid: $\overline{x} = 0$ by symmetry and

$$\overline{y} = \text{height of centroid} = \frac{\text{moment}}{\text{area}} = \frac{2/3}{\pi/2} = \frac{4}{3\pi}. \tag{5}$$

This is the "average height" of points inside the semicircle, found earlier in 8.5.

EXAMPLE 7 Integrate $\int_{y=0}^{y=1}\int_{x=y}^{x=1} \cos\ x^2 dx\, dy$ avoiding the impossible $\int \cos\ x^2\, dx$.

This is a famous example where reversing the order makes the calculation possible. The base R is the triangle in Figure 14.4 (note that x goes from y to 1). ***In the opposite order*** y ***goes from*** 0 ***to*** x. Then $\int \cos x^2 dy = x \cos x^2$ contains the factor x that we need:

$$\text{outer integral: } \int_0^1 x \cos x^2 dx = \left[\tfrac{1}{2}\sin x^2\right]_0^1 = \tfrac{1}{2}\sin 1.$$

14.1 EXERCISES

Read-through questions

The double integral $\iint_R f(x,y)dA$ gives the volume between R and __a__. The base is first cut into small __b__ of area ΔA. The volume above the ith piece is approximately __c__. The limit of the sum __d__ is the volume integral. Three properties of double integrals are __e__ (linearity) and __f__ and __g__.

If R is the rectangle $0 \leqslant x \leqslant 4, 4 \leqslant y \leqslant 6$, the integral $\iint x\,dA$ can be computed two ways. One is $\iint x\,dy\,dx$, when the inner integral is __h__$]_4^6 =$ __i__. The outer integral gives __j__$]_0^4 =$ __k__. When the x integral comes first it equals $\int x\,dx =$ __l__$]_0^4 =$ __m__. Then the y integral equals __n__. This is the volume between __o__ (describe V).

The area of R is $\iint$ __p__ $dy\,dx$. When R is the triangle between $x=0, y=2x$, and $y=1$, the inner limits on y are __q__. This is the length of a __r__ strip. The (outer) limits on x are __s__. The area is __t__. In the opposite order, the (inner) limits on x are __u__. Now the strip is __v__ and the outer integral is __w__. When the density is $\rho(x,y)$, the total mass in the region R is $\iint$ __x__. The moments are $M_y =$ __y__ and $M_x =$ __z__. The centroid has $\bar{x} = M_y/M$.

Compute the double integrals 1–4 by two integrations.

1 $\int_{y=0}^{1}\int_{x=0}^{2} x^2 dx\,dy$ and $\int_{y=0}^{1}\int_{x=0}^{2} y^2 dx\,dy$

2 $\int_{y=2}^{2e}\int_{x=1}^{e} 2xy\,dx\,dy$ and $\int_{y=2}^{2e}\int_{x=1}^{e} dx\,dy/xy$

3 $\int_{0}^{\pi/2}\int_{0}^{\pi/4} \sin(x+y)\,dx\,dy$ and $\int_{1}^{2}\int_{0}^{2} dy\,dx/(x+y)^2$

4 $\int_{0}^{1}\int_{1}^{2} ye^{xy} dx\,dy$ and $\int_{-1}^{1}\int_{0}^{3} dy\,dx/\sqrt{3+2x+y}$

In 5–10, draw the region and compute the area.

5 $\int_{x=1}^{2}\int_{y=1}^{2x} dy\,dx$

6 $\int_{0}^{1}\int_{x^3}^{x} dy\,dx$

7 $\int_{0}^{\infty}\int_{e^{-2x}}^{e^{-x}} dy\,dx$

8 $\int_{-1}^{1}\int_{x^2-1}^{1-x^2} dy\,dx$

9 $\int_{-1}^{1}\int_{y^2}^{1} dx\,dy$

10 $\int_{-1}^{1}\int_{x=y}^{|y|} dx\,dy$

In 11–16 reverse the order of integration (and find the new limits) in 5–10 respectively.

In 17–24 find the limits on $\iint dy\,dx$ and $\iint dx\,dy$. Draw R and compute its area.

17 $R=$ triangle inside the lines $x=0, y=1, y=2x$.

18 $R=$ triangle inside the lines $x=-1, y=0, x+y=0$.

19 $R=$ triangle inside the lines $y=x, y=-x, y=3$.

20 $R=$ triangle inside the lines $y=x, y=2x, y=4$.

21 $R=$ triangle with vertices $(0,0),(4,4),(4,8)$.

22 $R=$ triangle with vertices $(0,0),(-2,-1),(1,-2)$.

23 $R=$ triangle with vertices $(0,0),(2,0),(1,b)$. Here $b>0$.

*__24__ $R=$ triangle with vertices $(0,0),(a,b),(c,d)$. The sides are $y=bx/a, y=dx/c$, and $y=b+(x-a)(d-b)/(c-a)$. Find $A=\iint dy\,dx$ when $0<a<c, 0<d<b$.

25 Evaluate $\int_0^b\int_0^a \partial^2 f/\partial x\partial y\,dx\,dy$.

26 Evaluate $\int_0^b\int_0^a \partial f/\partial x\,dx\,dy$.

In 27–28, divide the unit square R into triangles S and T and verify $\iint_R f\,dA = \iint_S f\,dA + \iint_T f\,dA$.

27 $f(x,y)=2x-3y+1$

28 $f(x,y)=xe^y-ye^x$

29 The area under $y=f(x)$ is a single integral from a to b or a double integral (*find the limits*):

$$\int_a^b f(x)\,dx = \iint 1\,dy\,dx.$$

30 Find the limits and the area under $y=1-x^2$:

$$\int (1-x^2)\,dx \text{ and } \iint 1\,dx\,dy \text{ (reversed from 29).}$$

31 A city inside the circle $x^2+y^2=100$ has population density $\rho(x,y)=10(100-x^2-y^2)$. Integrate to find its population.

32 Find the volume bounded by the planes $x=0, y=0, z=0$, and $ax+by+cz=1$.

In 33–34 the rectangle with corners $(1,1),(1,3),(2,1),(2,3)$ has density $\rho(x,y)=x^2$. The moments are $M_y=\iint x\rho\,dA$ and $M_x=\iint y\rho\,dA$.

33 Find the mass.

34 Find the center of mass.

In 35–36 the region is a circular wedge of radius 1 between the lines $y=x$ and $y=-x$.

35 Find the area.

36 Find the centroid $(\bar{x},\bar{y})$.

37 Write a program to compute $\int_0^1\int_0^1 f(x,y)dx\,dy$ by the midpoint rule (*midpoints of n^2 small squares*). Which $f(x,y)$ are integrated exactly by your program?

38 Apply the midpoint code to integrate x^2 and xy and y^2. The errors decrease like what power of $\Delta x=\Delta y=1/n$?

Use the program to compute the volume under $f(x,y)$ in 39–42. Check by integrating exactly or doubling n.

39 $f(x,y)=3x+4y+5$

40 $f(x,y)=1/\sqrt{x^2+y^2}$

41 $f(x,y)=x^y$

42 $f(x,y)=e^x\sin\pi y$

43 In which order is $\iint x^y \, dx \, dy = \iint x^y \, dy \, dx$ easier to integrate over the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$? By reversing order, integrate $(x-1)/\ln x$ from 0 to 1—its antiderivative is unknown.

44 Explain in your own words the definition of the double integral of $f(x, y)$ over the region R.

45 $\sum y_i \Delta A$ might not approach $\iint y \, dA$ if we only know that $\Delta A \to 0$. In the square $0 \leqslant x, y \leqslant 1$, take rectangles of sides Δx and 1 (not Δx and Δy). If (x_i, y_i) is a point in the rectangle where $y_i = 1$, then $\sum y_i \Delta A =$ ______. But $\iint y \, dA =$ ______.

14.2 Change to Better Coordinates

You don't go far with double integrals before wanting to ***change variables***. Many regions simply do not fit with the x and y axes. Two examples are in Figure 14.5, a tilted square and a ring. Those are excellent shapes—***in the right coordinates***.

We have to be able to answer basic questions like these:

$$\text{Find the area} \iint dA \text{ and moment} \iint x\, dA \text{ and moment of inertia} \int\}$$

The problem is: *What is* dA? We are leaving the xy variables where $dA = dx\,dy$.

The reason for changing is this: The limits of integration in the y direction are miserable. I don't know them and I don't want to know them. For every x we would need the entry point P of the line $x =$ constant, and the exit point Q. The heights of P and Q are the limits on $\int dy$, the inner integral. The geometry of the square and ring are totally missed, if we stick rigidly to x and y.

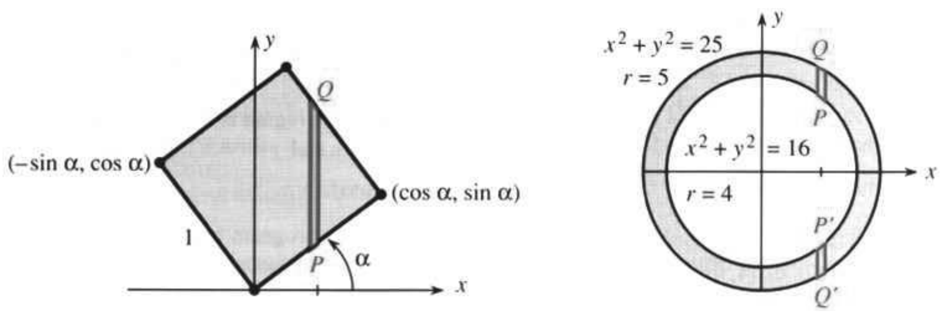

Fig. 14.5 Unit square turned through angle α. Ring with radii 4 and 5.

Which coordinates are better? Any sensible person agrees that the area of the tilted square is 1. "Just turn it and the area is obvious." But that sensible person may not know the moment or the center of gravity or the moment of inertia. So we actually have to do the turning.

The new coordinates u and v are in Figure 14.6a. The limits of integration on v are 0 and 1. So are the limits on u. *But when you change variables, you don't just change limits*. Two other changes come with new variables:

1. The small area $dA = dx\,dy$ becomes $dA =$ _____ $du\,dv$.
2. The integral of x becomes the integral of _____ .

Substituting $u = \sqrt{x}$ in a single integral, we make the same changes. Limits $x = 0$ and $x = 4$ become $u = 0$ and $u = 2$. Since x is u^2, dx is $2u\,du$. The purpose of the change is to find an antiderivative. For double integrals, the usual purpose is to improve the limits—but we have to accept the whole package.

To turn the square, there are formulas connecting x and y to u and v. The geometry is clear—***rotate axes by*** α—but it has to be converted into algebra:

$$\begin{aligned} u &= x\cos\alpha + y\sin\alpha \\ v &= -x\sin\alpha + y\cos\alpha \end{aligned} \quad \text{and in reverse} \quad \begin{aligned} x &= u\cos\alpha - v\sin\alpha \\ y &= u\sin\alpha + v\cos\alpha. \end{aligned} \qquad (1)$$

Figure 14.6 shows the rotation. As points move, the whole square turns. A good way to remember equation (1) is to follow the corners as they become $(1, 0)$ and $(0, 1)$.

The change from $\iint x\,dA$ to $\iint$ _____ $du\,dv$ is partly decided by equation (1). It gives x as a function of u and v. We also need dA. For a pure rotation the first guess is correct: ***The area*** $dx\,dy$ ***equals the area*** $du\,dv$. ***For most changes of variable this is false***. The general formula for dA comes after the examples.

