CHAPTER 7

Techniques of Integration

Chapter 5 introduced the integral as a limit of sums. The calculation of areas was started—by hand or computer. Chapter 6 opened a different door. Its new functions e^x and $\ln x$ led to differential equations. You might say that all along we have been solving the special differential equation $df/dx = v(x)$. *The solution is* $f = \int v(x)\,dx$. But the step to $dy/dx = cy$ was a breakthrough.

The truth is that we are able to do remarkable things. ***Mathematics has a language, and you are learning to speak it***. A short time ago the symbols dy/dx and $\int v(x)\,dx$ were a mystery. (My own class was not too sure about $v(x)$ itself—the symbol for a function.) It is easy to forget how far we have come, in looking ahead to what is next.

I do want to look ahead. For integrals there are two steps to take—more functions and more applications. ***By using mathematics we make it live***. The applications are most complete when we know the integral. This short chapter will widen (very much) the range of functions we can integrate. A computer with symbolic algebra widens it more.

Up to now, integration depended on recognizing derivatives. If $v(x) = \sec^2 x$ then $f(x) = \tan x$. To integrate $\tan x$ we use a substitution:

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln u = -\ln \cos x.$$

What we need now are ***techniques for other integrals***, to change them around until we can attack them. Two examples are $\int x \cos x\,dx$ and $\int \sqrt{1-x^2}\,dx$, which are not immediately recognizable. With integration by parts, and a new substitution, they become simple.

Those examples indicate where this chapter starts and stops. With reasonable effort (and the help of tables, which is fair) you can integrate important functions. With intense effort you could integrate even more functions. In older books that extra exertion was made—it tended to dominate the course. They had integrals like $\int (x+1)\,dx/\sqrt{2x^2-6x+4}$, which we could work on if we had to. ***Our time is too valuable for that***! Like long division, the ideas are for us and their intricate elaboration is for the computer.

Integration by parts comes first. Then we do new substitutions. Partial fractions is a useful idea (already applied to the logistic equation $y' = cy - by^2$). In the last section x goes to infinity or $y(x)$ goes to infinity—but the area stays finite. These improper integrals are quite common. Chapter 8 brings the applications.

7.1 Integration by Parts

There are two major ways to manipulate integrals (with the hope of making them easier). Substitutions are based on the chain rule, and more are ahead. Here we present the other method, based on the ***product rule***. The reverse of the product rule, to find integrals not derivatives, is ***integration by parts***.

We have mentioned $\int \cos^2 x\,dx$ and $\int \ln x\,dx$. Now is the right time to compute them (plus more examples). You will see how $\int \ln x\,dx$ is exchanged for $\int 1\,dx$—a definite improvement. Also $\int xe^x\,dx$ is exchanged for $\int e^x\,dx$. The difference between the harder integral and the easier integral is a known term—that is the point.

One note before starting: Integration by parts is *not just a trick* with no meaning. On the contrary, it expresses basic physical laws of equilibrium and force balance. It is a foundation for the theory of differential equations (and even delta functions). The final paragraphs, which are completely optional, illustrate those points too.

We begin with the product rule for the derivative of $u(x)$ times $v(x)$:

$$u(x)\frac{dv}{dx}+v(x)\frac{du}{dx}=\frac{d}{dx}(u(x)v(x)). \tag{1}$$

Integrate both sides. On the right, integration brings back $u(x)v(x)$. On the left are two integrals, and one of them moves to the other side (with a minus sign):

$$\int u(x)\frac{dv}{dx}\,dx=u(x)v(x)-\int v(x)\frac{du}{dx}\,dx. \tag{2}$$

That is the key to this section—not too impressive at first, but very powerful. It is ***integration by parts*** (u and v are the parts). In practice we write it without x's:

7A The integration by parts formula is $\int u\,dv=uv-\int v\,du$. (3)

The problem of integrating $u\,dv/dx$ is changed into the problem of integrating $v\,du/dx$. There is a minus sign to remember, and there is the "integrated term" $u(x)v(x)$. In the *definite integral*, that product $u(x)v(x)$ is evaluated at the end-points a and b:

$$\int_a^b u\frac{dv}{dx}\,dx=u(b)v(b)-u(a)v(a)-\int_a^b v\frac{du}{dx}\,dx. \tag{4}$$

The key is in choosing u ***and*** v. The goal of that choice is to make $\int v\,du$ easier than $\int u\,dv$. This is best seen by examples.

EXAMPLE 1 For $\int \ln x\,dx$ choose $u=\ln x$ and $dv=dx$ (so $v=x$):

$$\int \ln x\,dx=uv-\int v\,du=x\ln x-\int x\frac{1}{x}\,dx.$$

I used the basic formula (3). Instead of working with $\ln x$ (searching for an antiderivative), we now work with the right hand side. There x times $1/x$ is 1. The integral of 1 is x. Including the minus sign and the integrated term $uv=x\ln x$ and the constant C, the answer is

$$\int \ln x\,dx=x\ln x-x+C. \tag{5}$$

For safety, ***take the derivative***. The product rule gives $\ln x+x(1/x)-1$, which is $\ln x$. The area under $y=\ln x$ from 2 to 3 is $3\ln 3-3-2\ln 2+2$.

To repeat: We exchanged the integral of $\ln x$ for the integral of 1.

EXAMPLE 2 For $\int x\cos x\,dx$ choose $u = x$ and $dv = \cos x\,dx$ (so $v(x) = \sin x$):

$$\int x\cos x\,dx = uv - \int v\,du = x\sin x - \int \sin x\,dx. \tag{6}$$

Again the right side has a simple integral, which completes the solution:

$$\int x\cos x\,dx = x\sin x + \cos x + C. \tag{7}$$

Note The new integral is not always simpler. We could have chosen $u = \cos x$ and $dv = x\,dx$. Then $v = \frac{1}{2}x^2$. Integration using those parts give the true but useless result

$$\int x\cos x\,dx = uv - \int v\,du = \tfrac{1}{2}x^2\cos x + \int \tfrac{1}{2}x^2\sin x\,dx.$$

The last integral is harder instead of easier (x^2 is worse than x). In the forward direction this is no help. But in the opposite direction it simplifies $\int \frac{1}{2}x^2\sin x\,dx$. The idea in choosing u and v is this: ***Try to give u a nice derivative and dv a nice integral***.

EXAMPLE 3 For $\int(\cos x)^2\,dx$ choose $u = \cos x$ and $dv = \cos x\,dx$ (so $v = \sin x$):

$$\int(\cos x)^2\,dx = uv - \int v\,du = \cos x\sin x + \int(\sin x)^2\,dx.$$

The integral of $(\sin x)^2$ is no better and no worse than the integral of $(\cos x)^2$. But we never see $(\sin x)^2$ without thinking of $1 - (\cos x)^2$. So substitute for $(\sin x)^2$:

$$\int(\cos x)^2\,dx = \cos x\,\sin x + \int 1\,dx - \int(\cos x)^2\,dx.$$

The last integral on the right joins its twin on the left, and $\int 1\,dx = x$:

$$2\int(\cos x)^2\,dx = \cos x\,\sin x + x.$$

Dividing by 2 gives the answer, which is definitely not $\frac{1}{3}(\cos x)^3$. Add any C:

$$\int(\cos x)^2\,dx = \tfrac{1}{2}(\cos x\,\sin x + x) + C. \tag{8}$$

Question Integrate $(\cos x)^2$ from 0 to 2π. Why should the area be π ?

Answer The definite integral is $\frac{1}{2}(\cos x\,\sin x + x)\Big]_0^{2\pi}$. This does give π. That area can also be found by common sense, starting from $(\cos x)^2 + (\sin x)^2 = 1$. The area under 1 is 2π. The areas under $(\cos x)^2$ and $(\sin x)^2$ are the same. So each one is π.

EXAMPLE 4 Evaluate $\int \tan^{-1}x\,dx$ by choosing $u = \tan^{-1}x$ and $v = x$:

$$\int \tan^{-1}x\,dx = uv - \int v\,du = x\tan^{-1}x - \int x\frac{dx}{1+x^2}. \tag{9}$$

The last integral has $w = 1 + x^2$ below and almost has $dw = 2x\,dx$ above:

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln w = \frac{1}{2}\ln(1+x^2).$$

Substituting back into (9) gives $\int \tan^{-1}x\,dx$ as $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$. All the familiar inverse functions can be integrated by parts (take $v = x$, and add "$+C$" at the end).

Our final example shows how ***two integrations by parts*** may be needed, when the first one only simplifies the problem half way.

EXAMPLE 5 For $\int x^2 e^x\,dx$ choose $u = x^2$ and $dv = e^x\,dx$ (so $v = e^x$):

$$\int x^2 e^x\,dx = uv - \int v\,du = x^2 e^x - \int e^x(2x\,dx). \tag{10}$$

The last integral involves xe^x. This is better than $x^2 e^x$, but it still needs work:

$$\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx \quad (\text{now } u = x). \tag{11}$$

Finally e^x is alone. After *two* integrations by parts, we reach $\int e^x\,dx$. In equation (11), ***the integral of*** xe^x ***is*** $xe^x - e^x$. Substituting back into (10),

$$\int x^2 e^x\,dx = x^2 e^x - 2[xe^x - e^x] + C. \tag{12}$$

These five examples are in the list of prime candidates for integration by parts:

$$x^n e^x,\ x^n \sin x,\ x^n \cos x,\ x^n \ln x,\ e^x \sin x,\ e^x \cos x,\ \sin^{-1} x,\ \tan^{-1} x,\ \ldots.$$

This concludes the presentation of the method—brief and straightforward. Figure 7.1a shows how the areas $\int u\,dv$ and $\int v\,du$ fill out the difference between the big area $u(b)v(b)$ and the smaller area $u(a)v(a)$.

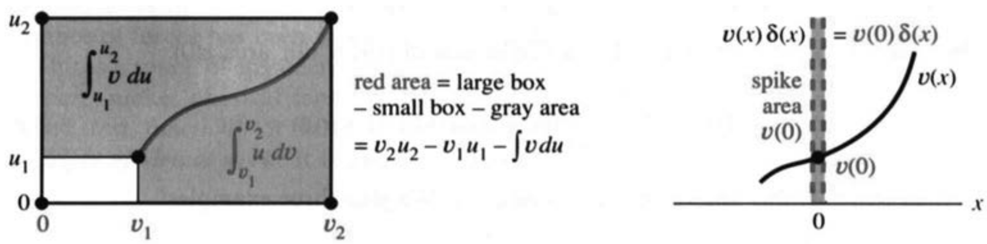


Fig. 7.1 The geometry of integration by parts. Delta function (area 1) multiplies $v(x)$ at $x = 0$.

In the movie *Stand and Deliver*, the Los Angeles teacher Jaime Escalante computed $\int x^2 \sin x\,dx$ with two integrations by parts. His success was through exercises—plus insight in choosing u and v. (Notice the difference from $\int x \sin x^2\,dx$. That falls the other way—to a substitution.) The class did extremely well on the Advanced Placement Exam. If you saw the movie, you remember that the examiner didn't believe it was possible. I spoke to him long after, and he confirms that practice was the key.

THE DELTA FUNCTION

From the most familiar functions we move to the least familiar. ***The delta function is the derivative of a step function***. The step function $U(x)$ jumps from 0 to 1 at $x = 0$. We write $\delta(x) = dU/dx$, recognizing as we do it that there is no genuine derivative at the jump. The delta function is the limit of higher and higher spikes—from the "burst of speed" in Section 1.2. They approach an infinite spike concentrated at a single point (where U jumps). This "non-function" may be unconventional—*it is certainly optional*—but it is important enough to come back to.

The slope dU/dx is zero except at $x = 0$, where the step function jumps. Thus $\delta(x) = 0$ except at that one point, where the delta function has a "spike." We cannot give a value for δ at $x = 0$, but ***we know its integral across the jump***. On every interval from $-A$ to A, the integral of dU/dx brings back U:

$$\int_{-A}^{A} \delta(x)\,dx = \int_{-A}^{A} \frac{dU}{dx}\,dx = U(x)\Big]_{-A}^{A} = 1. \tag{13}$$

"The area under the infinitely tall and infinitely thin spike $\delta(x)$ equals 1."

So far so good. The integral of $\delta(x)$ is $U(x)$. We now integrate by parts for a crucial purpose—***to find the area under*** $v(x)\delta(x)$. This is an ordinary function times the delta function. In some sense $v(x)$ times $\delta(x)$ equals $v(0)$ times $\delta(x)$—because away from $x=0$ the product is always zero. Thus $e^x\delta(x)$ equals $\delta(x)$, and $\sin x\,\delta(x)=0$.

The area under $v(x)\delta(x)$ is $v(0)$—which integration by parts will prove:

7B The integral of $v(x)$ times $\delta(x)$ is $\int_{-A}^{A} v(x)\delta(x)dx = v(0)$.

The area is $v(0)$ because the spike is multiplied by $v(0)$—the value of the smooth function $v(x)$ *at the spike*. But multiplying infinity is dangerous, to say the least. (Two times infinity is infinity). We cannot deal directly with the delta function. *It is only known by its integrals*! As long as the applications produce integrals (as they do), we can avoid the fact that δ is not a true function.

The integral of $v(x)\delta(x)=v(x)dU/dx$ is computed "by parts:"

$$\int_{-A}^{A} v(x)\delta(x)\,dx = v(x)U(x)\Big]_{-A}^{A} - \int_{-A}^{A} U(x)\frac{dv}{dx}\,dx. \tag{14}$$

Remember that $U=0$ or $U=1$. The right side of (14) is our area $v(0)$:

$$v(A)\cdot 1 - \int_0^A 1\frac{dv}{dx}\,dx = v(A)-(v(A)-v(0)) = v(0). \tag{15}$$

When $v(x)=1$, this answer matches $\int\delta\,dx=1$. We give three examples:

$$\int_{-2}^{2}\cos x\,\delta(x)\,dx=1 \qquad \int_{-5}^{6}(U(x)+\delta(x))dx=7 \qquad \int_{-1}^{1}(\delta(x))^2dx=\infty.$$

A nightmare question occurs to me. *What is the derivative of the delta function*?

INTEGRATION BY PARTS IN ENGINEERING

Physics and engineering and economics frequently involve *products*. Work is force times distance. Power is voltage times current. Income is price times quantity. When there are several forces or currents or sales, we add the products. When there are infinitely many, we integrate (probably by parts).

I start with differential equations for the displacement u at point x in a bar:

$$-\frac{dv}{dx} = f(x) \text{ with } v(x) = k\frac{du}{dx}. \tag{16}$$

This describes a hanging bar pulled down by a force $f(x)$. Each point x moves through a distance $u(x)$. The top of the bar is fixed, so $u(0)=0$. The stretching in the bar is du/dx. The internal force created by stretching is $v=k\,du/dx$. (This is Hooke's law.) Equation (16) is a ***balance of forces*** on the small piece of the bar in Figure 7.2.

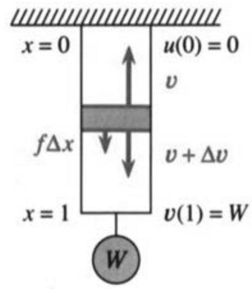


Fig. 7.2 Difference in internal force balances external force
$-\Delta v = f\Delta x$ or $-dv/dx = f(x)$
$v = W$ at $x = 1$ balances hanging weight

EXAMPLE 6 Suppose $f(x) = F$, a constant force per unit length. We can solve (16):

$$v(x) = -Fx + C \quad \text{and} \quad ku(x) = -\tfrac{1}{2}Fx^2 + Cx + D. \qquad (17)$$

The constants C and D are settled at the endpoints (as usual for integrals). At $x = 0$ we are given $u = 0$ so $D = 0$. At $x = 1$ we are given $v = W$ so $C = W + F$. Then $v(x)$ and $u(x)$ give force and displacement in the bar.

To see integration by parts, multiply $-dv/dx = f(x)$ by $u(x)$ and integrate:

$$\int_0^1 f(x)u(x)\,dx = -\int_0^1 \frac{dv}{dx}u(x)\,dx = -u(x)v(x)\Big]_0^1 + \int_0^1 v(x)\frac{du}{dx}\,dx. \qquad (18)$$

The left side is force times displacement, or *external work*. The last term is internal force times stretching—or *internal work*. The integrated term has $u(0) = 0$—the fixed support does no work. It also has $-u(1)W$, the work by the hanging weight. The balance of forces has been replaced by a ***balance of work***.

This is a touch of engineering mathematics, and here is the main point. Integration by parts makes physical sense! When $-dv/dx = f$ is multiplied by other functions—called *test functions* or virtual displacements—then equation (18) becomes ***the principle of virtual work***. It is absolutely basic to mechanics.

7.1 EXERCISES

Read-through questions

Integration by parts is the reverse of the __a__ rule. It changes $\int u\,dv$ into __b__ minus __c__. In case $u = x$ and $dv = e^{2x}dx$, it changes $\int xe^{2x}dx$ to __d__ minus __e__. The definite integral $\int_0^2 xe^{2x}dx$ becomes __f__ minus __g__.

In choosing u and dv, the __h__ of u and the __i__ of dv/dx should be as simple as possible. Normally $\ln x$ goes into __j__ and e^x goes into __k__. Prime candidates are $u = x$ or x^2 and $v = \sin x$ or __l__ or __m__. When $u = x^2$ we need __n__ integrations by parts. For $\int \sin^{-1} x\,dx$, the choice $dv = dx$ leads to __o__ minus __p__.

If U is the unit step function, $dU/dx = \delta$ is the unit __q__ function. The integral from $-A$ to A is $U(A) - U(-A) =$ __r__. The integral of $v(x)\delta(x)$ equals __s__. The integral $\int_{-1}^{1} \cos x\,\delta(x)\,dx$ equals __t__. In engineering, the balance of forces $-dv/dx = f$ is multiplied by a displacement $u(x)$ and integrated to give a balance of __u__.

Integrate 1–16, usually by parts (sometimes twice).

1 $\int x \sin x\,dx$

2 $\int x\,e^{4x}\,dx$

3 $\int x\,e^{-x}\,dx$

4 $\int x \cos 3x\,dx$

5 $\int x^2 \cos x\,dx$ (use Problem 1)

6 $\int x \ln x\,dx$

7 $\int \ln(2x+1)dx$

8 $\int x^2 e^{4x}\,dx$ (use Problem 2)

9 $\int e^x \sin x\,dx$

10 $\int e^x \cos x\,dx$

[9 and 10 need two integrations. I think e^x can be u or v.]

11 $\int e^{ax}\sin bx\,dx$

12 $\int xe^{-x^2}dx$

13 $\int \sin(\ln x)\,dx$

14 $\int \cos(\ln x)\,dx$

15 $\int (\ln x)^2 dx$

16 $\int x^2 \ln x\,dx$

17 $\int \sin^{-1} x\,dx$

18 $\int \cos^{-1}(2x)\,dx$

19 $\int x\tan^{-1} x\,dx$

20 $\int x^2 \sin x\,dx$ (from the movie)

21 $\int x^3\cos x\,dx$

22 $\int x^3 \sin x\,dx$

23 $\int x^3 e^x\,dx$

24 $\int x\sec^{-1} x\,dx$

25 $\int x\sec^2 x\,dx$

26 $\int x\cosh x\,dx$

Compute the definite integrals 27–34.

27 $\int_0^1 \ln x\,dx$

28 $\int_0^1 e^{\sqrt{x}}dx$ (let $u=\sqrt{x}$)

29 $\int_0^1 x\,e^{-2x}dx$

30 $\int_1^e \ln(x^2)dx$

31 $\int_0^\pi x\cos x\,dx$

32 $\int_{-\pi}^{\pi} x\sin x\,dx$

33 $\int_0^3 \ln(x^2+1)dx$

34 $\int_0^{\pi/2} x^2\sin x\,dx$

In 35–40 derive "reduction formulas" from higher to lower powers.

35 $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x dx$

36 $\int x^n e^{ax}\,dx =$ ______

37 $\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$

38 $\int x^n \sin x\,dx =$ ______

39 $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1}dx$

40 $\int x(\ln x)^n\,dx =$ ______

41 How would you compute $\int x\sin x\,e^x dx$ using Problem 9? Not necessary to do it.

42 How would you compute $\int x\,e^x\tan^{-1}x\,dx$? Don't do it.

43 (a) Integrate $\int x^3\sin x^2 dx$ by substitution and parts.
(b) The integral $\int x^n \sin x^2 dx$ is possible if n is ______.

44–54 are about optional topics at the end of the section.

44 For the delta function $\delta(x)$ find these integrals:
(a) $\int_{-1}^{1} e^{2x}\delta(x)dx$ (b) $\int_{-1}^{3} v(x)\delta(x)dx$ (c) $\int_2^4 \cos x\,\delta(x)dx$.

45 Solve $dy/dx = 3\delta(x)$ and $dy/dx = 3\delta(x)+y(x)$.

46 Strange fact: $\delta(2x)$ *is different from* $\delta(x)$. Integrate them both from -1 to 1.

47 The integral of $\delta(x)$ is the unit step $U(x)$. Graph the next integrals $R(x)=\int U(x)dx$ and $Q(x)=\int R(x)dx$. The ramp R and quadratic spline Q are zero at $x=0$.

48 In $\delta(x-\frac12)$, the spike shifts to $x=\frac12$. It is the derivative of the shifted step $U(x-\frac12)$. The integral of $v(x)\delta(x-\frac12)$ equals the value of v at $x=\frac12$. Compute
(a) $\int_0^1 \delta(x-\frac12)dx$; (b) $\int_0^1 e^x\delta(x-\frac12)dx$;
(c) $\int_{-1}^{1}\delta(x)\delta(x-\frac12)dx$.

49 The derivative of $\delta(x)$ is extremely singular. It is a "dipole" known by its integrals. Integrate by parts in (b) and (c):

$$\text{(a)}\int_{-1}^{1}\frac{d\delta}{dx}dx \quad \text{(b)}\int_{-1}^{1}x\frac{d\delta}{dx}dx \quad \text{(c)}\int_{-1}^{1}v(x)\frac{d\delta}{dx}dx = -v'(0).$$

50 Why is $\int_{-1}^{1}U(x)\delta(x)dx$ equal to $\frac12$? (By parts.)

51 Choose limits of integration in $v(x)=\int f(x)dx$ so that $dv/dx=-f(x)$ and $v=0$ at $x=1$.

52 Draw the graph of $v(x)$ if $v(1)=0$ and $-dv/dx=f(x)$:
(a) $f=x$; (b) $f=U(x-\frac12)$; (c) $f=\delta(x-\frac12)$.

53 What integral $u(x)$ solves $k\,du/dx=v(x)$ with end condition $u(0)=0$? Find $u(x)$ for the three v's (not f's) in Problem 52, and graph the three u's.

54 Draw the graph of $\Delta U/\Delta x = [U(x+\Delta x)-U(x)]/\Delta x$. What is the area under this graph?

Problems 55–62 need more than one integration.

55 Two integrations by parts lead to $V=$ integral of v:

$$\int uv'dx = uv - Vu' + \int Vu''dx.$$

Test this rule on $\int x^2\sin x\,dx$.

56 After n integrations by parts, $\int u(dv/dx)dx$ becomes

$$uv - u^{(1)}v_{(1)} + u^{(2)}v_{(2)} - \cdots + (-1)^n\int u^{(n)}v_{(n-1)}dx.$$

$u^{(n)}$ is the nth derivative of u, and $v_{(n)}$ is the nth integral of v. Integrate the last term by parts to stretch this formula to $n+1$ integrations.

57 Use Problem 56 to find $\int x^3 e^x dx$.

58 From $f(x)-f(0)=\int_0^x f'(t)dt$, integrate by parts (notice dt not dx) to reach $f(x)=f(0)+f'(0)x+\int_0^x f''(t)(x-t)dt$. Continuing as in Problem 56 produces *Taylor's formula*:

$$f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\cdots+\int_0^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt.$$

59 What is the difference between $\int_0^1 uw''dx$ and $\int_0^1 u''w\,dx$?

60 Compute the areas $A=\int_1^e \ln x\,dx$ and $B=\int_0^1 e^y dy$. Mark them on the rectangle with corners $(0,0),(e,0),(e,1),(0,1)$.

61 ***Find the mistake.*** I don't believe $e^x\cosh x = e^x\sinh x$:

$$\begin{aligned}\int e^x\sinh x\,dx &= e^x\cosh x - \int e^x\cosh x\,dx\\ &= e^x\cosh x - e^x\sinh x + \int e^x\sinh x\,dx.\end{aligned}$$

62 Choose C and D to make the derivative of $C\,e^{ax}\cos bx + D\,e^{ax}\sin bx$ equal to $e^{ax}\cos bx$. Is this easier than integrating $e^{ax}\cos bx$ twice by parts?

7.2 Trigonometric Integrals

The next section will put old integrals into new forms. For example $\int x^2\sqrt{1-x^2}\,dx$ will become $\int \sin^2\theta\cos^2\theta\,d\theta$. That looks simpler because the square root is gone. But still $\sin^2\theta\cos^2\theta$ has to be integrated. ***This brief section integrates any product of sines and cosines and secants and tangents***.

There are two methods to choose from. One uses integration by parts, the other is based on trigonometric identities. Both methods try to make the integral easy (but that may take time). We follow convention by changing the letter θ back to x.

Notice that $\sin^4 x\,\cos x\,dx$ *is easy to integrate*. It is $u^4\,du$. This is the goal in Example 1—to separate out $\cos x\,dx$. It becomes du, and $\sin x$ is u.

EXAMPLE 1 $\int \sin^2 x\cos^3 x\,dx$ (the exponent 3 is *odd*)

Solution Keep $\cos x\,dx$ as du. Convert the other $\cos^2 x$ to $1-\sin^2 x$:

$$\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x(1-\sin^2 x)\cos x\,dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

EXAMPLE 2 $\int \sin^5 x\,dx$ (the exponent 5 is *odd*)

Solution Keep $\sin x\,dx$ and convert everything else to cosines. ***The conversion is always based on*** $\sin^2 x + \cos^2 x = 1$:

$$\int (1-\cos^2 x)^2 \sin x\,dx = \int (1-2\cos^2 x + \cos^4 x)\sin x\,dx.$$

Now $\cos x$ is u and $-\sin x\,dx$ is du. We have $\int(-1+2u^2-u^4)\,du$.

General method for $\int \sin^m x\cos^n x\,dx$, ***when*** m ***or*** n ***is odd***

If n is odd, separate out a single $\cos x\,dx$. That leaves an even number of cosines.
Convert them to sines. Then $\cos x\,dx$ is du and the sines are u's.
If m is odd, separate out a single $\sin x\,dx$ as du. Convert the rest to cosines.
If m and n are both odd, use either method.
If m and n are both even, a new method is needed. Here are two examples.

EXAMPLE 3 $\int \cos^2 x\,dx \quad (m=0,\ n=2,\ \text{both even})$

There are two good ways to integrate $\cos^2 x$, but substitution is not one of them. If u equals $\cos x$, then du is not here. The successful methods are integration by parts and double-angle formulas. Both answers are in equation (2) below—I don't see either one as the obvious winner.

Integrating $\cos^2 x$ by parts was Example 3 of Section 7.1. The other approach, by double angles, is based on these formulas from trigonometry:

$$\cos^2 x = \tfrac{1}{2}\big(1+\cos 2x\big) \qquad \sin^2 x = \tfrac{1}{2}\big(1-\cos 2x\big) \tag{1}$$

The integral of $\cos 2x$ is $\frac{1}{2}\sin 2x$. So these formulas can be integrated directly. They give the only integrals you should memorize—either the integration by parts form, or the result from these double angles:

$$\int \cos^2 x\,dx \ \textbf{\textit{equals}}\ \ \tfrac{1}{2}\big(x+\sin x\,\cos x\big)\ \ \textbf{\textit{or}}\ \ \tfrac{1}{2}x+\tfrac{1}{4}\sin 2x \quad (\text{plus } C). \tag{2}$$

$$\int \sin^2 x\,dx \ \textbf{\textit{equals}}\ \ \tfrac{1}{2}\big(x-\sin x\,\cos x\big)\ \ \textbf{\textit{or}}\ \ \tfrac{1}{2}x-\tfrac{1}{4}\sin 2x \quad (\text{plus } C). \tag{3}$$

EXAMPLE 4 $\int \cos^4 x\,dx \quad (m=0, n=4$, both are even)

Changing $\cos^2 x$ to $1-\sin^2 x$ gets us nowhere. All exponents stay even. Substituting $u=\sin x$ won't simplify $\sin^4 x\,dx$, without du. Integrate by parts or switch to $2x$.