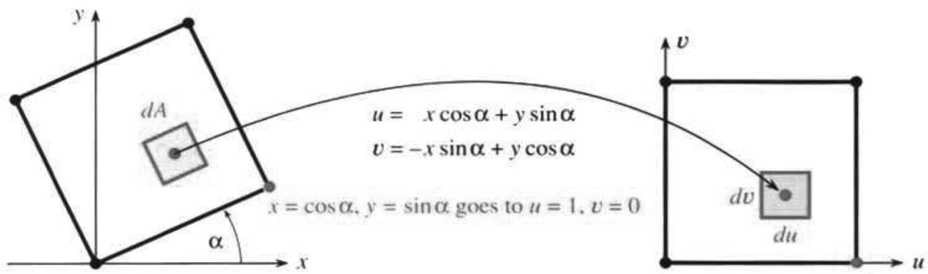

Fig. 14.6 Change of coordinates—axes turned by α. For rotation dA is $du\,dv$.

EXAMPLE 1 Find $\iint dA$ and $\iint x\,dA$ and $\overline{x}$ and also $\iint x^2\,dA$ for the tilted square.

Solution The area of the square is $\int_0^1\int_0^1 du\,dv = 1$. Notice the good limits. Then

$$\iint x\,dA = \int_0^1\int_0^1 (u\cos\alpha - v\sin\alpha)du\,dv = \tfrac{1}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha. \qquad (2)$$

This is the *moment around the y axis*. The factors $\frac{1}{2}$ come from $\frac{1}{2}u^2$ and $\frac{1}{2}v^2$. The x coordinate of the center of gravity is

$$\overline{x} = \iint x\,dA \Big/ \iint dA = \left(\tfrac{1}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha\right)/1.$$

Similarly the integral of y leads to $\overline{y}$. The answer is no mystery—the point $(\overline{x},\overline{y})$ is at the center of the square! Substituting $x = u\cos\alpha - v\sin\alpha$ made $x\,dA$ look worse, but the limits 0 and 1 are much better.

The moment of inertia I_y around the y axis is also simplified:

$$\iint x^2\,dA = \int_0^1\int_0^1 (u\cos\alpha - v\sin\alpha)^2 du\,dv = \frac{\cos^2\alpha}{3} - \frac{\cos\alpha\,\sin\alpha}{2} + \frac{\sin^2\alpha}{3}. \qquad (3)$$

You know this next fact but I will write it anyway: *The answers don't contain u or v*. Those are dummy variables like x and y. The answers do contain α, because the square has turned. (The area is fixed at 1.) The moment of inertia $I_x = \iint y^2\,dA$ is the same as equation (3) but with all plus signs.

Question The sum $I_x + I_y$ simplifies to $\frac{2}{3}$ (a constant). Why no dependence on α?
Answer $I_x + I_y$ equals I_0. This moment of inertia around $(0,0)$ is unchanged by rotation. We are turning the square around one of its corners.

CHANGE TO POLAR COORDINATES

The next change is to r and θ. A small area becomes $dA = r\,dr\,d\theta$ (definitely not $dr\,d\theta$). Area always comes from multiplying two lengths, and $d\theta$ is not a length. Figure 14.7 shows the crucial region—a "polar rectangle" cut out by rays and circles. Its area ΔA is found in two ways, both leading to $r\,dr\,d\theta$:

(***Approximate***) The straight sides have length Δr. The circular arcs are *close to* $r\Delta\theta$. The angles are 90°. So ΔA is close to $(\Delta r)(r\Delta\theta)$.

(***Exact***) A wedge has area $\frac{1}{2}r^2\Delta\theta$. The difference between wedges is ΔA:

$$\Delta A = \frac{1}{2}\left(r+\frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r-\frac{\Delta r}{2}\right)^2 \Delta\theta = r\,\Delta r\,\Delta\theta.$$

The exact method places r dead center (see figure). The approximation says: Forget the change in $r\Delta\theta$ as you move outward. Keep only the first-order terms.

A third method is coming, which requires no picture and no geometry. Calculus always has a third method! The change of variables $x = r\cos\theta$, $y = r\sin\theta$ will go into a general formula for dA, and out will come the area $r\,dr\,d\theta$.

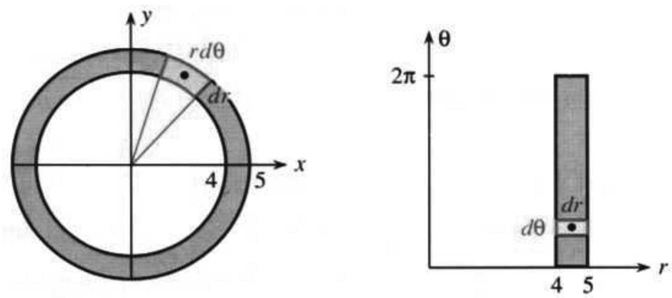

Fig. 14.7 Ring and polar rectangle in xy and $r\theta$, with stretching factor $r = 4.5$.

EXAMPLE 2 Find the area and center of gravity of the ring. Also find $\iint x^2 dA$.

Solution The limits on r are 4 and 5. The limits on θ are 0 and 2π. *Polar coordinates are perfect for a ring*. Compared with limits like $x = \sqrt{25-y^2}$, the change to $r\,dr\,d\theta$ is a small price to pay:

$$\text{area} = \int_0^{2\pi}\int_4^5 r\,dr\,d\theta = 2\pi\left[\tfrac{1}{2}r^2\right]_4^5 = \pi 5^2 - \pi 4^2 = 9\pi.$$

The θ integral is 2π (full circle). Actually the ring is a giant polar rectangle. We could have used the exact formula $r\,\Delta r\,\Delta\theta$, with $\Delta\theta = 2\pi$ and $\Delta r = 5-4$. When the radius r is centered at 4.5, the product $r\,\Delta r\,\Delta\theta$ is $(4.5)(1)(2\pi) = 9\pi$ as above.

Since the ring is symmetric around $(0,0)$, the integral of $x\,dA$ must be *zero*:

$$\iint_R x\,dA = \int_0^{2\pi}\int_4^5 (r\cos\theta) r\,dr\,d\theta = \left[\tfrac{1}{3}r^3\right]_4^5 \left[\sin\theta\right]_0^{2\pi} = 0.$$

Notice $r\cos\theta$ from x—the other r is from dA. The moment of inertia is

$$\iint_R x^2\,dA = \int_0^{2\pi}\int_4^5 r^2\cos^2\theta\, r\,dr\,d\theta = \left[\tfrac{1}{4}r^4\right]_4^5 \int_0^{2\pi} \cos^2\theta\,d\theta = \tfrac{1}{4}(5^4-4^4)\pi.$$

This θ integral is π not 2π, because the average of $\cos^2\theta$ is $\frac{1}{2}$ not 1.

For reference here are the moments of inertia when the density is $\rho(x,y)$:

$$I_y = \iint x^2\rho\,dA \quad I_x = \iint y^2\rho\,dA \quad I_0 = \iint r^2\rho\,dA = \textbf{polar moment} = I_x + I_y. \tag{4}$$

EXAMPLE 3 Find masses and moments for semicircular plates: $\rho = 1$ and $\rho = 1 - r$.

Solution The semicircles in Figure 14.8 have $r = 1$. The angle goes from 0 to π (the upper half-circle). Polar coordinates are best. ***The mass is the integral of the density*** ρ:

$$M = \int_0^{\pi}\int_0^{1} r\, dr\, d\theta = (\tfrac{1}{2})(\pi) \quad \text{and} \quad M = \int_0^{\pi}\int_0^{1}(1-r)r\, dr\, d\theta = (\tfrac{1}{6})(\pi).$$

The first mass $\pi/2$ equals the area (because $\rho = 1$). The second mass $\pi/6$ is smaller (because $\rho < 1$). Integrating $\rho = 1$ is the same as finding a volume when the height is $z = 1$ (part of a cylinder). Integrating $\rho = 1 - r$ is the same as finding a volume when the height is $z = 1 - r$ (part of a cone). Volumes of cones have the extra factor $\frac{1}{3}$. The center of gravity involves the moment $M_x = \iint y\rho\, dA$. ***The distance from the*** x ***axis is*** y, ***the mass of a small piece is*** $\rho\, dA$, ***integrate to add mass times distance***. Polar coordinates are still best, with $y = r \sin\theta$. Again $\rho = 1$ and $\rho = 1 - r$:

$$\iint y\, dA = \int_0^{\pi}\int_0^{1} r \sin\theta\, r\, dr\, d\theta = \tfrac{2}{3} \qquad \iint y(1-r)\, dA = \int_0^{\pi}\int_0^{1} r \sin\theta(1-r)r\, dr\, d\theta = \tfrac{1}{6}.$$

The height of the center of gravity is $\overline{y} = M_x/M =$ ***moment divided by mass***:

$$\overline{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \quad \text{when } \rho = 1 \qquad \overline{y} = \frac{1/6}{\pi/6} = \frac{1}{\pi} \quad \text{when } \rho = 1 - r.$$

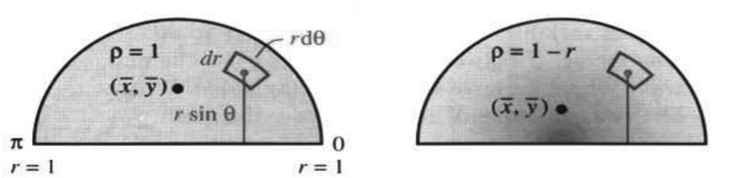

Fig. 14.8 Semicircles with density piled above them.

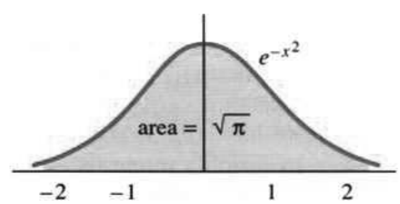

Fig. 14.9 Bell-shaped curve.

Question Compare $\overline{y}$ for $\rho = 1$ and $\rho =$ other positive constants and $\rho = 1 - r$.
Answer Any constant ρ gives $\overline{y} = 4/3\pi$. Since $1 - r$ is dense at $r = 0$, $\overline{y}$ drops to $1/\pi$.

Question How is $\overline{y} = 4/3\pi$ related to the "average" of y in the semicircle?
Answer They are identical. This is the point of $\overline{y}$. Divide the integral by the area:

$$\textit{The average value of a function is} \iint f(x,y)\,dA \Big/ \iint dA. \qquad (5)$$

The integral of f is divided by the integral of 1 (the area). In one dimension $\int_a^b v(x)\, dx$ was divided by $\int_a^b 1\, dx$ (the length $b - a$). That gave the average value of $v(x)$ in Section 5.6. Equation (5) is the same idea for $f(x,y)$.

EXAMPLE 4 Compute $A=\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$ from $A^2=\int_{-\infty}^{\infty} e^{-x^2}\,dx\int_{-\infty}^{\infty} e^{-y^2}\,dy=\pi$.

A is the area under a "bell-shaped curve"—see Figure 14.9. This is the most important definite integral in the study of probability. It is difficult because a factor $2x$ is not present. Integrating $2xe^{-x^2}$ gives $-e^{-x^2}$, but integrating e^{-x^2} is impossible—except approximately by a computer. How can we hope to show that A is exactly $\sqrt{\pi}$? The trick is to go from an area integral A to a volume integral A^2. This is unusual (and hard to like), but the end justifies the means:

$$A^2=\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dy\,dx=\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta. \qquad (6)$$

The double integrals cover the whole plane. The r^2 comes from x^2+y^2, and the key factor r appears in polar coordinates. It is now possible to substitute $u=r^2$. The r integral is $\frac{1}{2}\int_0^{\infty} e^{-u}du=\frac{1}{2}$. The θ integral is 2π. The double integral is $(\frac{1}{2})(2\pi)$. Therefore $A^2=\pi$ and the single integral is $A=\sqrt{\pi}$.

EXAMPLE 5 Apply Example 4 to the "***normal distribution***" $p(x)=e^{-x^2/2}/\sqrt{2\pi}$.

Section 8.4 discussed probability. It emphasized the importance of this particular $p(x)$. At that time we could not verify that $\int p(x)dx=1$. Now we can:

$$x=\sqrt{2}y \quad \text{yields} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}dy=1. \qquad (7)$$

Question Why include the 2's in $p(x)$? The integral of $e^{-x^2}/\sqrt{\pi}$ also equals 1.
Answer With the 2's the "***variance***" is $\int x^2 p(x)dx=1$. This is a convenient number.

CHANGE TO OTHER COORDINATES

A third method was promised, to find $r\,dr\,d\theta$ without a picture and without geometry. The method works directly from $x=r\cos\theta$ and $y=r\sin\theta$. It also finds the 1 in $du\,dv$, after a rotation of axes. Most important, this new method finds the factor J in the area $dA=J\,du\,dv$, for any change of variables. The change is from xy to uv.

For single integrals, the "***stretching factor***" J between the original dx and the new du is (not surprisingly) the ratio dx/du. Where we have dx, we write $(dx/du)du$. Where we have $(du/dx)dx$, we write du. That was the idea of substitutions—the main way to simplify integrals.

For double integrals the stretching factor appears in the area: $dx\,dy$ becomes $|J|\,du\,dv$. ***The old and new variables are related by*** $x=x(u,v)$ ***and*** $y=y(u,v)$. The point with coordinates u and v comes from the point with coordinates x and y. A whole region S, full of points in the uv plane, comes from the region R full of corresponding points in the xy plane. A small piece with area $|J|\,du\,dv$ comes from a small piece with area $dx\,dy$. *The formula for* J *is a two-dimensional version of* dx/du.

14B The stretching factor for area is the 2 by 2 ***Jacobian determinant*** $J(u,v)$:

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}. \tag{8}$$

An integral over R in the xy plane becomes an integral over S in the uv plane:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v))\,|J|\,du\,dv. \tag{9}$$

The determinant J is often written $\partial(x,y)/\partial(u,v)$, as a reminder that this stretching factor is like dx/du. *We require* $J \neq 0$. That keeps the stretching and shrinking under control.

You naturally ask: Why take the absolute value $|J|$ in equation (9)? Good question—it wasn't done for single integrals. The reason is in the limits of integration. The single integral $\int_0^1 dx$ is $\int_0^{-1}(-du)$ after changing x to $-u$. *We keep the minus sign and allow single integrals to run backward*. Double integrals could too, but normally they go left to right and down to up. We use the absolute value $|J|$ and run forward.

EXAMPLE 6 Polar coordinates have $x = u\cos v = r\cos\theta$ and $y = u\sin v = r\sin\theta$.

With no geometry:
$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r. \tag{10}$$

EXAMPLE 7 Find J for the *linear change* to $x = au + bv$ and $y = cu + dv$.

Ordinary determinant:
$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{11}$$

Why make this simple change, in which a, b, c, d are all constant? It straightens parallelograms into squares (and rotates those squares). Figure 14.10 is typical.

Common sense indicated $J = 1$ for pure rotation—no change in area. Now $J = 1$ comes from equations (1) and (11), because $ad - bc$ is $\cos^2\alpha + \sin^2\alpha$.

In practice, xy rectangles generally go into uv rectangles. The sides can be curved (as in polar rectangles) but the angles are often 90°. The change is "*orthogonal*." The next example has angles that are not 90°, and J still gives the answer.

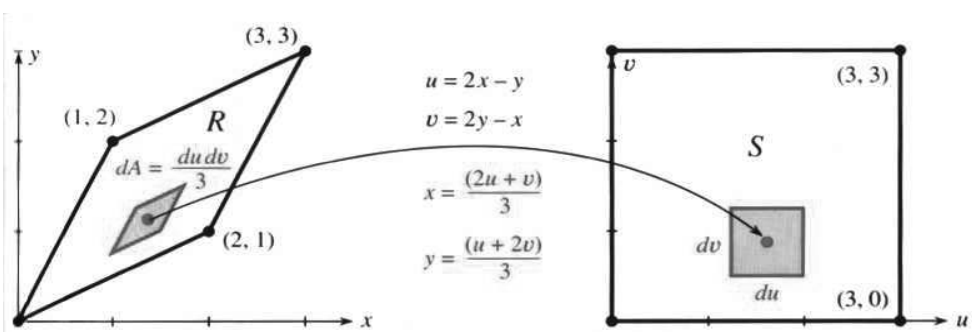

Fig. 14.10 Change from xy to uv has $J = \frac{1}{3}$.

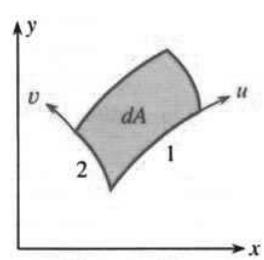

Fig. 14.11 Curved areas are also $dA = |J|\,du\,dv$.

EXAMPLE 8 Find the area of R in Figure 14.10. Also compute $\iint_R e^x\,dx\,dy$.

Solution The figure shows $x = \frac{2}{3}u + \frac{1}{3}v$ and $y = \frac{1}{3}u + \frac{2}{3}v$. The determinant is

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}.$$

The area of the xy parallelogram becomes an integral over the uv square:

$$\iint_R dx\,dy = \iint_S |J|\,du\,dv = \int_0^3\int_0^3 \tfrac{1}{3}\,du\,dv = \tfrac{1}{3}\cdot 3\cdot 3 = 3.$$

The square has area 9, the parallelogram has area 3. I don't know if $J = \frac{1}{3}$ is a stretching factor or a shrinking factor. The other integral $\iint e^x\,dx\,dy$ is

$$\int_0^3\int_0^3 e^{2u/3+v/3}\frac{1}{3}du\,dv = \left[\frac{3}{2}\,e^{2u/3}\right]_0^3\left[3e^{v/3}\right]_0^3\frac{1}{3} = \frac{3}{2}(e^2-1)(e-1).$$

Main point: The change to u and v makes the limits easy (just 0 and 3).

Why is the stretching factor J a determinant? With straight sides, this goes back to Section 11.3 on vectors. *The area of a parallelogram is a determinant*. Here the sides are curved, but that only produces $(du)^2$ and $(dv)^2$, which we ignore.

A change du gives one side of Figure 14.11—it is $(\partial x/\partial u\,\mathbf{i} + \partial y/\partial u\,\mathbf{j})du$. Side 2 is $(\partial x/\partial v\,\mathbf{i} + \partial y/\partial v\,\mathbf{j})dv$. The curving comes from second derivatives. The area (the cross product of the sides) is $|J|du\,dv$.

Final remark I can't resist looking at the change in the reverse direction. Now the rectangle is in xy and the parallelogram is in uv. In all formulas, exchange x for u and y for v:

$$\textbf{\textit{new}}\ J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\textbf{old}\ J}. \qquad (12)$$

This is exactly like $du/dx = 1/(dx/du)$. It is the derivative of the inverse function. The product of slopes is 1—stretch out, shrink back. From xy to uv we have 2 by 2 matrices, and the identity matrix I takes the place of 1:

$$\frac{dx}{du}\frac{du}{dx} = 1 \quad \text{becomes} \quad \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix}\begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (13)$$

The first row times the first column is $(\partial x/\partial u)(\partial u/\partial x) + (\partial x/\partial v)(\partial v/\partial x) = \partial x/\partial x = 1$. The first row times the second column is $(\partial x/\partial u)(\partial u/\partial y) + (\partial x/\partial v)(\partial v/\partial y) = \partial x/\partial y = 0$. ***The matrices are inverses of each other***. The determinants of a matrix and its inverse obey our rule: old J times new $J = 1$. Those J's cannot be zero, just as dx/du and du/dx were not zero. (Inverse functions increase steadily or decrease steadily.)

In two dimensions, an area $dx\,dy$ goes to $J\,du\,dv$ and comes back to $dx\,dy$.

14.2 EXERCISES

Read-through questions

We change variables to improve the ___a___ of integration. The disk $x^2+y^2 \leqslant 9$ becomes the rectangle $0 \leqslant r \leqslant$ ___b___, $0 \leqslant \theta \leqslant$ ___c___. The inner limits on $\iint dy\,dx$ are $y = \pm$ ___d___. In polar coordinates this area integral becomes $\iint$ ___e___ = ___f___.

A polar rectangle has sides dr and ___g___. Two sides are not ___h___ but the angles are still ___i___. The area between the circles $r=1$ and $r=3$ and the rays $\theta=0$ and $\theta=\pi/4$ is ___j___. The integral $\iint x\,dy\,dx$ changes to $\iint$ ___k___. This is the ___l___ around the ___m___ axis. Then $\overline{x}$ is the ratio ___n___. This is the x coordinate of the ___o___, and it is the ___p___ value of x.

In a rotation through α, the point that reaches (u,v) starts at $x = u\cos\alpha - v\sin\alpha$, $y =$ ___q___. A rectangle in the uv plane comes from a ___r___ in xy. The areas are ___s___ so the stretching factor is $J =$ ___t___. This is the determinant of the matrix ___u___ containing $\cos\alpha$ and $\sin\alpha$. The moment of inertia $\iint x^2 dx\,dy$ changes to $\iint$ ___v___ $du\,dv$.

For single integrals dx changes to ___w___ du. For double integrals $dx\,dy$ changes to $J\,du\,dv$ with $J =$ ___x___. The stretching factor J is the determinant of the 2 by 2 matrix ___y___. The functions $x(u,v)$ and $y(u,v)$ connect an xy region R to a uv region S, and $\iint_R dx\,dy = \iint_S$ ___z___ = area of ___A___. For polar coordinates $x =$ ___B___, $y =$ ___C___. For $x=u, y=u+4v$ the 2 by 2 determinant is $J =$ ___D___. A square in the uv plane comes from a ___E___ in xy. In the opposite direction the change has $u=x$ and $v=\frac{1}{4}(y-x)$ and a new $J =$ ___F___. This J is constant because this change of variables is ___G___.

In 1–12 R is a pie-shaped wedge: $0 \leqslant r \leqslant 1$ **and** $\pi/4 \leqslant \theta \leqslant 3\pi/4$.

1 What is the area of R? Check by integration in polar coordinates.

2 Find limits on $\iint dy\,dx$ to yield the area of R, and integrate. *Extra credit*: Find limits on $\iint dx\,dy$.

3 Equation (1) with $\alpha = \pi/4$ rotates R into the uv region $S =$ ______. Find limits on $\iint du\,dv$.

4 Compute the centroid height $\overline{y}$ of R by changing $\iint y\,dx\,dy$ to polar coordinates. Divide by the area of R.

5 The region R has $\overline{x}=0$ because ______. After rotation through $\alpha=\pi/4$, the centroid $(\overline{x},\overline{y})$ of R becomes the centroid ______ of S.

6 Find the centroid of any wedge $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant b$.

7 Suppose R^* is the wedge R moved up so that the sharp point is at $x=0$, $y=1$.

(a) Find limits on $\iint dy\,dx$ to integrate over R^*.

(b) With $x^* = x$ and $y^* = y-1$, the xy region R^* corresponds to what region in the x^*y^* plane?