First solution Integrate by parts. Take $u=\cos^3 x$ and $dv=\cos x\,dx$:

$$\int(\cos^3 x)(\cos x\,dx)=uv-\int v\,du=\cos^3 x\,\sin x-\int(\sin x)(-3\cos^2 x\sin x\,dx).$$

The last integral has even powers $\sin^2 x$ and $\cos^2 x$. This looks like no progress. Replacing $\sin^2 x$ by $1-\cos^2 x$ produces $\cos^4 x$ on the right-hand side also:

$$\int \cos^4 x\,dx=\cos^3 x\sin x+3\int \cos^2 x(1-\cos^2 x)dx.$$

Always even powers in the integrals. But now move $3\int \cos^4 x\,dx$ to the left side:

Reduction $$4\int \cos^4 x\,dx=\cos^3 x\sin x+3\int \cos^2 x\,dx. \tag{4}$$

Partial success—the problem is *reduced* from $\cos^4 x$ to $\cos^2 x$. Still an even power, but a lower power. The integral of $\cos^2 x$ is already known. Use it in equation (4):

$$\int \cos^4 x\;dx=\tfrac{1}{4}\cos^3 x\sin x+\tfrac{3}{4}\cdot\tfrac{1}{2}\big(x+\sin x\,\cos x\big)+C. \tag{5}$$

Second solution Substitute the double-angle formula $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x$:

$$\int \cos^4 x\;dx=\int(\tfrac{1}{2}+\tfrac{1}{2}\cos 2x)^2dx=\tfrac{1}{4}\int(1+2\cos 2x+\cos^2 2x)dx.$$

Certainly $\int dx=x$. Also $2\int \cos 2x\;dx=\sin 2x$. That leaves the cosine squared:

$$\int \cos^2 2x=\int \tfrac{1}{2}(1+\cos 4x)dx=\tfrac{1}{2}x+\tfrac{1}{8}\sin 4x+C.$$

The integral of $\cos^4 x$ using double angles is

$$\tfrac{1}{4}\left[x+\sin 2x+\tfrac{1}{2}x+\tfrac{1}{8}\sin 4x\right]+C. \tag{6}$$

That solution looks different from equation (5), but it can't be. There all angles were x, here we have $2x$ and $4x$. We went from $\cos^4 x$ to $\cos^2 2x$ to $\cos 4x$, which was integrated immediately. The powers were cut in half as the angle was doubled.

Double-angle method for $\int \sin^m x\cos^n x\;dx$, ***when*** m ***and*** n ***are even***.

Replace $\sin^2 x$ by $\frac{1}{2}(1-\cos 2x)$ and $\cos^2 x$ by $\frac{1}{2}(1+\cos 2x)$. The exponents drop to $m/2$ and $n/2$. If those are even, repeat the idea ($2x$ goes to $4x$). If $m/2$ or $n/2$ is odd, switch to the "general method" using substitution. With an odd power, we have du.

EXAMPLE 5 (Double angle) $\int \sin^2 x\cos^2 x\;dx=\int \frac{1}{4}(1-\cos 2x)(1+\cos 2x)dx.$

This leaves $1-\cos^2 2x$ in the last integral. That is familiar but not necessarily easy. We can look it up (safest) or remember it (quickest) or use double angles again:

$$\frac{1}{4}\int(1-\cos^2 2x)dx=\frac{1}{4}\int\left(1-\frac{1}{2}-\frac{1}{2}\cos 4x\right)dx=\frac{x}{8}-\frac{\sin 4x}{32}+C.$$

Conclusion ***Every*** $\sin^m x\cos^n x$ ***can be integrated***. This includes negative m and n— see tangents and secants below. Symbolic codes like MACSYMA or *Mathematica* give the answer directly. Do they use double angles or integration by parts ?

You may prefer the answer from integration by parts (I usually do). It avoids $2x$ and $4x$. But it makes no sense to go through every step every time. Either a computer does the algebra, or we use a "reduction formula" from n to $n-2$:

Reduction $$n\int \cos^n x\, dx = \cos^{n-1}x\,\sin x + (n-1)\int \cos^{n-2}x\, dx. \qquad (7)$$

For $n=2$ this is $\int \cos^2 x\, dx$—the integral to learn. For $n=4$ the reduction produces $\cos^2 x$. The integral of $\cos^6 x$ goes to $\cos^4 x$. There are similar reduction formulas for $\sin^m x$ and also for $\sin^m x \cos^n x$. I don't see a good reason to memorize them.

INTEGRALS WITH ANGLES px AND qx

Instead of $\sin^8 x$ times $\cos^6 x$, suppose you have $\sin 8x$ times $\cos 6x$. How do you integrate? Separately a sine and cosine are easy. The new question is *the integral of the product*:

EXAMPLE 6 ***Find*** $\int_0^{2\pi} \sin 8x\,\cos 6x\, dx$. ***More generally find*** $\int_0^{2\pi} \sin px\,\cos qx\, dx$.

This is not for the sake of making up new problems. I believe these are the most important definite integrals in this chapter (p and q are $0, 1, 2, \ldots$). They may be the most important in all of mathematics, especially because the question has such a beautiful answer. ***The integrals are zero***. On that fact rests the success of Fourier series, and the whole industry of signal processing.

One approach (the slow way) is to replace $\sin 8x$ and $\cos 6x$ by powers of cosines. That involves $\cos^{14} x$. The integration is not fun. A better approach, which applies to all angles px and qx, is to use the identity

$$\sin px\,\cos qx = \tfrac{1}{2}\sin(p+q)x + \tfrac{1}{2}\sin(p-q)x. \qquad (8)$$

Thus $\sin 8x\,\cos 6x = \frac{1}{2}\sin 14x + \frac{1}{2}\sin 2x$. Separated like that, sines are easy to integrate:

$$\int_0^{2\pi} \sin 8x\,\cos 6x\, dx = \left[-\frac{1}{2}\frac{\cos 14x}{14} - \frac{1}{2}\frac{\cos 2x}{2}\right]_0^{2\pi} = 0.$$

Since $\cos 14x$ is periodic, it has the same value at 0 and 2π. Subtraction gives zero. The same is true for $\cos 2x$. The integral of sine times cosine is always zero over a complete period (like 0 to 2π).

What about $\sin px \sin qx$ and $\cos px\,\cos qx$? Their integrals are also zero, *provided p is different from q*. When $p=q$ we have a perfect square. There is no negative area to cancel the positive area. The integral of $\cos^2 px$ or $\sin^2 px$ is π.

EXAMPLE 7 $\int_0^{2\pi} \sin 8x \sin 7x\, dx = 0$ and $\int_0^{2\pi} \sin^2 8x\, dx = \pi$.

With two sines or two cosines (instead of sine times cosine), we go back to the addition formulas of Section 1.5. Problem 24 derives these formulas:

$$\sin px\,\sin qx = -\tfrac{1}{2}\cos(p+q)x + \tfrac{1}{2}\cos(p-q)x \qquad (9)$$

$$\cos px\,\cos qx = \tfrac{1}{2}\cos(p+q)x + \tfrac{1}{2}\cos(p-q)x. \qquad (10)$$

With $p=8$ and $q=7$, we get $\cos 15x$ and $\cos x$. Their definite integrals are zero. With $p=8$ and $q=8$, we get $\cos 16x$ and $\cos 0x$ (which is 1). Formulas (9) and (10) also give a factor $\frac{1}{2}$. The integral of $\frac{1}{2}$ is π:

$$\int_0^{2\pi} \sin 8x \, \sin 7x \; dx = -\tfrac{1}{2}\int_0^{2\pi} \cos 15x \; dx + \tfrac{1}{2}\int_0^{2\pi} \cos \quad x \; dx = 0 + 0$$
$$\int_0^{2\pi} \sin 8x \, \sin 8x \; dx = -\tfrac{1}{2}\int_0^{2\pi} \cos 16x \; dx + \tfrac{1}{2}\int_0^{2\pi} \cos 0x \; dx = 0 + \pi$$

The answer zero is memorable. The answer π appears constantly in Fourier series. No ordinary numbers are seen in these integrals. The case $p = q = 1$ brings back $\int \cos^2 x \; dx = \frac{1}{2} + \frac{1}{4}\sin 2x$.

SECANTS AND TANGENTS

When we allow *negative* powers m and n, the main fact remains true. All integrals $\int \sin^m x \cos^n x \; dx$ can be expressed by known functions. The novelty for negative powers is that *logarithms appear*. That happens right at the start, for $\sin x / \cos x$ and for $1/\cos x$ (tangent and secant):

$$\int \tan x \; dx = -\int du/u = -\ln|\cos x| \qquad \text{(here } u = \cos x\text{)}$$
$$\int \sec x \; dx = \int du/u = \ln|\sec x + \tan x| \qquad \text{(here } u = \sec x + \tan x\text{).}$$

For higher powers there is one key identity: $1 + \tan^2 x = \sec^2 x$. That is the old identity $\cos^2 x + \sin^2 x = 1$ in disguise (just divide by $\cos^2 x$). We switch tangents to secants just as we switched sines to cosines. Since $(\tan x)' = \sec^2 x$ and $(\sec x)' = \sec x \, \tan x$, nothing else comes in.

EXAMPLE 8 $\int \tan^2 x \; dx = \int (\sec^2 x - 1) dx = \tan x - x + C$.

EXAMPLE 9 $\int \tan^3 x \; dx = \int \tan x (\sec^2 x - 1) dx$.

The first integral on the right is $\int u \; du = \frac{1}{2}u^2$, with $u = \tan x$. The last integral is $-\int \tan x \; dx$. The complete answer is $\frac{1}{2}(\tan x)^2 + \ln|\cos x| + C$. By taking absolute values, a negative cosine is also allowed. Avoid $\cos x = 0$.

EXAMPLE 10 ***Reduction*** $$\int (\tan x)^m dx = \frac{(\tan x)^{m-1}}{m-1} - \int (\tan x)^{m-2} dx$$

Same idea—separate off $(\tan x)^2$ as $\sec^2 x - 1$. Then integrate $(\tan x)^{m-2}\sec^2 x \; dx$, which is $u^{m-2} du$. This leaves the integral on the right, with the exponent lowered by 2. Every power $(\tan x)^m$ is eventually reduced to Example 8 or 9.

EXAMPLE 11 $\int \sec^3 x \; dx = uv - \int v \; du = \sec x \, \tan x - \int \tan^2 x \, \sec x \; dx$

This was integration by parts, with $u = \sec x$ and $v = \tan x$. In the integral on the right, replace $\tan^2 x$ by $\sec^2 x - 1$ (this identity is basic):

$$\int \sec^3 x \; dx = \sec x \, \tan x - \int \sec^3 x \; dx + \int \sec x \; dx.$$

Bring $\int \sec^3 x \; dx$ to the left side. That reduces the problem from $\sec^3 x$ to $\sec x$.

I believe those examples make the point—***trigonometric integrals are computable***. Every product $\tan^m x \sec^n x$ can be reduced to one of these examples. If n is even we substitute $u = \tan x$. If m is odd we set $u = \sec x$. If m is even and n is odd, use a reduction formula (and always use $\tan^2 x = \sec^2 x - 1$.)

I mention very briefly a completely different substitution $u = \tan \frac{1}{2}x$. This seems to all students and instructors (quite correctly) to come out of the blue:

$$\sin x = \frac{2u}{1+u^2} \qquad \text{and} \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad \text{and} \qquad dx = \frac{2du}{1+u^2}. \tag{11}$$

The x-integral can involve sums as well as products—not only $\sin^m x \cos^n x$ but also $1/(5+\sin x - \tan x)$. (No square roots.) The u-integral is a *ratio of ordinary polynomials*. It is done by *partial fractions*.

Application of $\int \sec x \, dx$ *to distance on a map* (Mercator projection)

The strange integral $\ln(\sec x + \tan x)$ has an everyday application. It measures the distance from the equator to latitude x, on a Mercator map of the world.

All mapmakers face the impossibility of putting part of a sphere onto a flat page. You can't preserve distances, when an orange peel is flattened. But angles can be preserved, and Mercator found a way to do it. His map came before Newton and Leibniz. Amazingly, and accidentally, somebody matched distances on the map with a table of logarithms—and discovered $\int \sec x \, dx$ *before calculus*. You would not be surprised to meet $\sin x$, but who would recognize $\ln(\sec x + \tan x)$?

The map starts with strips at all latitudes x. The heights are dx, the lengths are proportional to $\cos x$. ***We stretch the strips by*** $1/\cos x$—then Figure 7.3c lines up evenly on the page. When dx is also divided by $\cos x$, angles are preserved—a small

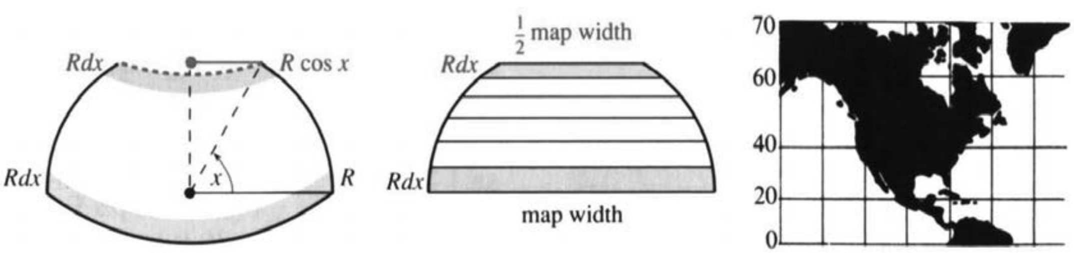


Fig. 7.3 Strips at latitude x are scaled by $\sec x$, making Greenland too large.

square becomes a bigger square. The distance north adds up the strip heights $dx/\cos x$. This gives $\int \sec x \, dx$.

The distance to the North Pole is infinite! Close to the Pole, maps are stretched totally out of shape. When sailors wanted to go from A to B at a constant angle with the North Star, they looked on Mercator's map to find the angle.

7.2 EXERCISES

Read-through questions

To integrate $\sin^4 x \cos^3 x$, replace $\cos^2 x$ by ___a___. Then $(\sin^4 x - \sin^6 x)\cos x \, dx$ is ___b___ du. In terms of $u = \sin x$ the integral is ___c___. This idea works for $\sin^m x \cos^n x$ if either m or n is ___d___.

If both m and n are ___e___, one method is integration by ___f___. For $\int \sin^4 x \, dx$, split off $dv = \sin x \, dx$. Then $-\int v \, du$ is ___g___. Replacing $\cos^2 x$ by ___h___ creates a new $\sin^4 x \, dx$ that combines with the original one. The result is a *reduction* to $\int \sin^2 x \, dx$, which is known to equal ___i___.

The second method uses the double-angle formula $\sin^2 x =$ ___j___. Then $\sin^4 x$ involves $\cos^2$ ___k___. Another doubling comes from $\cos^2 2x =$ ___l___. The integral contains the sine of ___m___.

To integrate $\sin 6x \cos 4x$, rewrite it as $\frac{1}{2}\sin 10x +$ __n__. The indefinite integral is __o__. The definite integral from 0 to 2π is __p__. The product $\cos px \cos qx$ is written as $\frac{1}{2}\cos(p+q)x+$ __q__. Its integral is also zero, except if __r__ when the answer is __s__.

With $u = \tan x$, the integral of $\tan^9 x \sec^2 x$ is __t__. Similarly $\int \sec^9 x \ (\sec x \tan x \ dx) =$ __u__. For the combination $\tan^m x \sec^n x$ we apply the identity $\tan^2 x =$ __v__. After reduction we may need $\int \tan x \ dx =$ __w__ and $\int \sec x \ dx =$ __x__.

Compute 1–8 by the "general method," when m or n is odd.

1 $\int \sin^3 x \ dx$

2 $\int \cos^3 x \ dx$

3 $\int \sin x \cos x \ dx$

4 $\int \cos^5 x \ dx$

5 $\int \sin^5 x \cos^2 x \ dx$

6 $\int \sin^3 x \cos^3 x \ dx$

7 $\int \sqrt{\sin x} \cos x \ dx$

8 $\int \sqrt{\sin x} \cos^3 x \ dx$

9 Repeat Problem 6 starting with $\sin x \cos x = \frac{1}{2}\sin 2x$.

10 Find $\int \sin^2 ax \cos ax \ dx$ and $\int \sin ax \cos ax \ dx$.

In 11–16 use the double-angle formulas (m, n even).

11 $\int_0^\pi \sin^2 x \ dx$

12 $\int_0^\pi \sin^4 x \ dx$

13 $\int \cos^2 3x \ dx$

14 $\int \sin^2 x \cos^2 x \ dx$

15 $\int \sin^2 x \ dx + \int \cos^2 x \ dx$

16 $\int \sin^2 x \cos^2 2x \ dx$

17 Use the reduction formula (7) to integrate $\cos^6 x$.

18 For $n > 1$ use formula (7) to prove

$$\int_0^{\pi/2} \cos^n x \ dx = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} x \ dx.$$

19 For $n = 2, 4, 6, \ldots$ deduce from Problem 18 that

$$\int_0^{\pi/2} \cos^n x \ dx = \frac{\pi}{2} \frac{(1)(3)\cdots(n-1)}{(2)(4)\cdots(n)}$$

20 For $n = 3, 5, 7, \ldots$ deduce from Problem 18 that

$$\int_0^{\pi/2} \cos^n x \ dx = \frac{(2)(4)\cdots(n-1)}{(3)(5)\cdots(n)}$$

21 (a) Separate $dv = \sin x \ dx$ from $u = \sin^{n-1} x$ and integrate $\int \sin^n x \ dx$ by parts.

(b) Substitute $1 - \sin^2 x$ for $\cos^2 x$ to find a reduction formula like equation (7).

22 For which n does symmetry give $\int_0^\pi \cos^n x \ dx = 0$?

23 Are the integrals (a)–(f) positive, negative, or zero?

(a) $\int_0^\pi \cos 3x \sin 3x \ dx$

(b) $\int_0^\pi \cos x \sin 2x \ dx$

(c) $\int_{-2\pi}^0 \cos x \sin x \ dx$

(d) $\int_0^\pi (\cos^2 x - \sin^2 x) \ dx$

(e) $\int_\pi^{3\pi} \cos px \sin qx \ dx$

(f) $\int_\pi^0 \cos^4 x \ dx$

24 Write down equation (9) for $p = q = 1$, and (10) for $p = 2$, $q = 1$. Derive (9) from the addition formulas for $\cos(s+t)$ and $\cos(s-t)$ in Section 1.5.

In 25–32 compute the indefinite integrals first, then the definite integrals.

25 $\int_0^{2\pi} \cos x \sin 2x \ dx$

26 $\int_0^\pi \sin 3x \sin 5x \ dx$

27 $\int_0^\pi \cos 99x \sin 101x \ dx$

28 $\int_{-\pi}^\pi \cos^2 3x \ dx$

29 $\int_0^\pi \cos 99x \cos 101x \ dx$

30 $\int_0^{2\pi} \sin x \sin 2x \sin 3x \ dx$

31 $\int_0^{4\pi} \cos x/2 \sin x/2 \ dx$

32 $\int_0^\pi x \cos x \ dx$ (by parts)

33 Suppose a *Fourier sine series* $A \sin x + B \sin 2x + C \sin 3x + \cdots$ adds up to x on the interval from 0 to π. Find A by multiplying all those functions (including x) by $\sin x$ and integrating from 0 to π. (B and C will disappear.)

34 Suppose a Fourier sine series $A \sin x + B \sin 2x + C \sin 3x + \cdots$ adds up to 1 on the interval from 0 to π. Find C by multiplying all functions (including 1) by $\sin 3x$ and integrating from 0 to π. (A and B will disappear.)

35 In 33, the series also equals x from $-\pi$ to 0, because all functions are odd. Sketch the "sawtooth function," which equals x from $-\pi$ to π and then has period 2π. What is the sum of the sine series at $x = \pi$?

36 In 34, the series equals -1 from $-\pi$ to 0, because sines are odd functions. Sketch the "square wave," which is alternately -1 and $+1$, and find A and B.

37 The area under $y = \sin x$ from 0 to π is positive. Which frequencies p have $\int_0^\pi \sin px \ dx = 0$?

38 Which frequencies q have $\int_0^\pi \cos qx \ dx = 0$?

39 For which p, q is $\int_0^\pi \sin px \cos qx \ dx = 0$?

40 Show that $\int_0^\pi \sin px \sin qx \ dx$ is always zero.

Compute the indefinite integrals 41–52.

41 $\int \sec x \tan x \ dx$

42 $\int \tan 5x \ dx$

43 $\int \tan^2 x \sec^2 x \ dx$

44 $\int \tan^2 x \sec x \ dx$

45 $\int \tan x \sec^3 x \ dx$

46 $\int \sec^4 x \ dx$

47 $\int \tan^4 x \ dx$

48 $\int \tan^5 x \ dx$

49 $\int \cot x \ dx$

50 $\int \csc x \ dx$

51 $\displaystyle\int \frac{\sin x}{\cos^3 x} \ dx$

52 $\displaystyle\int \frac{\sin^6 x}{\cos^3 x} \ dx$

53 Choose A so that $\cos x - \sin x = A\cos(x + \pi/4)$. Then integrate $1/(\cos x - \sin x)$.

54 Choose A so that $\cos x - \sqrt{3}\sin x = A\cos(x + \pi/3)$. Then integrate $1/(\cos x - \sqrt{3}\sin x)^2$.

55 Evaluate $\int_0^{2\pi} |\cos x - \sin x| \ dx$.

56 Show that $a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x-\alpha)$ and find the correct *phase angle* α.

57 If a square Mercator map shows 1000 miles at latitude $30°$, how many miles does it show at latitude $60°$?

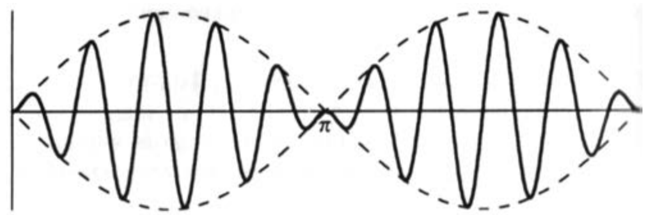
58 When lengths are scaled by $\sec x$, area is scaled by ______. Why is the area from the equator to latitude x proportional to $\tan x$?

59 Use substitution (11) to find $\int dx/(1+\cos x)$.

60 Explain from areas why $\int_0^\pi \sin^2 x\, dx = \int_0^\pi \cos^2 x\, dx$. These integrals add to $\int_0^\pi 1\, dx$, so they both equal ______.

61 What product $\sin px \sin qx$ is graphed below? Check that $(p\cos px \sin qx - q\sin px \cos qx)/(q^2-p^2)$ has this derivative.

62 Finish $\int \sec^3 x\, dx$ in Example 11. This is needed for the length of a parabola and a spiral (Problem 7.3.8 and Sections 8.2 and 9.3).



7.3 Trigonometric Substitutions

The most powerful tool we have, for integrating with pencil and paper and brain, is the *method of substitution*. To make it work, we have to think of good substitutions—which make the integral simpler. This section concentrates on the single most valuable collection of substitutions. They are the only ones you should memorize, and two examples are given immediately.

To integrate $\sqrt{1-x^2}$, ***substitute*** $x=\sin\theta$. ***Do not set*** $u=1-x^2$ $\left(\dfrac{du}{dx}\text{ is missing}\right)$

$$\int\sqrt{1-x^2}\,dx\to\int(\cos\theta)(\cos\theta\,d\theta)\qquad\int\frac{dx}{\sqrt{1-x^2}}\to\int\frac{\cos\theta\,d\theta}{\cos\theta}$$

The expression $\sqrt{1-x^2}$ is awkward as a function of x. It becomes graceful as a function of θ. We are practically invited to use the equation $1-(\sin\theta)^2=(\cos\theta)^2$. Then the square root is simply $\cos\theta$—provided this cosine is positive.

Notice the change in dx. *When* x *is* $\sin\theta$, dx *is* $\cos\theta\,d\theta$. Figure 7.4a shows the original area with new letters. Figure 7.4b shows an equal area, after rewriting $\int(\cos\theta)(\cos\theta\,d\theta)$ as $\int(\cos^2\theta)\,d\theta$. Changing from x to θ gives a new height and a new base. There is no change in area—that is the point of substitution.

To put it bluntly: If we go from $\sqrt{1-x^2}$ to $\cos\theta$, and forget the difference between dx and $d\theta$, and just compute $\int\cos\theta\,d\theta$, the answer is totally wrong.

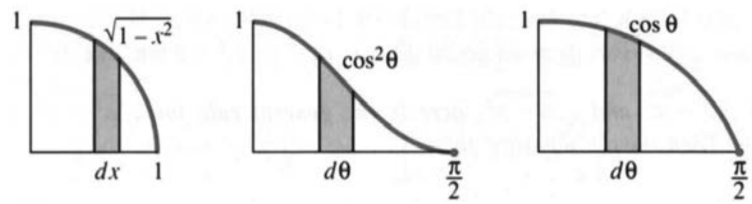


Fig. 7.4 Same area for $\sqrt{1-x^2}\,dx$ and $\cos^2\theta\,d\theta$. Third area is wrong: $dx\neq d\theta$

We still need the integral of $\cos^2\theta$. This was Example 3 of integration by parts, and also equation 7.2.6. It is worth memorizing. The example shows this θ integral, and returns to x:

EXAMPLE 1 $\int\cos^2\theta\,d\theta=\frac{1}{2}\sin\theta\cos\theta+\frac{1}{2}\theta$ is after substitution

$$\int\sqrt{1-x^2}\,dx=\tfrac{1}{2}x\sqrt{1-x^2}+\tfrac{1}{2}\sin^{-1}x\text{ is the original problem.}$$

We changed $\sin\theta$ back to x and $\cos\theta$ to $\sqrt{1-x^2}$. *Notice that* θ *is* $\sin^{-1}x$. The answer is trickier than you might expect for the area under a circular arc. Figure 7.5 shows how the two pieces of the integral are the areas of a pie-shaped wedge and a triangle.

EXAMPLE 2 $$\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{\cos\theta\,d\theta}{\cos\theta}=\theta+C=\sin^{-1}x+C.$$

Remember: We already know $\sin^{-1}x$. Its derivative $1/\sqrt{1-x^2}$ was computed in Section 4.4. That solves the example. But instead of matching this special problem

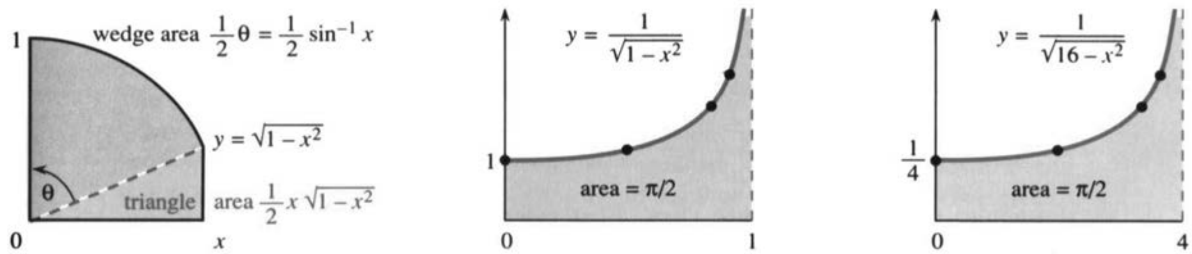


Fig. 7.5 $\int \sqrt{1-x^2}\,dx$ is a sum of simpler areas. Infinite graph but finite area.

with a memory from Chapter 4, the substitution $x = \sin\theta$ makes the solution automatic. From $\int d\theta = \theta$ we go back to $\sin^{-1} x$.