(c) After that change $dx\,dy$ equals ______ dx^*dy^*.

8 Find limits on $\iint r\,dr\,d\theta$ to integrate over R^* in Problem 7.

9 The right coordinates for R^* are r^* and θ^*, with $x = r^*\cos\theta^*$ and $y = r^*\sin\theta^* + 1$.

(a) Show that $J = r^*$ so $dA = r^*dr^*d\theta^*$.

(b) Find limits on $\iint r^*dr^*d\theta^*$ to integrate over R^*.

10 If the centroid of R is $(0,\overline{y})$, the centroid of R^* is ______. The centroid of the circle with radius 3 and center $(1,2)$ is ______. The centroid of the upper half of that circle is ______.

11 The moments of inertia I_x, I_y, I_0 of the original wedge R are ______.

12 The moments of inertia I_x, I_y, I_0 of the shifted wedge R^* are ______.

Problems 13–16 change four-sided regions to squares.

13 R has straight sides $y=2x$, $x=1$, $y=1+2x$, $x=0$. Locate its four corners and draw R. Find its area by geometry.

14 Choose a,b,c,d so that the change $x = au+bv$, $y = cu+dv$ takes the previous R into S, the unit square $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 1$. From the stretching factor $J = ad-bc$ find the area of R.

15 The region R has straight sides $x=0$, $x=1$, $y=0$, $y=2x+3$. Choose a,b,c so that $x=u$ and $y = au+bv+cuv$ change R to the unit square S.

16 A nonlinear term uv was needed in Problem 15. Which regions R could change to the square S with a linear $x=au+bv$, $y=cu+dv$?

Draw the xy region R that corresponds in 17–22 to the uv square S with corners $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. **Locate the corners of R and then its sides (like a jigsaw puzzle).**

17 $x = 2u+v$, $y = u+2v$

18 $x = 3u+2v$, $y = u+v$

19 $x = e^{2u+v}$, $y = e^{u+2v}$

20 $x = uv$, $y = v^2-u^2$

21 $x = u$, $y = v(1+u^2)$

22 $x = u\cos v$, $y = u\sin v$ (only three corners)

23 In Problems 17 and 19, compute J from equation (8). Then find the area of R from $\iint_S |J|\,du\,dv$.

24 In 18 and 20, find $J = \partial(x,y)/\partial(u,v)$ and the area of R.

25 If R lies between $x=0$ and $x=1$ under the graph of $y=f(x)>0$, then $x=u, y=vf(u)$ takes R to the unit square S. Locate the corners of R and the point corresponding to $u=\frac{1}{2}, v=1$. Compute J to prove what we know:

$$\text{area of } R = \int_0^1 f(x)dx = \int_0^1\int_0^1 J\,du\,dv.$$

26 From $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}(y/x)$, compute $\partial r/\partial x$, $\partial r/\partial y$, $\partial\theta/\partial x$, $\partial\theta/\partial y$, and the determinant $J=\partial(r,\theta)/\partial(x,y)$. How is this J related to the factor $r=\partial(x,y)/\partial(r,\theta)$ that enters $r\,dr\,d\theta$?

27 Example 4 integrated e^{-x^2} from 0 to ∞ (answer $\sqrt{\pi}$). Also $B=\int_0^1 e^{-x^2}dx$ leads to $B^2=\int_0^1 e^{-x^2}dx\int_0^1 e^{-y^2}dy$. Change this double integral over the unit square to r and θ—and find the limits on r that make exact integration impossible.

28 Integrate by parts to prove that the standard normal distribution $p(x)=e^{-x^2/2}/\sqrt{2\pi}$ has $\sigma^2=\int_{-\infty}^{\infty}x^2p(x)dx=1$.

29 Find the average distance from a point on a circle to the points inside. Suggestion: Let $(0,0)$ be the point and let $0\leqslant r\leqslant 2a\cos\theta$, $0\leqslant\theta\leqslant\pi$ be the circle (radius a). The distance is r, so the average distance is $\bar{r}=\iint$ _____ / $\iint$ _____ .

30 Draw the region $R: 0\leqslant x\leqslant 1$, $0\leqslant y\leqslant\infty$ and describe it with polar coordinates (limits on r and θ). Integrate $\iint_R(x^2+y^2)^{-3/2}dx\ dy$ in polar coordinates.

31 Using polar coordinates, find the volume under $z=x^2+y^2$ above the unit disk $x^2+y^2\leqslant 1$.

32 The end of Example 1 stated the moment of inertia $\iint y^2dA$. Check that integration.

33 In the square $-1\leqslant x\leqslant 2$, $-2\leqslant y\leqslant 1$, where could you distribute a unit mass (with $\iint\rho\,dxdy=1$) to maximize

(a) $\iint x^2\rho\,dA$ (b) $\iint y^2\rho\,dA$ (c) $\iint r^2\rho\,dA$?

34 *True or false*, with a reason:

(a) If the uv region S corresponds to the xy region R, then area of S = area of R.

(b) $\iint x\,dA\leqslant\iint x^2\,dA$

(c) The average value of $f(x,y)$ is $\iint f(x,y)dA$

(d) $\int_{-\infty}^{\infty}xe^{-x^2}dx=0$

(e) A polar rectangle has the same area as a straight-sided region with the same corners.

35 Find the mass of the tilted square in Example 1 if the density is $\rho=xy$.

36 Find the mass of the ring in Example 2 if the density is $\rho=x^2+y^2$. This is the same as which moment of inertia with which density ?

37 Find the polar moment of inertia I_0 of the ring in Example 2 if the density is $\rho=x^2+y^2$.

38 Give the following statement an appropriate name: $\iint_R f(x,y)dA=f(P)$ times (area of R), where P is a point in R. Which point P makes this correct for $f=x$ and $f=y$?

39 Find the xy coordinates of the top point in Figure 14.6a and check that it goes to $(u,v)=(1,1)$.

14.3 Triple Integrals

At this point in the book, I feel I can speak to you directly. You can guess what triple integrals are like. Instead of a small interval or a small rectangle, there is a small box. Instead of length dx or area $dx\,dy$, the box has volume $dV = dx\,dy\,dz$. That is length times width times height. The goal is to put small boxes together (by integration). The main problem will be to discover the correct limits on x, y, z.

We could dream up more and more complicated regions in three-dimensional space. But I don't think you can see the method clearly without seeing the region clearly. In practice six shapes are the most important:

box prism cylinder cone tetrahedron sphere.

The box is easiest and the sphere may be the hardest (but no problem in spherical coordinates). Circular cylinders and cones fall in the middle, where xyz coordinates are possible but $r\theta z$ are the best. I start with the box and prism and xyz.

EXAMPLE 1 By triple integrals find the volume of a box and a prism (Figure 14.12).

$$\iiint_{\text{box}} dV = \int_{z=0}^{1}\int_{y=0}^{3}\int_{x=0}^{2} dx\,dy\,dz \quad \text{and} \quad \iiint_{\text{prism}} dV = \int_{z=0}^{1}\int_{y=0}^{3-3z}\int_{x=0}^{2} dx\,dy\,dz$$

The inner integral for both is $\int dx = 2$. Lines in the x direction have length 2, cutting through the box and the prism. The middle integrals show the limits on y (since dy comes second):

$$\int_{y=0}^{3} 2\,dy = 6 \quad \text{and} \quad \int_{y=0}^{3-3z} 2\,dy = 6-6z.$$

After two integrations these are ***areas***. The first area 6 is for a plane section through the box. The second area $6-6z$ is cut through the prism. The shaded rectangle goes from $y=0$ to $y=3-3z$—we needed and used the equation $y+3z=3$ for the boundary of the prism. *At this point z is still constant*! But the area depends on z, because the prism gets thinner going upwards. The base area is $6-6z=6$, the top area is $6-6z=0$.

The outer integral multiplies those areas by dz, to give the volume of slices. They are horizontal slices because z came last. Integration adds up the slices to find the total volume:

$$\text{box volume} = \int_{z=0}^{1} 6\,dz = 6 \qquad \text{prism volume} = \int_{z=0}^{1}(6-6z)dz = \Big[6z-3z^2\Big]_0^1 = 3.$$

The box volume $2\cdot 3\cdot 1$ didn't need calculus. The prism is half of the box, so its volume was sure to be 3—but it is satisfying to see how $6z-3z^2$ gives the answer. Our purpose is to see how a triple integral works.

Question Find the prism volume in the order $dz\,dy\,dx$ (***six orders are possible***).

Answer $$\int_0^2\int_0^3\int_0^{(3-y)/3} dz\,dy\,dx = \int_0^2\int_0^3\left(\frac{3-y}{3}\right)dy\,dx = \int_0^2 \frac{3}{2}dx = 3.$$

To find those limits on the z integral, follow a line in the z direction. It enters the prism at $z=0$ and exits at the sloping face $y+3z=3$. That gives the upper limit $z=(3-y)/3$. It is the height of a thin stick as in Section 14.1. This section writes out $\int dz$ for the height, but a quicker solution starts at the double integral.

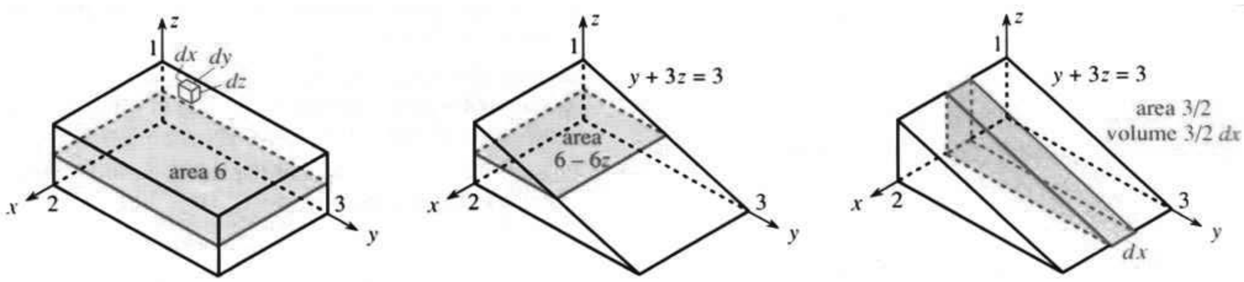

Fig. 14.12 Box with sides $2, 3, 1$. The prism is half of the box: volume $\int(6-6z)dz$ or $\int \frac{3}{2} dx$.

What is the number $\frac{3}{2}$ in the last integral? It is the ***area of a vertical slice***, cut by a plane $x =$ constant. The outer integral adds up slices.

$$\iiint f(x,y,z)\,dV \text{ is computed from three single integrals } \int\Big[\int\Big[\int f\,dx\Big]dy\Big]dz.$$

That step cannot be taken in silence—some basic calculus is involved. The triple integral is the limit of $\sum f_i \Delta V$, a sum over small boxes of volume ΔV. Here f_i is any value of $f(x,y,z)$ in the ith box. (In the limit, the boxes fit a curved region.) Now take those boxes *in a certain order*. Put them into lines in the x direction and put the lines of boxes into planes. The lines lead to the inner x integral, whose answer depends on y and z. The y integral combines the lines into planes. Finally the outer integral accounts for all planes and all boxes.

Example 2 is important because it displays more possibilities than a box or prism.

EXAMPLE 2 Find the volume of a tetrahedron (4-sided pyramid). Locate $(\overline{x}, \overline{y}, \overline{z})$.

Solution A tetrahedron has four flat faces, all triangles. The fourth face in Figure 14.13 is on the plane $x+y+z=1$. A line in the x direction enters at $x=0$ and exits at $x=1-y-z$. (The length depends on y and z. The equation of the boundary plane gives x.) Then those lines are put into plane slices by the y integral:

$$\int_{y=0}^{1-z}\int_{x=0}^{1-y-z} dx\,dy = \int_{y=0}^{1-z}(1-y-z)dy = \Big[y-\tfrac{1}{2}y^2-zy\Big]_0^{1-z} = \tfrac{1}{2}(1-z)^2.$$

What is this number $\frac{1}{2}(1-z)^2$? ***It is the area at height*** z. The plane at that height slices out a right triangle, whose legs have length $1-z$. The area is correct, but look at the limits of integration. ***If*** x ***goes to*** $1-y-z$, ***why does*** y ***go to*** $1-z$? Reason: We are assembling lines, not points. The figure shows a line at every y up to $1-z$.

Adding the slices gives the volume: $\int_0^1 \frac{1}{2}(1-z)^2 dz = \big[\frac{1}{6}(z-1)^3\big]_0^1 = \frac{1}{6}$. This agrees with $\frac{1}{3}$(base times height), the volume of a pyramid.

The height $\overline{z}$ of the centroid is "z_{average}." We compute $\iiint z\,dV$ and divide by the volume. Each horizontal slice is multiplied by its height z, and the limits of integration don't change:

$$\iiint z\,dV = \int_0^1\int_0^{1-y}\int_0^{1-y-z} z\,dx\,dy\,dz = \int_0^1 \frac{z(1-z)^2}{2}\,dz = \frac{1}{24}.$$

This is quick because z is constant in the x and y integrals. Each triangular slice contributes z times its area $\frac{1}{2}(1-z)^2$ times dz. Then the z integral gives the moment

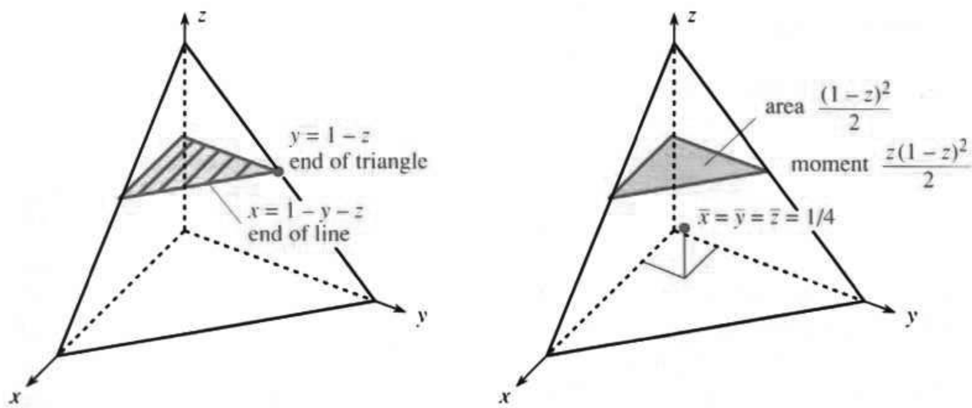

Fig. 14.13 Lines end at plane $x + y + z = 1$. Triangles end at edge $y + z = 1$. The average height is $\bar{z} = \iiint z\,dV / \iiint dV$.

$1/24$. To find the *average* height, divide $1/24$ by the volume:

$$\bar{z} = \text{ height of centroid } = \frac{\iiint z\,dV}{\iiint dV} = \frac{1/24}{1/6} = \frac{1}{4}.$$

By symmetry $\bar{x} = \frac{1}{4}$ and $\bar{y} = \frac{1}{4}$. The centroid is the point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Compare that with $(\frac{1}{3}, \frac{1}{3})$, the centroid of the standard right triangle. Compare also with $\frac{1}{2}$, the center of the unit interval. There must be a five-sided region in four dimensions centered at $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

For area and volume we meet another pattern. Length of standard interval is 1, area of standard triangle is $\frac{1}{2}$, volume of standard tetrahedron is $\frac{1}{6}$, hypervolume in four dimensions must be _____. The interval reaches the point $x = 1$, the triangle reaches the line $x + y = 1$, the tetrahedron reaches the plane $x + y + z = 1$. The four-dimensional region stops at the hyperplane _____ $= 1$.

EXAMPLE 3 Find the volume $\iiint dx\,dy\,dz$ inside the unit sphere $x^2 + y^2 + z^2 = 1$.

First question: What are the limits on x? If a needle goes through the sphere in the x direction, where does it enter and leave? Moving in the x direction, the numbers y and z stay constant. The inner integral deals only with x. The smallest and largest x are at the boundary where $x^2 + y^2 + z^2 = 1$. This equation does the work—we solve it for x. Look at the limits on the x integral:

$$\text{volume of sphere} = \int_?^? \int_?^? \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx\,dy\,dz = \int_?^? \int_?^? 2\sqrt{1-y^2-z^2}\,dy\,dz. \quad (1)$$

The limits on y are $-\sqrt{1-z^2}$ and $+\sqrt{1-z^2}$. You can use algebra on the boundary equation $x^2 + y^2 + z^2 = 1$. But notice that x is gone! We want the smallest and largest y, for each z. It helps very much to draw the plane at height z, slicing through the sphere in Figure 14.14. The slice is a circle of radius $r = \sqrt{1-z^2}$. So the area is πr^2, which must come from the y integral:

$$\int 2\sqrt{1-y^2-z^2}\,dy = \text{ area of slice } = \pi(1-z^2). \quad (2)$$

I admit that I didn't integrate. Is it cheating to use the formula πr^2? I don't think so. Mathematics is hard enough, and we don't have to work blindfolded. The goal is understanding, and if you know the area then use it. Of course the integral of $\sqrt{1-y^2-z^2}$ can be done if necessary—use Section 7.2.

The triple integral is down to a single integral. We went from one needle to a circle of needles and now to a sphere of needles. The volume is a sum of slices of area $\pi(1-z^2)$. The South Pole is at $z=-1$, the North Pole is at $z=+1$, and the integral is the volume $4\pi/3$ inside the unit sphere:

$$\int_{-1}^{1} \pi(1-z^2)dz = \pi\left(z-\frac{1}{3}z^3\right)\Big]_{-1}^{1} = \frac{2}{3}\pi - \left(-\frac{2}{3}\pi\right) = \frac{4}{3}\pi. \qquad (3)$$

Question 1 A cone also has circular slices. How is the last integral changed?

Answer The slices of a cone have radius $1-z$. Integrate $(1-z)^2$ not $\sqrt{1-z^2}$.

Question 2 How does this compare with a circular cylinder (height 1, radius 1)?

Answer Now all slices have radius 1. Above $z=0$, a cylinder has volume π and a half-sphere has volume $\frac{2}{3}\pi$ and a cone has volume $\frac{1}{3}\pi$.

For solids with equal surface area, the sphere has largest volume.

Question 3 What is the average height $\bar{z}$ in the cone and half-sphere and cylinder?

Answer $$\bar{z} = \frac{\int z(\text{slice area})dz}{\int(\text{slice area})dz} = \frac{1}{4} \quad\text{and}\quad \frac{3}{8} \quad\text{and}\quad \frac{1}{2}.$$

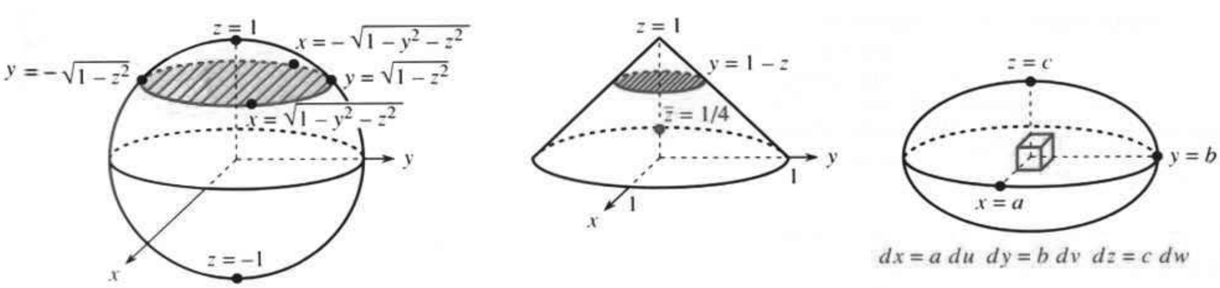

Fig. 14.14 $\int dx$ = length of needle, $\iint dx\,dy$ = area of slice. Ellipsoid is a stretched sphere.

EXAMPLE 4 Find the volume $\iiint dx\,dy\,dz$ inside the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

The limits on x are now $\pm\sqrt{1-y^2/b^2-z^2/c^2}$. The algebra looks terrible. The geometry is better—all slices are ellipses. A ***change of variable*** is absolutely the best.

Introduce $u=x/a$ and $v=y/b$ and $w=z/c$. Then the outer boundary becomes $u^2+v^2+w^2=1$. In these new variables the shape is a sphere. The triple integral for a sphere is $\iiint du\,dv\,dw = 4\pi/3$. But what volume dV in xyz space corresponds to a small box with sides du and dv and dw?

Every uvw box comes from an xyz box. The box is stretched with no bending or twisting. Since u is x/a, the length dx is $a\,du$. Similarly $dy=b\,dv$ and $dz=c\,dw$. The volume of the xyz box (Figure 14.14) is $dx\,dy\,dz=(abc)\,du\,dv\,dw$.

The ***stretching factor*** $J = abc$ is a constant, and the volume of the ellipsoid is

$$\overset{\text{bad limits}}{\underset{\text{ellipsoid}}{\iiint}} dx\,dy\,dz = \overset{\text{better limits}}{\underset{\text{sphere}}{\iiint}} (abc)\,du\,dv\,dw = \frac{4\pi}{3}abc. \qquad (4)$$

You realize that this is special—other volumes are much more complicated. The sphere and ellipsoid are curved, but the small xyz boxes are straight. The next section introduces spherical coordinates, and we can finally write "*good limits*." But then we need a different J.

14.3 EXERCISES

Read-through questions

Six important solid shapes are __a__. The integral $\iiint dx\,dy\,dz$ adds the volume __b__ of small __c__. For computation it becomes __d__ single integrals. The inner integral $\int dx$ is the __e__ of a line through the solid. The variables __f__ and __g__ are held constant. The double integral $\iint dx\,dy$ is the __h__ of a slice, with __i__ held constant. Then the z integral adds up the volumes of __j__.

If the solid region V is bounded by the planes $x=0$, $y=0$, $z=0$, and $x+2y+3z=1$, the limits on the inner x integral are __k__. The limits on y are __l__. The limits on z are __m__. In the new variables $u=x$, $v=2y$, $w=3z$, the equation of the outer boundary is __n__. The volume of the tetrahedron in uvw space is __o__. From $dx=du$ and $dy=dv/2$ and $dz=$ __p__, the volume of an xyz box is $dx\,dy\,dz=$ __q__ $du\,dv\,dw$. So the volume of V is __r__.

To find the average height $\bar{z}$ in V we compute __s__/__t__. To find the total mass in V if the density is $\rho = e^z$ we compute the integral __u__. To find the average density we compute __v__/__w__. In the order $\iiint dz\,dx\,dy$ the limits on the inner integral can depend on __x__. The limits on the middle integral can depend on __y__. The outer limits for the ellipsoid $x^2+2y^2+3z^2 \leqslant 8$ are __z__.

1 For the solid region $0 \leqslant x \leqslant y \leqslant z \leqslant 1$, find the limits in $\iiint dx\,dy\,dz$ and compute the volume.