The rest of this section is about other substitutions. They are more complicated than $x = \sin\theta$ (but closely related). A table will display the three main choices—$\sin\theta, \tan\theta, \sec\theta$—and their uses.

TRIGONOMETRIC SUBSTITUTIONS

After working with $\sqrt{1-x^2}$, the next step is $\sqrt{4-x^2}$. The change $x = \sin\theta$ simplified the first, but it does nothing for the second: $4 - \sin^2\theta$ is not familiar. Nevertheless a factor of 2 makes everything work. Instead of $x = \sin\theta$, ***the idea is to substitute*** $x = 2\sin\theta$:

$$\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta \quad \text{and} \quad dx = 2\cos\theta\, d\theta.$$

Notice both 2's. The integral is $4\int \cos^2\theta\, d\theta = 2\sin\theta\cos\theta + 2\theta$. But watch closely. This is not 4 times the previous $\int \cos^2\theta\, d\theta$! *Since x is $2\sin\theta$, θ is now* $\sin^{-1}(x/2)$.

EXAMPLE 3 $\int \sqrt{4-x^2}\,dx = 4\int \cos^2\theta\, d\theta = x\sqrt{1-(x/2)^2} + 2\sin^{-1}(x/2)$.

Based on $\sqrt{1-x^2}$ ***and*** $\sqrt{4-x^2}$, ***here is the general rule for*** $\sqrt{a^2-x^2}$. ***Substitute*** $x = a\sin\theta$. ***Then the*** a***'s separate out***:

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta \quad \text{and} \quad dx = a\cos\theta\, d\theta.$$

That is the automatic substitution to try, whenever the square root appears.

EXAMPLE 4 $$\int_{x=0}^{4} \frac{dx}{\sqrt{16-x^2}} = \int_{\theta=0}^{\pi/2} \frac{4\cos\theta\, d\theta}{\sqrt{4^2 - 4^2(\sin\theta)^2}} = \int_{\theta=0}^{\pi/2} d\theta = \frac{\pi}{2}.$$

Here $a^2 = 16$. Then $a = 4$ and $x = 4\sin\theta$. The integral has $4\cos\theta$ above and below, so it is $\int d\theta$. The antiderivative is just θ. For the definite integral notice that $x = 4$ means $\sin\theta = 1$, and this means $\theta = \pi/2$.

A table of integrals would hide that substitution. The table only gives $\sin^{-1}(x/4)$. There is no mention of $\int d\theta = \theta$. *But what if $16 - x^2$ changes to $x^2 - 16$?*

EXAMPLE 5 $$\int_{x=4}^{8} \frac{dx}{\sqrt{x^2-16}} = ?$$

Notice the two changes—the sign in the square root and the limits on x. Example 4 stayed *inside* the interval $|x| \leqslant 4$, where $16 - x^2$ has a square root. Example 5 stays *outside*, where x^2-16 has a square root. The new problem cannot use $x=4\sin\theta$, because we don't want the square root of $-\cos^2\theta$.

The new substitution is $x=4\sec\theta$. ***This turns the square root into*** $4\tan\theta$:

$x=4\sec\theta$ gives $dx=4\sec\theta\tan\theta\,d\theta$ and $x^2-16=16\sec^2\theta-16=16\tan^2\theta$.

This substitution solves the example, when the limits are changed to θ:

$$\int_0^{\pi/3}\frac{4\sec\theta\tan d\theta}{4\tan\theta}=\int_0^{\pi/3}\sec\theta\,d\theta=\ln(\sec\theta+\tan\theta)\Big]_0^{\pi/3}=\ln(2+\sqrt{3}).$$

I want to emphasize the three steps. First came the substitution $x=4\sec\theta$. An unrecognizable integral became $\int\sec\theta\,d\theta$. Second came the new limits ($\theta=0$ when $x=4$, $\theta=\pi/3$ when $x=8$). Then I integrated $\sec\theta$.

Example 6 has the same x^2-16. So the substitution is again $x=4\sec\theta$:

EXAMPLE 6 $$\int_{x=8}^{\infty}\frac{16\,dx}{(x^2-16)^{3/2}}=\int_{\theta=\pi/3}^{\pi/2}\frac{64\sec\theta\tan\theta\,d\theta}{(4\tan\theta)^3}=\int_{\pi/3}^{\pi/2}\frac{\cos\theta\,d\theta}{\sin^2\theta}.$$

Step one substitutes $x=4\sec\theta$. Step two changes the limits to θ. The upper limit $x=\infty$ becomes $\theta=\pi/2$, where the secant is infinite. The limit $x=8$ again means $\theta=\pi/3$. To get a grip on the integral, I also changed to sines and cosines.

The integral of $\cos\theta/\sin^2\theta$ needs another substitution! (Or else recognize $\cot\theta\csc\theta$.) With $u=\sin\theta$ we have $\int du/u^2=-1/u=-1/\sin\theta$:

Solution $$\int_{\pi/3}^{\pi/2}\frac{\cos\theta\,d\theta}{\sin^2\theta}=\frac{-1}{\sin\theta}\Big]_{\pi/3}^{\pi/2}=-1+\frac{2}{\sqrt{3}}.$$

Warning With lower limit $\theta=0$ (or $x=4$) this integral would be a disaster. It divides by $\sin 0$, which is zero. *This area is infinite.*

$(Warning)^2$ Example 5 also blew up at $x=4$, but the area was ***not*** infinite. To make the point directly, compare $x^{-1/2}$ to $x^{-3/2}$. Both blow up at $x=0$, but the first one has finite area:

$$\int_0^1\frac{1}{\sqrt{x}}\,dx=2\sqrt{x}\Big]_0^1=2\qquad\int_0^1\frac{1}{x^{3/2}}\,dx=\frac{-2}{\sqrt{x}}\Big]_0^1=\infty.$$

Section 7.5 separates finite areas (slow growth of $1/\sqrt{x}$) from infinite areas (fast growth of $x^{-3/2}$).

Last substitution Together with $16-x^2$ and x^2-16 comes the possibility $16+x^2$. (You might ask about $-16-x^2$, but for obvious reasons we don't take its square root.) This third form $16+x^2$ requires a third substitution $x=4\tan\theta$. Then $16+x^2=16+16\tan^2\theta=16\sec^2\theta$. Here is an example:

EXAMPLE 7 $$\int_{x=0}^{\infty}\frac{dx}{16+x^2}=\int_{\theta=0}^{\pi/2}\frac{4\sec^2\theta\,d\theta}{16\sec^2\theta}=\frac{1}{4}\theta\Big]_0^{\pi/2}=\frac{\pi}{8}.$$

Table of substitutions for $a^2-x^2,\ a^2+x^2,\ x^2-a^2$

$x=a\sin\theta$	replaces	a^2-x^2	by	$a^2\cos^2\theta$	and	dx by	$a\cos\theta\,d\theta$
$x=a\tan\theta$	replaces	a^2+x^2	by	$a^2\sec^2\theta$	and	dx by	$a\sec^2\theta\,d\theta$
$x=a\sec\theta$	replaces	x^2-a^2	by	$a^2\tan^2\theta$	and	dx by	$a\sec\theta\tan\theta\,d\theta$

Note There is a subtle difference between changing x to $\sin\theta$ and changing $\sin\theta$ to u:

in Example 1, dx was replaced by $\cos\theta\, d\theta$(new method)

in Example 6, $\cos\theta\, d\theta$ was already there and became du (old method).

The combination $\cos\theta\, d\theta$ was put into the first and pulled out of the second.

My point is that Chapter 5 needed du/dx inside the integral. Then $(du/dx)dx$ became du. ***Now it is not necessary to see so far ahead***. We can try any substitution. If it works, we win. In this section, $x = \sin\theta$ or $\sec\theta$ or $\tan\theta$ is bound to succeed.

NEW $\displaystyle\int \frac{dx}{1+x^2} = \int d\theta$ by trying $x = \tan\theta$ **OLD** $\displaystyle\int \frac{x\, dx}{1+x^2} = \int \frac{du}{2u}$ by seeing du

We mention the ***hyperbolic substitutions*** $\tanh\theta, \sinh\theta,$ and $\cosh\theta$. The table below shows their use. They give new forms for the same integrals. If you are familiar with hyperbolic functions the new form might look simpler—as it does in Example 8.

$x = a\tanh\theta$	replaces	$a^2 - x^2$	by	$a^2\,\text{sech}^2\theta$	and	dx	by	$a\,\text{sech}^2\theta\, d\theta$
$x = a\sinh\theta$	replaces	$a^2 + x^2$	by	$a^2\cosh^2\theta$	and	dx	by	$a\cosh\theta\, d\theta$
$x = a\cosh\theta$	replaces	$x^2 - a^2$	by	$a^2\sinh^2\theta$	and	dx	by	$a\sinh\theta\, d\theta$

EXAMPLE 8 $$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sinh\theta\, d\theta}{\sinh\theta} = \theta + C = \cosh^{-1}x + C.$$

$\int d\theta$ is simple. The bad part is $\cosh^{-1}x$ at the end. Compare with $x = \sec\theta$:

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\,\tan\theta\, d\theta}{\tan\theta} = \ln(\sec\theta + \tan\theta) + C = \ln(x + \sqrt{x^2-1}) + C.$$

This way looks harder, but most tables prefer that final logarithm. It is clearer than $\cosh^{-1}x$, even if it takes more space. All answers agree if Problem 35 is correct.

COMPLETING THE SQUARE

We have not said what to do for $\sqrt{x^2-2x+2}$ or $\sqrt{-x^2+2x}$. Those square roots contain a *linear term*—a multiple of x. The device for removing linear terms is worth knowing. It is called ***completing the square***, and two examples will begin to explain it:

$$x^2 - 2x + 2 = (x-1)^2 + 1 = u^2 + 1$$
$$-x^2 + 2x = -(x-1)^2 + 1 = 1 - u^2.$$

The idea has three steps. First, get the x^2 and x terms into one square. Here that square was $(x-1)^2 = x^2 - 2x + 1$. Second, fix up the constant term. Here we recover the original functions by adding 1. Third, set $u = x - 1$ to leave no linear term. Then the integral goes forward based on the substitutions of this section:

$$\int \frac{dx}{\sqrt{x^2-2x+1}} = \int \frac{du}{\sqrt{u^2+1}} \qquad \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

The same idea applies to any quadratic that contains a linear term $2bx$:

$$\text{rewrite} \quad x^2+2bx+c \quad \text{as} \quad (x+b)^2+C, \text{ with } C=c-b^2$$
$$\text{rewrite} \quad -x^2+2bx+c \quad \text{as} \quad -(x-b)^2+C, \text{ with } C=c+b^2$$

To match the quadratic with the square, we fix up the constant:

$$x^2+10x+16 = \quad (x+5)^2+C \text{ leads to } C=16-25=-9$$
$$-x^2+10x+16 = -(x-5)^2+C \text{ leads to } C=16+25=41.$$

EXAMPLE 9 $$\int\frac{dx}{x^2+10x+16}=\int\frac{dx}{(x+5)^2-9}=\int\frac{du}{u^2-9}.$$

Here $u=x+5$ and $du=dx$. Now comes a choice—struggle on with $u=3\sec\theta$ or look for $\int du/(u^2-a^2)$ inside the front cover. Then set $a=3$:

$$\int\frac{du}{u^2-9}=\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right|=\frac{1}{6}\ln\left|\frac{x+2}{x+8}\right|.$$

Note If the quadratic starts with $5x^2$ or $-5x^2$, factor out the 5 first:

$$5x^2-10x+25=5(x^2-2x+5)=(\text{complete the square})=5[(x-1)^2+4].$$

Now $u=x-1$ produces $5[u^2+4]$. This is ready for table lookup or $u=2\tan\theta$:

EXAMPLE 10 $$\int\frac{dx}{5x^2-10x+25}=\int\frac{du}{5[u^2+4]}=\int\frac{2\sec^2\theta\,d\theta}{5[4\sec^2\theta]}=\frac{1}{10}\int d\theta.$$

This answer is $\theta/10+C$. Now go backwards: $\theta/10=(\tan^{-1}\frac{1}{2}u)/10=(\tan^{-1}\frac{1}{2}(x-1))/10$. Nobody could see that from the start. A double substitution takes practice, from x to u to θ. Then go backwards from θ to u to x.

Final remark For u^2+a^2 we substitute $u=a\tan\theta$. For u^2-a^2 we substitute $u=a\sec\theta$. This big dividing line depends on whether the constant C (after completing the square) is positive or negative. We either have $C=a^2$ or $C=-a^2$. The same dividing line in the original $x^2+2bx+c$ is between $c>b^2$ and $c<b^2$. In between, $c=b^2$ yields the perfect square $(x+b)^2$— and no trigonometric substitution at all.

7.3 EXERCISES

Read-through questions

The function $\sqrt{1-x^2}$ suggests the substitution $x=$ __a__. The square root becomes __b__ and dx changes to __c__. The integral $\int(1-x^2)^{3/2}dx$ becomes $\int$ __d__ $d\theta$. The interval $\frac{1}{2}\leqslant x\leqslant 1$ changes to __e__ $\leqslant\theta\leqslant$ __f__.

For $\sqrt{a^2-x^2}$ the substitution is $x=$ __g__ with $dx=$ __h__. For x^2-a^2 we use $x=$ __i__ with $dx=$ __j__. Then $\int dx/(1+x^2)$ becomes $\int d\theta$, because $1+\tan^2\theta=$ __k__. The answer is $\theta=\tan^{-1}x$. We already knew that __l__ is the derivative of $\tan^{-1}x$.