2 Reverse the order in Problem 1 to $\iiint dz\,dy\,dx$ and find the limits of integration. The four faces of this tetrahedron are the planes $x=0$ and $y=x$ and ______ .

3 This tetrahedron and five others like it fill the unit cube. Change the inequalities in Problem 1 to describe the other five.

4 Find the centroid $(\bar{x}, \bar{y}, \bar{z})$ in Problem 1.

Find the limits of integration in $\iiint dx\,dy\,dz$ and the volume of solids 5–16. Draw a very rough picture.

5 A cube with sides of length 2, centered at $(0, 0, 0)$.

6 Half of that cube, the box above the xy plane.

7 Part of the same cube, the prism above the plane $z=y$.

8 Part of the same cube, above $z=y$ and $z=0$.

9 Part of the same cube, above $z=x$ and below $z=y$.

10 Part of the same cube, where $x \leqslant y \leqslant z$. What shape is this?

11 The tetrahedron bounded by planes $x=0$, $y=0$, $z=0$, and $x+y+2z=2$.

12 The tetrahedron with corners $(0,0,0)$, $(2,0,0)$, $(0,4,0)$, $(0,0,4)$. First find the plane through the last three corners.

13 The part of the tetrahedron in Problem 11 below $z=\frac{1}{2}$.

14 The tetrahedron in Problem 12 with its top sliced off by the plane $z=1$.

15 The volume above $z=0$ below the cone $\sqrt{x^2+y^2}=1-z$.

*__16__ The tetrahedron in Problem 12, after it falls across the x axis onto the xy plane.

In 17–20 find the limits in $\iiint dx\,dy\,dz$ *or* $\iiint dz\,dy\,dx$. Compute the volume.

17 A circular cylinder with height 6 and base $x^2+y^2 \leqslant 1$.

18 The part of that cylinder below the plane $z=x$. Watch the base. Draw a picture.

19 The volume shared by the cube (Problem 5) and cylinder.

20 The same cylinder lying along the x axis.

21 A cube is inscribed in a sphere: radius 1, both centers at $(0,0,0)$. What is the volume of the cube?

22 Find the volume and the centroid of the region bounded by $x=0$, $y=0$, $z=0$, and $x/a+y/b+z/c=1$.

23 Find the volume and centroid of the solid $0 \leqslant z \leqslant 4-x^2-y^2$.

24 Based on the text, what is the volume inside $x^2+4y^2+9z^2=16$? What is the "hypervolume" of the 4-dimensional pyramid that stops at $x+y+z+w=1$?

25 Find the partial derivatives $\partial I/\partial x$, $\partial I/\partial y$, $\partial^2 I/\partial y\,\partial z$ of

$$I = \int_0^z\int_0^y dx\,dy \text{ and } I = \int_0^z\int_0^y\int_0^x f(x,y,z)\,dx\,dy\,dz.$$

26 Define the average value of $f(x,y,z)$ in a solid V.

27 Find the moment of inertia $\iiint l^2 dV$ of the cube $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$ when l is the distance to

(a) the x axis (b) the edge $y = z = 1$ (c) the diagonal $x = y = z$.

28 Add upper limits to produce the volume of a unit cube from small cubes: $V = \sum_{i=1}\sum_{j=1}\sum_{k=1}(\Delta x)^3 = 1$.

*__29__ Find the limit as $\Delta x \to 0$ of $\sum_{i=1}^{3/\Delta x}\sum_{j=1}^{2/\Delta x}\sum_{k=1}^{j}(\Delta x)^3$.

30 The midpoint rule for an integral over the unit cube chooses the center value $f(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Which functions $f = x^m y^n z^p$ are integrated correctly?

31 The trapezoidal rule estimates $\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz$ as $\frac{1}{8}$ times the sum of $f(x,y,z)$ at 8 corners. This correctly integrates $x^m y^n z^p$ for which m, n, p?

32 Propose a 27-point "Simpson's Rule" for integration over a cube. If many small cubes fill a large box, why are there only 8 new points per cube?

14.4 Cylindrical and Spherical Coordinates

Cylindrical coordinates are good for describing solids that are ***symmetric around an axis***. The solid is three-dimensional, so there are three coordinates r, θ, z:

r: *out* from the axis $\quad$ θ: *around* the axis $\quad$ z: *along* the axis.

This is a mixture of polar coordinates $r\theta$ in a plane, plus z upward. You will not find $r\theta z$ difficult to work with. Start with a cylinder centered on the z axis:

solid cylinder : $0 \leqslant r \leqslant 1$ $\quad$ *flat bottom and top* : $0 \leqslant z \leqslant 3$ $\quad$ *half-cylinder* : $0 \leqslant \theta \leqslant \pi$

Integration over this half-cylinder is $\int_0^3\int_0^\pi\int_0^1 \underline{\quad ? \quad}\, dr\, d\theta\, dz$. These limits on r, θ, z are especially simple. Two other axially symmetric solids are almost as convenient:

cone : *integrate to* $r + z = 1$ $\quad$ ***sphere*** : *integrate to* $r^2 + z^2 = R^2$

I would not use cylindrical coordinates for a box. Or a tetrahedron.

The integral needs one thing more—the volume dV. The movements dr and $d\theta$ and dz give a "curved box" in xyz space, drawn in Figure 14.15c. The base is a polar rectangle, with area $r\ dr\ d\theta$. The new part is the height dz. ***The volume of the curved box is*** $r\ dr\ d\theta\ dz$. Then r goes in the blank space in the triple integral—the stretching factor is $J = r$. There are six orders of integration (we give two):

$$\text{volume} = \int_z\int_\theta\int_r r\ dr\ d\theta\ dz = \int_\theta\int_z\int_r r\ dr\ dz\ d\theta. \tag{1}$$

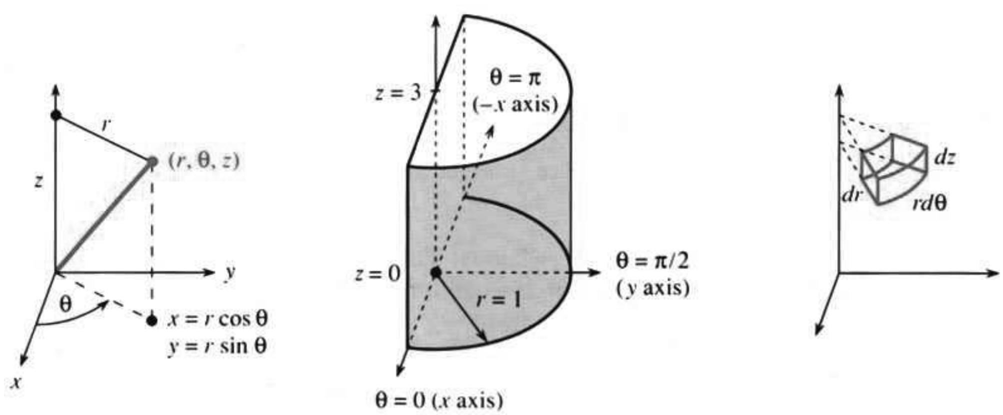

Fig. 14.15 Cylindrical coordinates for a point and a half-cylinder. Small volume $r\ dr\ d\theta\ dz$.

EXAMPLE 1 (Volume of the half-cylinder). The integral of $r\ dr$ from 0 to 1 is $\frac{1}{2}$. The θ integral is π and the z integral is 3. The volume is $3\pi/2$.

EXAMPLE 2 The surface $r = 1 - z$ encloses the cone in Figure 14.16. Find its volume.

First solution $\quad$ Since r goes out to $1 - z$, the integral of $r\ dr$ is $\frac{1}{2}(1-z)^2$. The θ integral is 2π (a full rotation). Stop there for a moment.

We have reached $\int\int r\ dr\ d\theta = \frac{1}{2}(1-z)^2\, 2\pi$. This is the ***area of a slice at height*** z. The slice is a circle, its radius is $1 - z$, its area is $\pi(1-z)^2$. The z integral adds those slices to give $\pi/3$. That is correct, but it is not the only way to compute the volume.

Second solution Do the z and θ integrals first. Since z goes up to $1-r$, and θ goes around to 2π, those integrals produce $\iint r\,dz\,d\theta = r(1-r)2\pi$. Stop again—this must be the area of something.

After the z and θ integrals we have a ***shell at radius*** r. The height is $1-r$ (the outer shells are shorter). This height times $2\pi r$ gives the area around the shell. The choice between shells and slices is exactly as in Chapter 8. ***Different orders of integration give different ways to cut up the solid***.

The volume of the shell is area times thickness dr. The volume of the complete cone is the integral of shell volumes: $\int_0^1 r(1-r)2\pi\,dr = \pi/3$.

Third solution Do the r and z integrals first: $\iint r\,dr\,dz = \frac{1}{6}$. Then the θ integral is $\int \frac{1}{6}\,d\theta$, which gives $\frac{1}{6}$ times 2π. This is the volume $\pi/3$—*but what is* $\frac{1}{6}\,d\theta$?

The third cone is cut into wedges. The volume of a wedge is $\frac{1}{6}\,d\theta$. It is quite common to do the θ integral last, especially when it just multiplies by 2π. It is not so common to think of wedges.

Question Is the volume $\frac{1}{6}\,d\theta$ equal to an area $\frac{1}{6}$ times a thickness $d\theta$?

Answer No! The triangle in the third cone has area $\frac{1}{2}$ not $\frac{1}{6}$. *Thickness is never* $d\theta$.

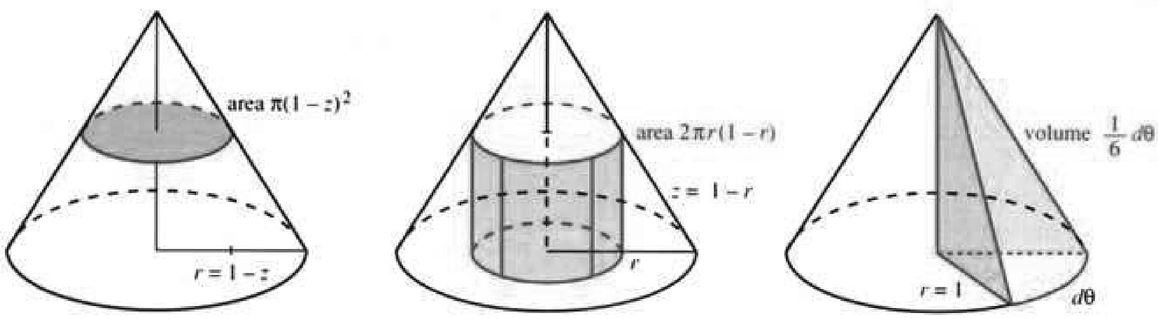

Fig. 14.16 A cone cut three ways: slice at height z, shell at radius r, wedge at angle θ.

This cone is typical of a ***solid of revolution***. The axis is in the z direction. The θ integral yields 2π, whether it comes first, second, or third. The r integral goes out to a radius $f(z)$, which is 1 for the cylinder and $1-z$ for the cone. The integral $\iint r\,dr\,d\theta$ is $\pi(f(z))^2 =$ area of circular slice. This leaves the z integral $\int \pi(f(z))^2 dz$. That is our old volume formula $\int \pi(f(x))^2 dx$ from Chapter 8, where the slices were cut through the x axis.

EXAMPLE 3 The ***moment of inertia*** around the z axis is $\iiint r^3 dr\,d\theta\,dz$. The extra r^2 is (distance to axis)^2. For the cone this triple integral is $\pi/10$.

EXAMPLE 4 The ***moment*** around the z axis is $\iiint r^2\,dr\,d\theta\,dz$. For the cone this is $\pi/6$. The *average distance* $\overline{r}$ is (moment)/(volume) $= (\pi/6)/(\pi/3) = \frac{1}{2}$.

EXAMPLE 5 A sphere of radius R has the boundary $r^2+z^2=R^2$, in cylindrical coordinates. The outer limit on the r integral is $\sqrt{R^2-z^2}$. That is not acceptable in difficult problems. To avoid it we now change to coordinates that are natural for a sphere.

SPHERICAL COORDINATES

The Earth is a solid sphere (or near enough). On its surface we use two coordinates—latitude and longitude. To dig inward or fly outward, there is a third coordinate—*the distance* ρ *from the center*. This Greek letter *rho* replaces r to avoid confusion with cylindrical coordinates. Where r is measured from the z axis, ρ is measured from the origin. Thus $r^2 = x^2 + y^2$ and $\rho^2 = x^2 + y^2 + z^2$.

The angle θ is the same as before. It goes from 0 to 2π. It is the longitude, which increases as you travel east around the Equator.

The angle ϕ is new. It equals 0 at the North Pole and π (not 2π) at the South Pole. It is the ***polar angle***, measured down from the z axis. The Equator has a latitude of 0 but a polar angle of $\pi/2$ (halfway down). Here are some typical shapes:

solid sphere (or ball): $0 \leqslant \rho \leqslant R$ surface of sphere: $\rho = R$

upper half-sphere: $0 \leqslant \phi \leqslant \pi/2$ eastern half-sphere: $0 \leqslant \theta \leqslant \pi$

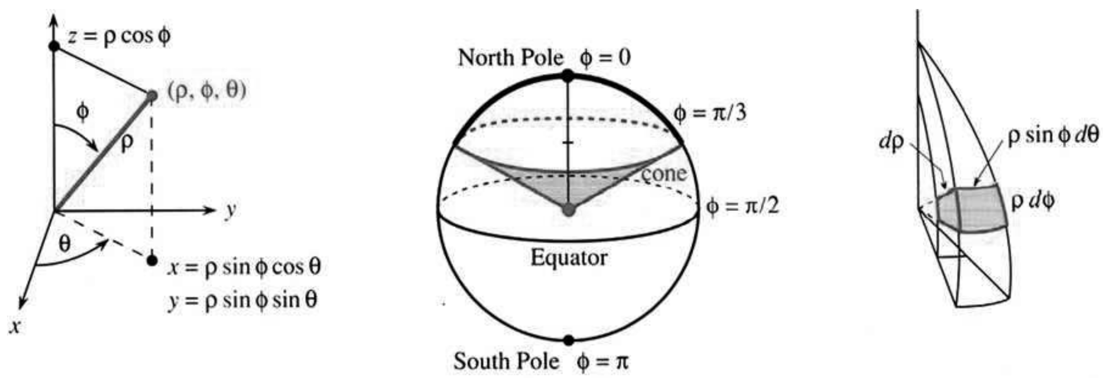

Fig. 14.17 Spherical coordinates $\rho\phi\theta$. The volume $dV = \rho^2 \sin\ \phi\ d\rho\ d\phi\ d\theta$ of a spherical box.

The angle ϕ is constant on a cone from the origin. It cuts the surface in a circle (Figure 14.17b), but not a great circle. The angle θ is constant along a half-circle from pole to pole. The distance ρ is constant on each inner sphere, starting at the center $\rho = 0$ and moving out to $\rho = R$.

In spherical coordinates the volume integral is $\iiint \rho^2 \sin\ \phi\ d\rho\ d\phi\ d\theta$. To explain that surprising factor $J = \rho^2 \sin\phi$, start with $x = r\cos\theta$ and $y = r\sin\theta$. In spherical coordinates r is $\rho\sin\phi$ and z is $\rho\cos\phi$—see the triangle in the figure. So substitute $\rho\sin\phi$ for r:

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi. \qquad (1)$$

Remember those two steps, $\rho\phi\theta$ to $r\theta z$ to xyz. We check that $x^2 + y^2 + z^2 = \rho^2$:

$$\rho^2(\sin^2\phi\ \cos^2\theta + \sin^2\phi\ \sin^2\ \theta + \cos^2\phi) = \rho^2(\sin^2\phi + \cos^2\phi) = \rho^2.$$

The volume integral is explained by Figure 14.17c. That shows a "*spherical box*" with right angles and curved edges. Two edges are $d\rho$ and $\rho d\phi$. The third edge is horizontal. The usual $rd\theta$ becomes $\rho\sin\phi\ d\theta$. Multiplying those lengths gives dV.

The volume of the box is $dV = \rho^2 \sin\ \phi\ d\rho\ d\phi\ d\theta$. This is a *distance cubed*, from $\rho^2 d\rho$.

EXAMPLE 6 A solid ball of radius R has known volume $V = \frac{4}{3}\pi R^3$. Notice the limits:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = \left[\tfrac{1}{3}\rho^3\right]_0^R \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} = (\tfrac{1}{3}R^3)(2)(2\pi).$$

Question What is the volume above the cone in Figure 14.17?
Answer The ϕ integral stops at $[-\cos\phi]_0^{\pi/3} = \frac{1}{2}$. The volume is $(\frac{1}{3}R^3)(\frac{1}{2})(2\pi)$.

EXAMPLE 7 The *surface area* of a sphere is $A = 4\pi R^2$. Forget the ρ integral:

$$A = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi\ d\phi\ d\theta = R^2\left[-\cos\phi\right]_0^{\pi}\left[\theta\right]_0^{2\pi} = R^2(2)(2\pi).$$

After those examples from geometry, here is the real thing from science. I want to compute one of the most important triple integrals in physics—“the gravitational attraction of a solid sphere.” For some reason Isaac Newton had trouble with this integral. He refused to publish his masterpiece on astronomy until he had solved it. I think he didn’t use spherical coordinates—and the integral is not easy even now.

The answer that Newton finally found is beautiful. ***The sphere acts as if all its mass were concentrated at the center***. At an outside point $(0, 0, D)$, the force of gravity is proportional to $1/D^2$. The force from a uniform solid sphere equals the force from a point mass, at every outside point P. That is exactly what Newton wanted and needed, to explain the solar system and to prove Kepler’s laws.

Here is the difficulty. Some parts of the sphere are closer than D, some parts are farther away. The actual distance q, from the outside point P to a typical inside point, is shown in Figure 14.18. The *average* distance $\overline{q}$ to all points in the sphere is not D. But what Newton needed was a different average, and by good luck or some divine calculus it works perfectly: ***The average of*** $1/q$ ***is*** $1/D$. This gives the potential energy:

$$\textbf{\textit{potential at point}}\ P = \iiint_{\text{sphere}} \frac{1}{q}\,dV = \frac{\text{Volume of sphere}}{D}. \qquad (2)$$

A small volume dV at the distance q contributes dV/q to the potential (Section 8.6, with density 1). The integral adds the contributions from the whole sphere. Equation (2) says that the potential at $r = D$ is not changed when the sphere is squeezed to the center. The potential equals the whole volume divided by the single distance D.

Important point: The average of $1/q$ is $1/D$ and not $1/\overline{q}$. The average of $\frac{1}{2}$ and $\frac{1}{4}$ is not $\frac{1}{3}$. Smaller point: I wrote “sphere” where I should have written “ball.” The sphere is solid: $0 \leqslant \rho \leqslant R, 0 \leqslant \phi \leqslant \pi, 0 \leqslant \theta \leqslant 2\pi$. What about the force? For the small volume it is proportional to dV/q^2 (this is the inverse square law). But *force is a vector*, pulling the outside point toward dV—not toward the center of the sphere. The figure shows the geometry and the symmetry. ***We want the*** z ***component of the force***. (By symmetry the overall x and y components are zero.) The angle between the force vector and the z axis is α, so for the z component we multiply by $\cos\alpha$. The total force comes from the integral that Newton discovered:

$$\textbf{\textit{force at point}}\ P = \iiint_{\text{sphere}} \frac{\cos\alpha}{q^2}\,dV = \frac{\text{volume of sphere}}{D^2}. \tag{3}$$

I will compute the integral (2) and leave you the privilege of solving (3). I mean that word seriously. If you have come this far, you deserve the pleasure of doing what at one time only Isaac Newton could do. Problem 26 offers a suggestion (just the law of cosines) but the integral is yours.

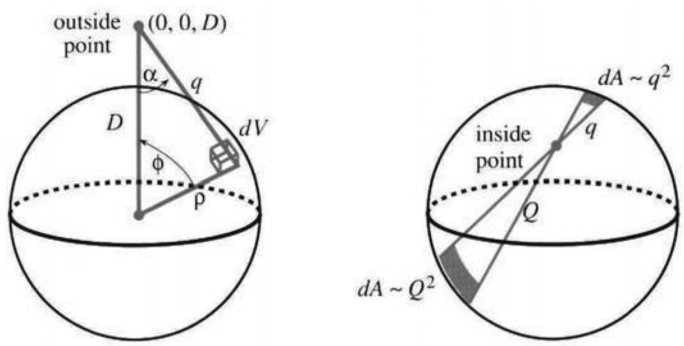

Fig. 14.18 Distance q from outside point to inside point. Distances q and Q to surface.

The law of cosines also helps with (2). For the triangle in the figure it gives $q^2 = D^2 - 2\rho D\cos\phi + \rho^2$. Call this whole quantity u. We do the surface integral first ($d\phi$ and $d\theta$ with ρ fixed). Then $q^2 = u$ and $q = \sqrt{u}$ and $du = 2\rho D\sin\phi\, d\phi$:

$$\int_0^{2\pi}\int_0^{\pi} \frac{\rho^2\ \sin\phi\ d\phi\ d\theta}{q} = \int \frac{2\pi\rho^2}{2\rho D}\frac{du}{\sqrt{u}} = \left[\frac{2\pi\rho}{D}\sqrt{u}\right]_{\phi=0}^{\phi=\pi}. \tag{4}$$

2π came from the θ integral. The integral of $du/\sqrt{u}$ is $2\sqrt{u}$. Since $\cos\phi = -1$ at the upper limit, u is $D^2 + 2\rho D + \rho^2$. *The square root of u is $D + \rho$.* At the lower limit $\cos\phi = +1$ and $u = D^2 - 2\rho D + \rho^2$. This is another perfect square—its square root is $D - \rho$. The surface integral (4) with fixed ρ is

$$\int\int \frac{dA}{q} = \frac{2\pi\rho}{D}\big[(D+\rho) - (D-\rho)\big] = \frac{4\pi\rho^2}{D}. \tag{5}$$

Last comes the ρ integral: $\int_0^R 4\pi\rho^2 d\rho/D = \frac{4}{3}\pi R^3/D$. This proves formula (2): ***potential equals volume of the sphere divided by*** D.

Note* 1** Physicists are also happy about equation (5). The average of $1/q$ is $1/D$ not only over the solid sphere but over each spherical shell of area $4\pi\rho^2$. The shells can have different densities, as they do in the Earth, and still Newton is correct. This also applies to the force integral (3)—each separate shell acts as if its mass were concentrated at the center***. Then the final ρ integral yields this property for the solid sphere.

***Note* 2** Physicists also know that force is minus the derivative of potential. The derivative of (2) with respect to D produces the force integral (3). Problem 27 explains this shortcut to Equation (3).