The quadratic $x^2+2bx+c$ contains a __m__ term $2bx$. To remove it we __n__ the square. This gives $(x+b)^2+C$ with $C=$ __o__. The example x^2+4x+9 becomes __p__. Then $u=x+2$. In case x^2 enters with a minus sign, $-x^2+4x+9$ becomes (__q__ $)^2+$ __r__. When the quadratic contains $4x^2$, start by factoring out __s__.

Integrate 1–20 by substitution. Change θ back to x.

1 $\displaystyle\int\frac{dx}{\sqrt{4-x^2}}$

2 $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}$

3 $\int \sqrt{4-x^2}\,dx$

4 $\int \frac{dx}{\sqrt{1+9x^2}}$

5 $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

6 $\int \frac{dx}{x^2\sqrt{1-x^2}}$

7 $\int \frac{dx}{(1+x^2)^2}$

8 $\int \sqrt{x^2+a^2}\,dx$ (see 7.2.62)

9 $\int \frac{\sqrt{x^2-25}}{x}\,dx$

10 $\int \frac{x^3\,dx}{\sqrt{9-x^2}}$

11 $\int \frac{dx}{\sqrt{x^6-x^4}}$

12 $\int \sqrt{x^6-x^8}\,dx$

13 $\int \frac{dx}{(1+x^2)^{3/2}}$

14 $\int \frac{dx}{(1-x^2)^{3/2}}$

15 $\int \frac{dx}{(x^2-9)^{3/2}}$

***16** $\int \frac{\sqrt{1+x^2}\,dx}{x}$

***17** $\int \frac{x^2\,dx}{\sqrt{x^2-1}}$

18 $\int \frac{x^2\,dx}{x^2+4}$

19 $\int \frac{dx}{x^2\sqrt{x^2+1}}$

***20** $\int \frac{x^2\,dx}{\sqrt{1+x^2}}$

21 (Important) This section started with $x=\sin\theta$ and

$$\int dx/\sqrt{1-x^2}=\int d\theta=\theta=\sin^{-1}x.$$

(a) Use $x=\cos\theta$ to get a different answer.

(b) How can the same integral give two answers?

22 Compute $\int dx/x\sqrt{x^2-1}$ with $x=\sec\theta$. Recompute with $x=\csc\theta$. How can both answers be correct?

23 Integrate $x/(x^2+1)$ with $x=\tan\theta$, and also directly as a logarithm. Show that the results agree.

24 Show that $\int dx/x\sqrt{x^4-1}=\frac{1}{2}\sec^{-1}(x^2)$.

Calculate the definite integrals 25–32.

25 $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx =$ area of ______

26 $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

27 $\int_{.5}^{1}\frac{dx}{\sqrt{1-x^2}}$

28 $\int_{1}^{4}\frac{dx}{\sqrt{x^2-1}}$

29 $\int_{2}^{\infty}\frac{dx}{(x^2-1)^{3/2}}$

30 $\int_{-1}^{1}\frac{x\,dx}{x^2+1}$

31 $\int_{-\infty}^{\infty}\frac{dx}{x^2+9}$

32 $\int_{1/2}^{1}\sqrt{1-x^2}\,dx =$ area of ______.

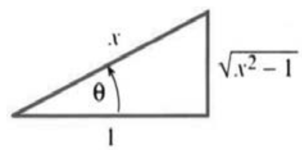
33 Combine the integrals to prove the reduction formula ($n\neq 0$):

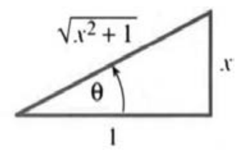
$$\int \frac{x^{n+1}}{x^2+1}dx=\frac{x^n}{n}-\int\frac{x^{n-1}}{x^2+1}dx.$$

34 Integrate $1/\cos x$ and $1/(1+\cos x)$ and $\sqrt{1+\cos x}$.

35 (a) $x=$ ______ gives $\int dx/\sqrt{x^2-1}=\ln(\sec\theta+\tan\theta)$.
(b) From the triangle, this answer is $f=\ln(x+\sqrt{x^2-1})$. Check that $df/dx=1/\sqrt{x^2-1}$.
(c) Verify that $\cosh f=\frac{1}{2}(e^f+e^{-f})=x$. Then $f=\cosh^{-1}x$, the answer in Example 8.

36 (a) $x=$ ______ gives $\int dx/\sqrt{x^2+1}=\ln(\sec\theta+\tan\theta)$.
(b) The second triangle converts this answer to $g=\ln(x+\sqrt{x^2+1})$. Check that $dg/dx=1/\sqrt{x^2+1}$.
(c) Verify that $\sinh g=\frac{1}{2}(e^g-e^{-g})=x$ so $g=\sinh^{-1}x$.
(d) Substitute $x=\sinh g$ directly into $\int dx/\sqrt{x^2+1}$ and integrate.





In 37–42 substitute $x=\sinh\theta$, $\cosh\theta$, or $\tanh\theta$. After integration change back to x.

37 $\int\frac{dx}{\sqrt{x^2-1}}$

38 $\int\frac{dx}{x\sqrt{1-x^2}}$

39 $\int\sqrt{x^2-1}\,dx$

40 $\int\frac{\sqrt{x^2-1}}{x^2}\,dx$

41 $\int\frac{dx}{1-x^2}$

42 $\int\frac{\sqrt{1+x^2}}{x^2}dx$

Rewrite 43–48 as $(x+b)^2+C$ or $-(x-b)^2+C$ by completing the square.

43 x^2-4x+8

44 $-x^2+2x+8$

45 x^2-6x

46 $-x^2+10$

47 x^2+2x+1

48 $x^2+4x-12$

49 For the three functions $f(x)$ in Problems 43, 45, 47 integrate $1/f(x)$.

50 For the three functions $g(x)$ in Problems 44, 46, 48 integrate $1/\sqrt{g(x)}$.

51 For $\int dx/(x^2+2bx+c)$ why does the answer have different forms for $b^2>c$ and $b^2<c$? What is the answer if $b^2=c$?

52 What substitution $u=x+b$ or $u=x-b$ will remove the linear term?

(a) $\int\frac{dx}{x^2-4x+c}$

(b) $\int\frac{dx}{3x^2+6x}$

(c) $\int\frac{dx}{-x^2+10x+c}$

(d) $\int\frac{dx}{2x^2-x}$

53 *Find the mistake.* With $x=\sin\theta$ and $\sqrt{1-x^2}=\cos\theta$, substituting $dx=\cos\theta\,d\theta$ changes

$$\int_0^{2\pi}\cos^2\theta\,d\theta \quad\text{into}\quad \int_0^0\sqrt{1-x^2}\,dx.$$

54 (a) If $x = \tan\theta$ then $\int \sqrt{1+x^2}dx = \int$ ______ $d\theta$.
(b) Convert $\frac{1}{2}[\sec\theta\ \tan\theta + \ln(\sec\theta + \tan\theta)]$ back to x.
(c) If $x = \sinh\theta$ then $\int \sqrt{1+x^2}\,dx = \int$ ______ $d\theta$.
(d) Convert $\frac{1}{2}[\sinh\theta\ \cosh\theta + \theta]$ back to x.

These answers agree. In Section 8.2 they will give the length of a parabola. Compare with Problem 7.2.62.

55 Rescale x and y in Figure 7.5b to produce the equal area $\int y\,dx$ in Figure 7.5c. What happens to y and what happens to dx?

56 Draw $y = 1/\sqrt{1-x^2}$ and $y = 1/\sqrt{16-x^2}$ to the same scale ($1''$ across and up; $4''$ across and $\frac{1}{4}''$ up).

57 What is wrong, if anything, with

$$\int_0^2 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 2?$$

7.4 Partial Fractions

This section is about rational functions $P(x)/Q(x)$. Sometimes their integrals are also rational functions (ratios of polynomials). More often they are not. It is very common for the integral of P/Q to involve logarithms. We meet logarithms immediately in the simple case $1/(x-2)$, whose integral is $\ln|x-2|+C$. We meet them again in a sum of simple cases:

$$\int\left[\frac{1}{x-2}+\frac{3}{x+2}-\frac{4}{x}\right]dx=\ln|x-2|+3\ln|x+2|-4\ln|x|+C.$$

Our plan is to split P/Q ***into a sum like this—and integrate each piece***.

Which rational function produced that particular sum? It was

$$\frac{1}{x-2}+\frac{3}{x+2}-\frac{4}{x}=\frac{(x+2)(x)+3(x-2)(x)-4(x-2)(x+2)}{(x-2)(x+2)(x)}=\frac{-4x+16}{(x-2)(x+2)(x)}.$$

This is P/Q. It is a ratio of polynomials, degree 1 over degree 3. The pieces of P are collected into $-4x+16$. The common denominator $(x-2)(x+2)(x)=x^3-4x$ is Q. But I kept these factors separate, for the following reason. When we start with P/Q, and break it into a sum of pieces, *the first things we need are the factors of* Q.

In the standard problem P/Q is given. To integrate it, we break it up. The goal of partial fractions is to find the pieces—*to prepare for integration*. That is the technique to learn in this section, and we start right away with examples.

EXAMPLE 1 Suppose P/Q has the same Q but a different numerator P:

$$\frac{P}{Q}=\frac{3x^2+8x-4}{(x-2)(x+2)(x)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x}. \tag{1}$$

Notice the form of those pieces! They are the "***partial fractions***" that add to P/Q. Each one is a constant divided by a factor of Q. We know the factors $x-2$ and $x+2$ and x. We don't know the constants A,B,C. In the previous case they were $1,3,-4$. In this and other examples, there are two ways to find them.

Method **3** (slow) Put the right side of (1) over the common denominator Q:

$$\frac{3x^2+8x-4}{Q}=\frac{A(x+2)(x)+B(x-2)(x)+C(x-2)(x+2)}{(x-2)(x+2)(x)} \tag{2}$$

Why is A multiplied by $(x+2)(x)$? Because canceling those factors will leave $A/(x-2)$ as in equation (1). Similarly we have $B/(x+2)$ and C/x. ***Choose the numbers*** A,B,C ***so that the numerators match***. As soon as they agree, the splitting is correct.

Method **4** (quicker) Multiply equation (1) by $x-2$. That leaves a space:

$$\frac{3x^2+8x-4}{(x+2)(x)}=A+\frac{B(x-2)}{x+2}+\frac{C(x-2)}{x}. \tag{3}$$

Now set $x=2$ ***and immediately you have*** A. The last two terms of (3) are zero, because $x-2$ is zero when $x=2$. On the left side, $x=2$ gives

$$\frac{3(2)^2+8(2)-4}{(2+2)(2)}=\frac{24}{8}=3. \quad \text{(which is } A\text{)}.$$

Notice how multiplying by $x-2$ produced a hole on the left side. Method **2** is the "***cover-up method***." *Cover up* $x-2$ *and then substitute* $x=2$. The result is $3=A+0+0$, just what we wanted.

In Method **1**, the numerators of equation (2) must agree. The factors that multiply B and C are again zero at $x=2$. That leads to the same A—but the cover-up method avoids the unnecessary step of writing down equation (2).

Calculation of B Multiply equation (1) by $x+2$, which covers up the $(x+2)$:

$$\frac{3x^2+8x-4}{(x-2)\qquad(x)}=\frac{A(x+2)}{x-2}+B+\frac{C(x+2)}{x}. \tag{4}$$

Now set $x=-2$, so A and C are multiplied by zero:

$$\frac{3(-2)^2+8(-2)-4}{(-2-2)\qquad(-2)}=\frac{-8}{8}=-1=B.$$

This is almost full speed, but (4) was not needed. *Just cover up in Q and give x the right value* (which makes the covered factor zero).

Calculation of C (***quickest***) In equation (1), cover up the factor (x) and set $x=0$:

$$\frac{3(0)^2+8(0)-4}{(0-2)(0+2)}=\frac{-4}{-4}=1=C. \tag{5}$$

To repeat: The same result $A=3$, $B=-1$, $C=1$ comes from Method **1**.

EXAMPLE 2
$$\frac{x+2}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}.$$

First cover up $(x-1)$ on the left and set $x=1$. Next cover up $(x+3)$ and set $x=-3$:

$$\frac{1+2}{(\qquad)(1+3)}=\frac{3}{4}=A \qquad\qquad \frac{-3+2}{(-3-1)(\qquad)}=\frac{-1}{-4}=B.$$

The integral is $\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+3|+C$.

EXAMPLE 3 This was needed for the logistic equation in Section 6.5:

$$\frac{1}{y(c-by)}=\frac{A}{y}+\frac{B}{c-by}. \tag{6}$$

First multiply by y. That covers up y in the first two terms and changes B to By. Then set $y=0$. The equation becomes $1/c=A$.

To find B, multiply by $c-by$. That covers up $c-by$ in the outside terms. In the middle, A times $c-by$ will be zero at $y=c/b$. That leaves B on the right equal to $1/y=b/c$ on the left. Then $A=1/c$ and $B=b/c$ give the integral announced in Equation 6.5.9:

$$\int\frac{dy}{cy-by^2}=\int\frac{dy}{cy}+\int\frac{b\,dy}{c(c-by)}=\frac{\ln y}{c}-\frac{\ln(c-by)}{c}. \tag{7}$$

It is time to admit that the general method of partial fractions can be very awkward. First of all, it requires the factors of the denominator Q. When Q is a quadratic ax^2+bx+c, we can find its roots and its factors. In theory a cubic or a quartic can also be factored, but in practice only a few are possible—for example $x^4 - 1$ is $(x^2-1)(x^2+1)$. Even for this good example, *two of the roots are imaginary*. We can split x^2-1 into $(x+1)(x-1)$. We cannot split x^2+1 without introducing i.

The method of partial fractions can work directly with x^2+1, as we now see.

EXAMPLE 4 $\displaystyle\int \frac{3x^2+2x+7}{x^2+1}\,dx$ (a quadratic over a quadratic).

This has another difficulty. The degree of P equals the degree of $Q(=2)$. ***Partial fractions cannot start until P has lower degree***. Therefore I divide the leading term x^2 into the leading term $3x^2$. That gives 3, which is separated off by itself:

$$\frac{3x^2+2x+7}{x^2+1} = 3 + \frac{2x+4}{x^2+1}. \tag{8}$$

Note how 3 really used $3x^2+3$ from the original numerator. That left $2x+4$. ***Partial fractions will accept a linear factor*** $2x+4$ (or $Ax+B$, not just A) ***above a quadratic***.