EXAMPLE 8 ***Everywhere inside a hollow sphere the force of gravity is zero.***

When D is smaller than ρ, the lower limit $\sqrt{u}$ in the integral (4) changes from $D-\rho$ to $\rho - D$. That way the square root stays positive. This changes the answer in (5) to $4\pi\rho^2/\rho$, so the potential no longer depends on D. *The potential is constant inside the hollow shell*. Since the force comes from its derivative, the force is zero.

A more intuitive proof is in the second figure. The infinitesimal areas on the surface are proportional to q^2 and Q^2. But the distances to those areas are q and Q, so the forces involve $1/q^2$ and $1/Q^2$ (the inverse square law). Therefore the two areas exert equal and opposite forces on the inside point, and they cancel each other. The total force from the shell is zero.

I believe this zero integral is the reason that the inside of a car is safe from lightning. Of course a car is not a sphere. But electric charge distributes itself to keep the surface at constant potential. The potential stays constant inside—therefore no force. The tires help to prevent conduction of current (and electrocution of driver).

P.S. Don't just step out of the car. Let a metal chain conduct the charge to the ground. Otherwise you could be the conductor.

CHANGE OF COORDINATES—STRETCHING FACTOR J

Once more we look to calculus for a formula. We need the volume of a small curved box in any uvw coordinate system. The $r\theta z$ box and the $\rho\phi\theta$ box have right angles, and their volumes were read off from the geometry (stretching factors $J = r$ and $J = \rho^2 \sin\phi$ in Figures 14.15 and 14.17). Now we change from xyz to other coordinates uvw—which are chosen to fit the problem.

Going from xy to uv, the area $dA = J\ du\ dv$ was a 2 by 2 determinant. In three dimensions the determinant is 3 by 3. The matrix is always the "Jacobian matrix," containing first derivatives. There were four derivatives from xy to uv, now there are nine from xyz to uvw.

14C Suppose x, y, z are given in terms of u, v, w. Then a small box in uvw space (sides du, dv, dw) comes from a volume $dV = J\ du\ dv\ dw$ in xyz space:

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \text{stretching factor}\, \frac{\partial(x,y,z)}{\partial(u,v,w)}. \qquad (6)$$

The volume integral $\iiint dx\ dy\ dz$ becomes $\iiint |J| du\ dv\ dw$, with limits on uvw.

Remember that a 3 by 3 determinant is the sum of six terms (Section 11.5). One term in J is $(\partial x/\partial u)(\partial y/\partial v)(\partial z/\partial w)$, along the main diagonal. This comes from pure stretching, and the other five terms allow for rotation. The best way to exhibit the formula is for spherical coordinates—where the nine derivatives are easy but the determinant is not:

EXAMPLE 9 Find the factor J for $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$.

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}.$$

The determinant has six terms, but two are zero—because of the zero in the corner. The other four terms are $\rho^2\sin\phi\cos^2\phi\sin^2\theta$ and $\rho^2\sin\phi\cos^2\phi\cos^2\theta$ and $\rho^2\sin^3\phi\sin^2\theta$ and $\rho^2\sin^3\phi\cos^2\theta$. Add the first two (note $\sin^2\theta+\cos^2\theta$) and separately add the second two. Then add the sums to reach $J=\rho^2\sin\phi$.

Geometry already gave this answer. For most uvw variables, use the determinant.

14.4 EXERCISES

Read-through questions

The three __a__ coordinates are $r\theta z$. The point at $x=y=z=1$ has $r=$ __b__, $\theta=$ __c__, $z=$ __d__. The volume integral is $\iiint$ __e__. The solid region $1\leqslant r\leqslant 2, 0\leqslant\theta\leqslant 2\pi, 0\leqslant z\leqslant 4$ is a __f__. Its volume is __g__. From the r and θ integrals the area of a __h__ equals __i__. From the z and θ integrals the area of a __j__ equals __k__. In $r\theta z$ coordinates the shapes of __l__ are convenient, while __m__ are not.

The three __n__ coordinates are $\rho\phi\theta$. The point at $x=y=z=1$ has $\rho=$ __o__, $\phi=$ __p__, $\theta=$ __q__. The angle ϕ is measured from __r__. θ is measured from __s__. ρ is the distance to __t__, where r was the distance to __u__. If $\rho\phi\theta$ are known then $x=$ __v__, $y=$ __w__, $z=$ __x__. The stretching factor J is a 3 by 3 __y__, and volume is $\iiint$ __z__.

The solid region $1\leqslant\rho\leqslant 2, 0\leqslant\phi\leqslant\pi, 0\leqslant\theta\leqslant 2\pi$ is a __A__. Its volume is __B__. From the ϕ and θ integrals the area of a __C__ at radius ρ equals __D__. Newton discovered that the outside gravitational attraction of a __E__ is the same as for an equal mass located at __F__.

Convert the xyz coordinates in 1–4 to $r\theta z$ and $\rho\phi\theta$.

1 $(D,0,0)$ **2** $(0,-D,0)$

3 $(0,0,D)$(watch θ) **4** $(3,4,5)$

Convert the spherical coordinates in 5–7 to xyz and $r\theta z$.

5 $\rho=4,\ \phi=\pi/4,\ \theta=-\pi/4$

6 $\rho=2,\ \phi=\pi/3,\ \theta=\pi/6$

7 $\rho=1,\ \phi=\pi,\ \theta=$ anything.

8 Where does $x=r$ and $y=\theta$?

9 Find the polar angle ϕ for the point with cylindrical coordinates $r\theta z$.

10 What are $x(t),y(t),z(t)$ on the great circle from $\rho=1, \phi=\pi/2, \theta=0$ with speed 1 to $\rho=1,\phi=\pi/4,\theta=\pi/2$?

From the limits of integration describe each region in 11–20 and find its volume. The inner integral has the inner limits.

11 $\int_{\theta=0}^{2\pi}\int_{r=0}^{1/\sqrt{2}}\int_{z=r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

12 $\int_0^{\pi}\int_0^1\int_0^{1+r^2} r\,dz\,dr\,d\theta$

13 $\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{2-z} r\,dr\,dz\,d\theta$

14 $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} r\,d\theta\,dr\,dz$

15 $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

16 $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

17 $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

18 $\int_0^{2\pi}\int_0^{\pi/4}\int_1^3 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

19 $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

20 $\int_0^1\int_0^1\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

21 Example 5 gave the volume integral for a sphere in $r\theta z$ coordinates. What is the area of the circular slice at height z ? What is the area of the cylindrical shell at radius r ? Integrate over slices (dz) and over shells (dr) to reach $4\pi R^3/3$.

22 Describe the solid with $0\leqslant\rho\leqslant 1-\cos\phi$ and find its volume.

23 A cylindrical tree has radius a. A saw cuts horizontally, ending halfway in at the x axis. Then it cuts on a sloping plane (angle α with the horizontal), also ending at the x axis. What is the volume of the wedge that falls out?

24 Find the mass of a planet of radius R, if its density at each radius ρ is $\delta = (\rho+1)/\rho$. Notice the infinite density at the center, but finite mass $M = \iiint \delta \, dV$. Here ρ is radius, not density.

25 For the cone out to $r = 1-z$, the average distance from the z axis is $\overline{r} = \frac{1}{2}$. For the triangle out to $r = 1-z$ the average is $\overline{r} = \frac{1}{3}$. How can they be different when rotating the triangle produces the cone?

Problems 26–32, on the attraction of a sphere, use Figure 14.18 and the law of cosines $q^2 = D^2 - 2\rho D \cos\phi + \rho^2 = u$.

26 ***Newton's achievement*** Show that $\iiint (\cos\alpha) dV/q^2$ equals volume$/D^2$. *One hint only*: Find $\cos\alpha$ from a second law of cosines $\rho^2 = D^2 - 2qD\cos\alpha + q^2$. The ϕ integral should involve $1/q$ and $1/q^3$. Equation (2) integrates $1/q$, leaving $\iiint dV/q^3$ still to do.

27 Compute $\partial q/\partial D$ in the first cosine law and show from Figure 14.18 that it equals $\cos\alpha$. Then the derivative of equation (2) with respect to D is a shortcut to Newton's equation (3).

28 The lines of length D and q meet at the angle α. Move the meeting point up by ΔD. Explain why the other line stretches by $\Delta q \approx \Delta D \cos\alpha$. So $\partial q/\partial D = \cos\alpha$ as before.

29 Show that the average distance is $\overline{q} = 4R/3$, from the North Pole $(D = R)$ to points on the Earth's surface $(\rho = R)$. To compute: $\overline{q} = \iint qR^2 \sin\phi \; d\phi \; d\theta/(\text{area } 4\pi R^2)$. Use the same substitution u.

30 Show as in Problem 29 that the average distance is $\overline{q} = D + \frac{1}{3}\rho^2/D$, from the outside point $(0,0,D)$ to points on the shell of radius ρ. Then integrate $\iiint q \, dV$ and divide by $4\pi R^3/3$ to find $\overline{q}$ for the solid sphere.

31 In Figure 14.18b, it is not true that the areas on the surface are exactly proportional to q^2 and Q^2. Why not? What happens to the second proof in Example 8?

32 For two solid spheres attracting each other (sun and planet), can we concentrate *both* spheres into point masses at their centers?

*__33__ Compute $\iiint \cos\alpha \; dV/q^3$ to find the force of gravity at $(0,0,D)$ from a *cylinder* $x^2 + y^2 \leqslant a^2, 0 \leqslant z \leqslant h$. Show from a figure why $q^2 = r^2 + (D-z)^2$ and $\cos\alpha = (D-z)/q$.

34 A *linear* change of variables has $x = au + bv + cw$, $y = du + ev + fw$, and $z = gu + hv + iw$. Write down the six terms in the determinant J. Three terms have minus signs.

35 A pure stretching has $x = au$, $y = bv$, and $z = cw$. Find the 3 by 3 matrix and its determinant J. What is special about the xyz box in this case?

36 (a) The matrix in Example 9 has three columns. Find the lengths of those three vectors (sum of squares, then square root). Compare with the edges of the box in Figure 14.17.

(b) Take the dot product of every column in J with every other column. Zero dot products mean right angles in the box. So J is the product of the column lengths.

37 Find the stretching factor J for cylindrical coordinates from the matrix of first derivatives.

38 Follow Problem 36 for cylindrical coordinates—find the length of each column in J and compare with the box in Figure 14.15.

39 Find the moment of inertia around the z axis of a spherical shell (radius ρ, density 1). The distance from the axis to a point on the shell is $r =$ ______. Substitute for r to find

$$I(\rho) = \int_0^{2\pi} \int_0^{\pi} r^2 \rho^2 \sin\phi \; d\phi \; d\theta.$$

Divide by mr^2 (which is $4\pi\rho^4$) to compute the number J for a hollow ball in the rolling experiment of Section 8.5.

40 The moment of inertia of a solid sphere (radius R, density 1) adds up the hollow spheres of Problem 39: $I = \int_0^R I(\rho) d\rho =$ ______. Divide by mR^2 (which is $\frac{4}{3}\pi R^5$) to find J in the rolling experiment. A solid ball rolls faster than a hollow ball because ______.

41 Inside the Earth, the force of gravity is proportional to the distance ρ from the center. Reason: The inner ball of radius ρ has mass proportional to ______ (assume constant density). The force is proportional to that mass divided by ______. The rest of the Earth (sphere with hole) exerts no force because ______.

42 Dig a tunnel through the center to Australia. Drop a ball in the tunnel at $y = R$; Australia is $y = -R$. The force of gravity is $-cy$ by Problem 41. Newton's law is $my'' = -cy$. What does the ball do when it reaches Australia?