This example contains $2x/(x^2+1)$, which integrates to $\ln(x^2+1)$. The final $4/(x^2+1)$ integrates to $4\tan^{-1}x$. When the denominator is x^2+x+1 we complete the square before integrating. The point of Sections 7.2 and 7.3 was to make that integration possible. This section gets the fraction ready—in parts.

The essential point is that we never have to go higher than quadratics. ***Every denominator Q can be split into linear factors and quadratic factors***. There is no magic way to find those factors, and most examples begin by giving them. They go into their own fractions, and they have their own numerators—which are the A and B and $2x+4$ we have been computing.

The one remaining question is ***what to do if a factor is repeated***. This happens in Example 5.

EXAMPLE 5 $\displaystyle\frac{2x+3}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}.$

The key is the new term $B/(x-1)^2$. That is the right form to expect. With $(x-1)(x-2)$ this term would have been $B/(x-2)$. But when $(x-1)$ is repeated, something new is needed. To find B, *multiply through by* $(x-1)^2$ *and set* $x=1$:

$$2x+3 = A(x-1)+B \quad \text{becomes} \quad 5 = B \quad \text{when} \quad x=1.$$

This cover-up method gives B. Then $A=2$ is easy, and the integral is $2\ln|x-1| - 5/(x-1)$. The fraction $5/(x-1)^2$ has an integral without logarithms.

EXAMPLE 6 $\displaystyle\frac{2x^3+9x^2+4}{x^2(x^2+4)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} + \frac{E}{x-1}.$

This final example has almost everything! It is more of a game than a calculus problem. In fact calculus doesn't enter until we integrate (and nothing is new there). Before computing A, B, C, D, E, we write down the overall rules for partial fractions:

1. The degree of P must be less than the degree of Q. Otherwise divide their leading terms as in equation (8) to lower the degree of P. Here $3 < 5$.
2. Expect the fractions illustrated by Example 6. The linear factors x and $x+1$ (*and the repeated* x^2) are underneath constants. The quadratic x^2+4 is under a linear term. A repeated $(x^2+4)^2$ would be under a new $Fx+G$.

3. Find the numbers $A, B, C, \ldots$ by any means, including cover-up.
4. Integrate each term separately and add.

We could prove that this method always works. It makes better sense to show that it works once, in Example 6.

To find E, cover up $(x-1)$ on the left and substitute $x=1$. Then $E=3$. To find B, cover up x^2 on the left and set $x=0$. Then $B=4/(0+4)(0-1)=-1$. The cover-up method has done its job, and there are several ways to find A, C, D. Compare the numerators, after multiplying through by the common denominator Q:

$$2x^3+9x^2+4=Ax(x^2+4)(x-1)-(x^2+4)(x-1)+(Cx+D)(x^2)(x-1)+3x^2(x^2+4).$$

The known terms on the right, from $B=-1$ and $E=3$, can move to the left:

$$-3x^4+3x^3-4x^2+4x=Ax(x^2+4)(x-1)+(Cx+D)x^2(x-1).$$

We can divide through by x and $x-1$, which checks that B and E were correct:

$$-3x^2-4=A(x^2+4)+(Cx+D)x.$$

Finally $x=0$ yields $A=-1$. This leaves $-2x^2=(Cx+D)x$. Then $C=-2$ and $D=0$.

You should never have to do such a problem! I never intend to do another one. It completely depends on expecting the right form and matching the numerators. They could also be matched by comparing coefficients of $x^4, x^3, x^2, x, 1$—to give five equations for A, B, C, D, E. That is an invitation to human error. Cover-up is the way to start, and usually the way to finish. With repeated factors and quadratic factors, match numerators at the end.

7.4 EXERCISES

Read-through questions

The idea of __a__ fractions is to express $P(x)/Q(x)$ as a __b__ of simpler terms, each one easy to integrate. To begin, the degree of P should be __c__ the degree of Q. Then Q is split into __d__ factors like $x-5$ (possibly repeated) and quadratic factors like x^2+x+1 (possibly repeated). The quadratic factors have two __e__ roots, and do not allow real linear factors.

A factor like $x-5$ contributes a fraction $A/$__f__. Its integral is __g__. To compute A, cover up __h__ in the denominator of P/Q. Then set $x=$__i__, and the rest of P/Q becomes A. An equivalent method puts all fractions over a common denominator (which is __j__). Then match the __k__. At the same point $x=$__l__ this matching gives A.

A repeated linear factor $(x-5)^2$ contributes not only $A/(x-5)$ but also $B/$__m__. A quadratic factor like x^2+x+1 contributes a fraction __n__$/(x^2+x+1)$ involving C and D. A repeated quadratic factor or a triple linear factor would bring in $(Ex+F)/(x^2+x+1)^2$ or $G/(x-5)^3$. The conclusion is that any P/Q can be split into partial __o__, which can always be integrated.

1 Find the numbers A and B to split $1/(x^2-x)$:

$$\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}.$$

Cover up x and set $x=0$ to find A. Cover up $x-1$ and set $x=1$ to find B. Then integrate.

2 Find the numbers A and B to split $1/(x^2-1)$:

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}.$$

Multiply by $x-1$ and set $x=1$. Multiply by $x+1$ and set $x=-1$. *Integrate*. Then find A and B again by method 1—with numerator $A(x+1)+B(x-1)$ equal to 1.

Express the rational functions 3–16 as partial fractions:

3 $\dfrac{1}{(x-3)(x-2)}$

4 $\dfrac{x}{(x-3)(x-2)}$

5 $\dfrac{x^2+1}{(x)(x+1)(x+2)}$

6 $\dfrac{1}{x^3-x}$

7 $\dfrac{3x+1}{x^2}$

8 $\dfrac{3x+1}{(x-1)^2}$

9 $\dfrac{3x^2}{x^2+1}$ (divide first)

10 $\dfrac{1}{(x-1)(x^2+1)}$

11 $\dfrac{1}{x^2(x-1)}$

12 $\dfrac{x}{x^2-4}$

13 $\dfrac{1}{x(x-1)(x-2)(x-3)}$

14 $\dfrac{x^2+1}{x+1}$ (divide first)

15 $\dfrac{1}{x^4-1} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

16 $\dfrac{1}{x^2(x-1)}$ $\left(\text{remember the } \dfrac{A}{x} \text{ term}\right)$

17 Apply Method **1** (matching numerators) to Example 3:

$$\frac{1}{cy-by^2} = \frac{A}{y} + \frac{B}{c-by} = \frac{A(c-by)+By}{y(c-by)}.$$

Match the numerators on the far left and far right. Why does $Ac = 1$? Why does $-bA + B = 0$? What are A and B ?

18 What goes wrong if we look for A and B so that

$$\frac{x^2}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}\,?$$

Over a common denominator, try to match the numerators. What to do first?

19 Split $\dfrac{3x^2}{x^3-1} = \dfrac{3x^2}{(x-1)(x^2+x+1)}$ into $\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

(a) Cover up $x-1$ and set $x = 1$ to find A.

(b) Subtract $A/(x-1)$ from the left side. Find $Bx+C$.

(c) Integrate all terms. Why do we already know

$$\ln(x^3-1) = \ln(x-1) + \ln(x^2+x+1)\,?$$

20 Solve $dy/dt = 1-y^2$ by separating $\int dy/1-y^2 = \int dt$. Then

$$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)} = \frac{1/2}{1-y} + \frac{1/2}{1+y}$$

Integration gives $\frac{1}{2}\ln$ ______ $= t + C$. With $y_0 = 0$ the constant is $C =$ ______. Taking exponentials gives ______. The solution is $y =$ ______. ***This is the S-curve***.

By substitution change 21–28 to integrals of rational functions. Problem 23 integrates $1/\sin\theta$ with no special trick.

21 $\displaystyle\int \frac{e^x\,dx}{e^{2x}-e^x}$

22 $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

23 $\displaystyle\int \frac{\sin\theta\,d\theta}{1-\cos^2\theta}$

24 $\displaystyle\int \frac{dt}{(e^t-e^{-t})^2}$

25 $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

26 $\displaystyle\int \frac{\sqrt[3]{x-8}}{x}\,dx$

27 $\displaystyle\int \frac{dx}{1+\sqrt{x+1}}$

28 $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

29 Multiply this partial fraction by $x-a$. Then let $x \to a$:

$$\frac{1}{Q(x)} = \frac{A}{x-a} + \cdots.$$

Show that $A = 1/Q'(a)$. When $x = a$ is a double root this fails because $Q'(a) =$ ______.

30 Find A in $\dfrac{1}{x^8-1} = \dfrac{A}{x-1} + \cdots$. Use Problem 29.

31 (for instructors only) Which rational functions P/Q are the derivatives of other rational functions (no logarithms)?

7.5 Improper Integrals

"***Improper***" means that some part of $\int_a^b y(x)\,dx$ becomes ***infinite***. It might be b or a or the function y. The region under the graph reaches infinitely far—to the right or left or up or down. (Those come from $b=\infty$ and $a=-\infty$ and $y\to\infty$ and $y\to-\infty$.) Nevertheless the integral may "*converge*." Just because the ***region*** is infinite, it is not automatic that the ***area*** is infinite. That is the point of this section—to decide when improper integrals have proper answers.

The first examples show finite area when $b=\infty$, then $a=-\infty$, then $y=1/\sqrt{x}$ at $x=0$. The areas in Figure 7.6 are $1, 1, 2$:

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^\infty = 1 \qquad \int_{-\infty}^0 e^x\,dx = e^x\Big]_{-\infty}^0 = 1 \qquad \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_0^1 = 2.$$

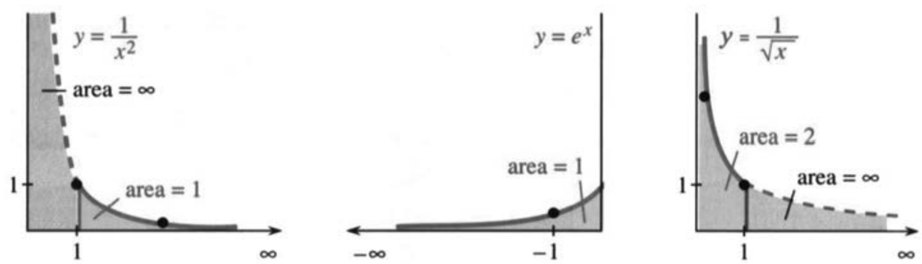


Fig. 7.6 The shaded areas are finite but the regions go to infinity.

In practice we substitute the dangerous limits and watch what happens. When the integral is $-1/x$, substituting $b=\infty$ gives "$-1/\infty=0$." When the integral is e^x, substituting $a=-\infty$ gives "$e^{-\infty}=0$." I think that is fair, and I know it is successful. But it is not completely precise.

The strict rules involve a limit. Calculus sneaks up on $1/\infty$ and $e^{-\infty}$ just as it sneaks up on $0/0$. Instead of swallowing an infinite region all at once, the formal definitions push out to the limit:

DEFINITION $$\int_a^\infty y(x)dx = \lim_{b\to\infty}\int_a^b y(x)dx \qquad \int_{-\infty}^b y(x)dx = \lim_{a\to-\infty}\int_a^b y(x)dx.$$

The conclusion is the same. The first examples converged to $1, 1, 2$. Now come two more examples going out to $b=\infty$:

The area under $1/x$ is *infinite*: $$\int_1^\infty \frac{dx}{x} = \ln x\Big]_1^\infty = \infty \tag{1}$$

The area under $1/x^p$ is *finite* if $p>1$: $$\int_1^\infty \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big]_1^\infty = \frac{1}{p-1}. \tag{2}$$

The area under $1/x$ is like $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, which is also infinite. In fact the sum approximates the integral—the curved area is close to the rectangular area. They go together (slowly to infinity).

A larger p brings the graph more quickly to zero. Figure 7.7a shows a finite area $1/(p-1)=100$. The region is still infinite, but we can cover it with strips cut out of a square! The borderline for finite area is $p=1$. I call it the borderline, but $p=1$ is strictly on the side of divergence.

The borderline is also $p = 1$ ***when the function climbs the*** y ***axis***. At $x = 0$, the graph of $y = 1/x^p$ goes to infinity. For $p = 1$, the area under $1/x$ is again infinite. But at $x = 0$ it is a *small* p (meaning $p < 1$) that produces finite area:

$$\int_0^1 \frac{dx}{x} = \ln x\Big]_0^1 = \infty \qquad \int_0^1 \frac{dx}{x^p} = \frac{x^{1-p}}{1-p}\Big]_0^1 = \frac{1}{1-p} \quad \text{if } p < 1. \tag{3}$$

Loosely speaking "$-\ln 0 = \infty$." Strictly speaking we integrate from the point $x = a$ *near* zero, to get $\int_a^1 dx/x = -\ln a$. As a approaches zero, the area shows itself as infinite. For $y = 1/x^2$, which blows up faster, the area $-1/x]_0^1$ is again infinite.

For $y = 1/\sqrt{x}$, the area from 0 to 1 is 2. In that case $p = \frac{1}{2}$. For $p = 99/100$ the area is $1/(1-p) = 100$. Approaching $p = 1$ the borderline in Figure 7.7 seems clear. *But that cutoff is not as sharp as it looks*.

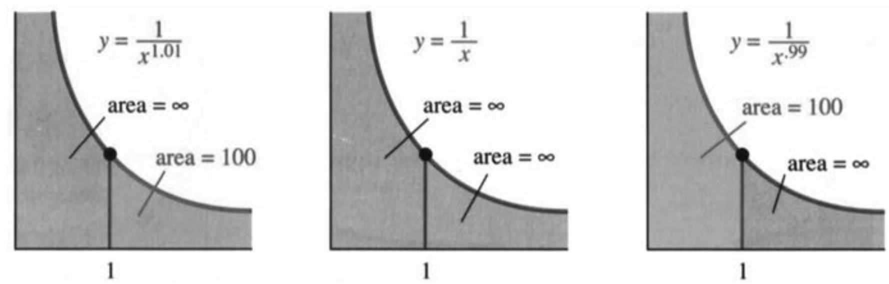


Fig. 7.7 Graphs of $1/x^p$ on both sides of $p = 1$. I drew the same curves!

Narrower borderline Under the graph of $1/x$, the area is infinite. When we divide by $\ln x$ or $(\ln x)^2$, the borderline is somewhere in between. One has infinite area (going out to $x = \infty$), the other area is finite:

$$\int_e^\infty \frac{dx}{x(\ln x)} = \ln(\ln x)\Big]_e^\infty = \infty \qquad \int_e^\infty \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Big]_e^\infty = 1. \tag{4}$$

The first is $\int du/u$ with $u = \ln x$. The logarithm of $\ln x$ does eventually make it to infinity. At $x = 10^{10}$, the logarithm is near 23 and $\ln(\ln x)$ is near 3. That is slow! Even slower is $\ln(\ln(\ln x))$ in Problem 11. No function is *exactly* on the borderline.

The second integral in equation (4) is convergent (to 1). It is $\int du/u^2$ with $u = \ln x$. At first I wrote it with x going from zero to infinity. That gave an answer I couldn't believe:

$$\int_0^\infty \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Big]_0^\infty = 0\ (??)$$

There must be a mistake, because we are integrating a positive function. The area can't be zero. It is true that $1/\ln b$ goes to zero as $b \to \infty$. It is also true that $1/\ln a$ goes to zero as $a \to 0$. *But there is another infinity in this integral*. The trouble is at $x = 1$, where $\ln x$ is zero and the area is infinite.

EXAMPLE 1 The factor e^{-x} overrides any power x^p (but only as $x \to \infty$).

$$\int_0^\infty x^{50} e^{-x}\, dx = 50! \quad \text{but} \quad \int_0^\infty x^{-1} e^{-x}\, dx = \infty.$$

The first integral is $(50)(49)(48)\cdots(1)$. It comes from fifty integrations by parts (not recommended). Changing 50 to $\frac{1}{2}$, the integral defines "$\frac{1}{2}$ *factorial*." The product

$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots$ has no way to stop, but somehow $\frac{1}{2}!$ is $\frac{1}{2}\sqrt{\pi}$. See Problem 28. The integral $\int_0^\infty x^0 e^{-x}\,dx = 1$ is the reason behind "*zero factorial*" $= 1$. That seems the most surprising of all.

The area under e^{-x}/x is $(-1)! = \infty$. The factor e^{-x} is absolutely no help at $x = 0$. That is an example (the first of many) in which we do not know an antiderivative—but still we get a decision. To integrate e^{-x}/x we need a computer. But to decide that an improper integral is infinite (in this case) or finite (in other cases), we rely on the following *comparison test*:

> **7C** (***Comparison test***) Suppose that $0 \leqslant u(x) \leqslant v(x)$. Then the area under $u(x)$ is smaller than the area under $v(x)$:
>
> $\int u(x)dx < \infty$ if $\int v(x)dx < \infty$ if $\int u(x)dx = \infty$ then $\int v(x)dx = \infty$.

Comparison can decide if the area is finite. We don't get the exact area, but we learn about one function from the other. The trick is to construct a simple function (like $1/x^p$) which is on one side of the given function—and stays close to it:

EXAMPLE 2 $\displaystyle\int_1^\infty \frac{dx}{x^2+4x}$ converges by comparison with $\displaystyle\int_1^\infty \frac{dx}{x^2} = 1.$

EXAMPLE 3 $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+1}$ diverges by comparison with $\displaystyle\int_1^\infty \frac{dx}{2\sqrt{x}} = \infty.$

EXAMPLE 4 $\displaystyle\int_0^1 \frac{dx}{x^2+4x}$ diverges by comparison with $\displaystyle\int_0^1 \frac{dx}{5x} = \infty.$

EXAMPLE 5 $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}+1}$ converges by comparison with $\displaystyle\int_0^1 \frac{dx}{1} = 1.$

In Examples 2 and 5, the integral on the right is larger than the integral on the left. Removing $4x$ and $\sqrt{x}$ increased the area. Therefore the integrals on the left are somewhere between 0 and 1.

In Examples 3 and 4, we *increased* the denominators. The integrals on the right are smaller, but still they diverge. So the integrals on the left diverge. The idea of *comparing functions* is seen in the next examples and Figure 7.8.

EXAMPLE 6 $\displaystyle\int_0^\infty e^{-x^2}dx$ is below $\displaystyle\int_0^1 1\,dx + \int_1^\infty e^{-x}dx = 1+1.$

EXAMPLE 7 $\displaystyle\int_1^e \frac{dx}{\ln x}$ is above $\displaystyle\int_1^e \frac{dx}{x\ln x} = \infty.$

EXAMPLE 8 $\displaystyle\int_0^1 \frac{dx}{\sqrt{x-x^2}}$ is below $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} + \int_0^1 \frac{dx}{\sqrt{1-x}} = 2+2.$

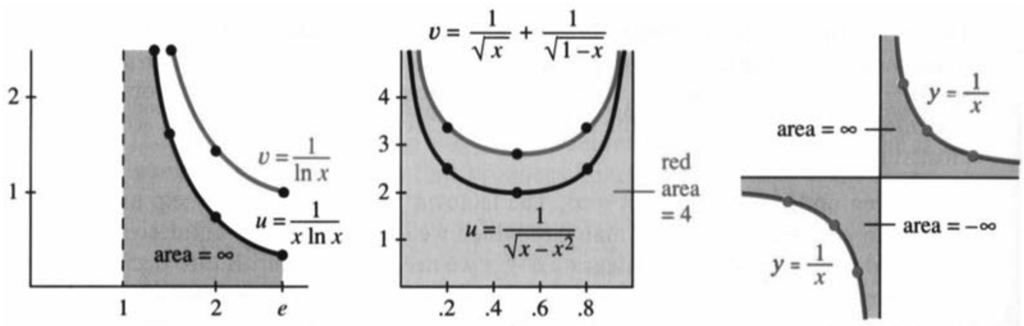


Fig. 7.8 Comparing $u(x)$ to $v(x)$: $\int_1^e dx/\ln x = \infty$ and $\int_0^1 dx/\sqrt{x-x^2} < 4$. But $\infty - \infty \neq 0$.

There are two situations not yet mentioned, and both are quite common. The first is an integral all the way from $a = -\infty$ to $b = +\infty$. That is split into two parts, and ***each part must converge***. By definition, the limits at $-\infty$ and $+\infty$ are kept *separate*:

$$\int_{-\infty}^{\infty} y(x)\,dx = \int_{-\infty}^{0} y(x)\,dx + \int_{0}^{\infty} y(x)\,dx = \lim_{a\to-\infty}\int_a^0 y(x)\,dx + \lim_{b\to\infty}\int_0^b y(x)\,dx.$$

The bell-shaped curve $y = e^{-x^2}$ covers a finite area (exactly $\sqrt{\pi}$). The region extends to infinity in both directions, and the separate areas are $\frac{1}{2}\sqrt{\pi}$. But notice:

$\int_{-\infty}^{\infty} x\,dx$ ***is not defined even though*** $\int_{-b}^{b} x\,dx = 0$ ***for every*** b.

The area under $y = x$ is $+\infty$ on one side of zero. The area is $-\infty$ on the other side. ***We cannot accept*** $\infty - \infty = 0$. The two areas must be separately finite, and in this case they are not.

EXAMPLE 9 $1/x$ has balancing regions left and right of $x = 0$. Compute $\int_{-1}^{1} dx/x$.

This integral does not exist. There is no answer, even for the region in Figure 7.8c. (They are mirror images because $1/x$ is an odd function.) You may feel that the combined integral from -1 to 1 should be zero. Cauchy agreed with that—his "principal value integral" is zero. But the rules say no: $\infty - \infty$ is not zero.

7.5 EXERCISES

Read-through questions

An improper integral $\int_a^b y(x)\,dx$ has lower limit $a =$ __a__ or upper limit $b =$ __b__ or y becomes __c__ in the interval $a \leqslant x \leqslant b$. The example $\int_1^\infty dx/x^3$ is improper because __d__. We should study the limit of $\int_1^b dx/x^3$ as __e__. In practice we work directly with $-\frac{1}{2}x^{-2}]_1^\infty =$ __f__. For $p > 1$ the improper integral __g__ is finite. For $p < 1$ the improper integral __h__ is finite. For $y = e^{-x}$ the integral from 0 to ∞ is __i__.

Suppose $0 \leqslant u(x) \leqslant v(x)$ for all x. The convergence of __j__ implies the convergence of __k__. The divergence of $\int u(x)dx$ __l__ the divergence of $\int v(x)dx$. From $-\infty$ to ∞, the integral of $1/(e^x + e^{-x})$ converges by comparison with __m__. Strictly speaking we split $(-\infty, \infty)$ into (__n__ , 0) and (0, __o__). Changing to $1/(e^x - e^{-x})$ gives divergence, because __p__. Also $\int_{-\pi}^{\pi} dx/\sin x$ diverges by comparison with __q__. The regions left and right of zero don't cancel because $\infty - \infty$ is ___r___.

Decide convergence or divergence in 1–16. Compute the integrals that converge.

1 $\displaystyle\int_1^\infty \frac{dx}{x^e}$

2 $\displaystyle\int_0^1 \frac{dx}{x^\pi}$

3 $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$

4 $\displaystyle\int_0^1 \frac{dx}{1-x}$

5 $\displaystyle\int_{-\infty}^0 \frac{dx}{x^2+1}$

6 $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

7 $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$

8 $\displaystyle\int_{-\infty}^\infty \sin x\,dx$

9 $\displaystyle\int_0^e \ln x\,dx$ (by parts)

10 $\displaystyle\int_0^\infty x\,e^{-x}\,dx$ (by parts)

11 $\displaystyle\int_{100}^\infty \frac{dx}{x(\ln x)(\ln\ln x)}$

12 $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2-1)^2}$

13 $\displaystyle\int_0^\infty \cos^2 x\,dx$

14 $\displaystyle\int_0^{\pi/2} \tan x\,dx$

15 $\displaystyle\int_0^\infty \frac{dx}{x^p}$

16 $\displaystyle\int_0^\infty \frac{e^x\,dx}{(e^x-1)^p}$

In 17–26, find a larger integral that converges or a smaller integral that diverges.

17 $\displaystyle\int_1^\infty \frac{dx}{x^6+1}$

18 $\displaystyle\int_0^1 \frac{dx}{x^6+1}$

19 $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1}$

20 $\displaystyle\int_0^1 \frac{e^{-x}dx}{1-x}$

21 $\displaystyle\int_1^\infty e^{-x}\sin x\,dx$

22 $\displaystyle\int_1^\infty x^{-x}dx$

23 $\displaystyle\int_0^\infty e^{2x}e^{-x^2}\,dx$

24 $\displaystyle\int_0^1 \sqrt{-\ln x}\,dx$

25 $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

26 $\displaystyle\int_1^\infty \left(\frac{1}{x}-\frac{1}{1+x}\right)dx$

27 If $p>0$, integrate by parts to show that

$$\int_0^\infty x^p\,e^{-x}\,dx = p\int_0^\infty x^{p-1}e^{-x}\,dx.$$

The first integral is the definition of $p!$ So the equation is $p! =$ ______. In particular $0! =$ ______. Another notation for $p!$ is $\Gamma(p+1)$—using the *gamma function* emphasizes that p need not be an integer.

28 Compute $\left(-\frac{1}{2}\right)!$ by substituting $x=u^2$:

$$\int_0^\infty x^{-1/2}e^{-x}\,dx = \text{\_\_\_\_\_\_} = \sqrt{\pi}\ \text{(known)}.$$

Then apply Problem 27 to find $\left(\frac{1}{2}\right)!$

29 Integrate $\int_0^\infty x^2\,e^{-x^2}\,dx$ by parts.

30 The *beta function* $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is finite when m and n are greater than ______.

31 A *perpetual annuity* pays s dollars a year forever. With continuous interest rate c, its present value is $y_0=\int_0^\infty se^{-ct}dt$. To receive \$1000/year at $c=10\%$, you deposit $y_0 =$ ______.

32 In a perpetual annuity that pays once a year, the present value is $y_0 = s/a + s/a^2 + \cdots =$ ______. To receive \$1000/year at 10% (now $a=1.1$) you again deposit $y_0 =$ ______. Infinite sums are like improper integrals.

33 The work to move a satellite (mass m) infinitely far from the Earth (radius R, mass M) is $W=\int_R^\infty GMm\;dx/x^2$. Evaluate W. What *escape velocity* at liftoff gives an energy $\frac{1}{2}mv_0^2$ that equals W?

34 The escape velocity for a black hole exceeds the speed of light: $v_0 > 3\cdot 10^8$ m/sec. The Earth has $GM = 4\cdot 10^{14}\text{m}^3/\text{sec}^2$. If it were compressed to radius $R=$ ______, the Earth would be a black hole.

35 Show how the area under $y=1/2^x$ can be covered (draw a graph) by rectangles of area $1+\frac{1}{2}+\frac{1}{4}+\cdots = 2$. What is the exact area from $x=0$ to $x=\infty$?

36 Explain this paradox:

$$\int_{-b}^b \frac{x\,dx}{1+x^2} = 0 \text{ for every } b \text{ but } \int_{-\infty}^\infty \frac{x\,dx}{1+x^2} \text{ diverges.}$$

37 Compute the area between $y=\sec x$ and $y=\tan x$ for $0\leqslant x\leqslant \pi/2$. What is improper?

*38 Compute any of these integrals found by geniuses:

$$\int \frac{x^{-1/2}\,dx}{1+x} = \pi \qquad \int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx = \ln 2$$

$$\int_0^\infty x\,e^{-x}\cos x\,dx = 0 \qquad \int_0^\infty \cos x^2 dx = \sqrt{\pi/8}.$$

39 For which p is $\displaystyle\int_0^\infty \frac{dx}{x^p+x^{-p}} = \infty$?

40 Explain from Figure 7.6c why the red area is 2, when Figure 7.6a has red area 1.