CHAPTER 4

Derivatives by the Chain Rule

4.1 The Chain Rule

You remember that the derivative of $f(x)g(x)$ is not $(df/dx)(dg/dx)$. The derivative of sin x times x^2 is not cos x times $2x$. The product rule gave two terms, not one term. But there is another way of combining the sine function f and the squaring function g into a single function. The derivative of that new function does involve the cosine times $2x$ (but with a certain twist). We will first explain the new function, and then find the "***chain rule***" for its derivative.

May I say here that the chain rule is important. It is easy to learn, and you will use it often. I see it as the third basic way to find derivatives of new functions from derivatives of old functions. (So far the old functions are x^n, $\sin x$, and $\cos x$. Still ahead are e^x and $\log x$.) When f and g are added and multiplied, derivatives come from the *sum rule* and *product rule*. This section combines f and g in a third way.

The new function is $\sin(x^2)$—***the sine of*** x^2. It is created out of the two original functions: if $x = 3$ then $x^2 = 9$ and $\sin(x^2) = \sin 9$. There is a "chain" of functions, combining $\sin x$ and x^2 into the composite function $\sin(x^2)$. You start with x, *then find* $g(x)$, ***then find*** $f(g(x))$:

The squaring function gives $y = x^2$. This is $g(x)$.
The sine function produces $z = \sin y = \sin(x^2)$. This is $f(g(x))$.

The "***inside function***" $g(x)$ gives y. *This is the input to the* "***outside function***" $f(y)$. That is called ***composition***. It starts with x and ends with z. The composite function is sometimes written $f \circ g$ (the circle shows the difference from an ordinary product fg). More often you will see $f(g(x))$:

$$z(x) = f \circ g\,(x) = f(g(x)). \tag{1}$$

Other examples are $\cos 2x$ and $(2x)^3$, with $g = 2x$. *On a calculator you input x, then push the "g" button, then push the "f" button*:

From x **compute** $y = g(x)$ **From** y **compute** $z = f(y)$.

There is not a button for every function! But the squaring function and sine function are on most calculators, and they are used ***in that order***. Figure 4.1a shows how squaring will stretch and squeeze the sine function.

That graph of $\sin x^2$ is a crazy FM signal (the Frequency is Modulated). The wave goes up and down like $\sin x$, but not at the same places. Changing to $\sin g(x)$ moves the peaks left and right. Compare with a product $g(x)\sin x$, which is an AM signal (the Amplitude is Modulated).

Remark $f(g(x))$ is usually different from $g(f(x))$. ***The order of f and g is usually important***. For $f(x) = \sin x$ and $g(x) = x^2$, the chain in the opposite order $g(f(x))$ gives something different:

First apply the sine function: $y = \sin x$
Then apply the squaring function: $z = (\sin x)^2$.

That result is often written $\sin^2 x$, to save on parentheses. It is never written $\sin x^2$, which is totally different. Compare them in Figure 4.1.

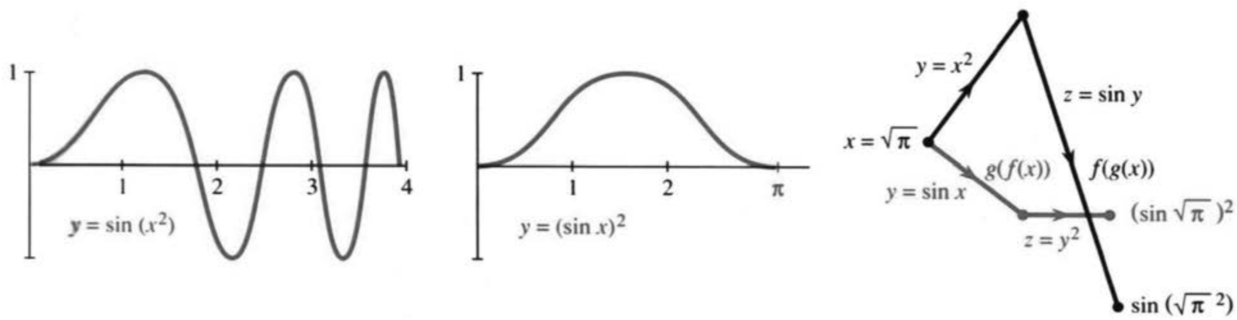


Fig. 4.1 $f(g(x))$ is different from $g(f(x))$. Apply g then f, or f then g.

EXAMPLE 1 The composite function $f \circ g$ can be deceptive. If $g(x) = x^3$ and $f(y) = y^4$, how does $f(g(x))$ differ from the ordinary product $f(x)g(x)$? The ordinary product is x^7. The chain starts with $y = x^3$, and then $z = y^4 = x^{12}$. The composition of x^3 and y^4 gives $f(g(x)) = x^{12}$.

EXAMPLE 2 In Newton's method, $F(x)$ is composed with itself. This is ***iteration***. Every output x_n is fed back as input, to find $x_{n+1} = F(x_n)$. The example $F(x) = \frac{1}{2}x + 4$ has $F(F(x)) = \frac{1}{2}\left(\frac{1}{2}x + 4\right) + 4$. That produces $z = \frac{1}{4}x + 6$.

The derivative of $F(x)$ is $\frac{1}{2}$. The derivative of $z = F(F(x))$ is $\frac{1}{4}$, which is $\frac{1}{2}$ times $\frac{1}{2}$. ***We multiply derivatives***. This is a special case of the chain rule.

An extremely special case is $f(x) = x$ and $g(x) = x$. The ordinary product is x^2. The chain $f(g(x))$ produces only x! The output from the "*identity function*" is $g(x) = x$.† When the second identity function operates on x it produces x again. The derivative is 1 times 1. I can give more composite functions in a table:

$y = g(x)$	$z = f(y)$	$z = f(g(x))$
$x^2 - 1$	$\sqrt{y}$	$\sqrt{x^2 - 1}$
$\cos x$	y^3	$(\cos x)^3$
2^x	2^y	2^{2^x}
$x + 5$	$y - 5$	x

† A calculator has no button for the identity function. It wouldn't do anything.

The last one adds 5 to get y. Then it subtracts 5 to reach z. So $z = x$. Here output equals input: $f(g(x)) = x$. These "***inverse functions***" are in Section 4.3. The other examples create new functions $z(x)$ and we want their derivatives.

THE DERIVATIVE OF $f(g(x))$

What is the derivative of $z = \sin x^2$? It is the limit of $\Delta z/\Delta x$. Therefore we look at a nearby point $x + \Delta x$. That change in x produces a change in $y = x^2$—which moves to $y + \Delta y = (x + \Delta x)^2$. From this change in y, there is a change in $z = f(y)$. It is a "domino effect," in which each changed input yields a changed output: Δx ***produces*** Δy ***produces*** Δz. We have to connect the final Δz to the original Δx.

The key is to write $\Delta z/\Delta x$ ***as*** $\Delta z/\Delta y$ ***times*** $\Delta y/\Delta x$. Then let Δx approach zero. In the limit, dz/dx is given by the "chain rule":

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \textit{ becomes the chain rule } \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{2}$$

As Δx goes to zero, the ratio $\Delta y/\Delta x$ approaches dy/dx. Therefore Δy must be going to zero, and $\Delta z/\Delta y$ approaches dz/dy. The limit of a product is the product of the separate limits (end of quick proof). *We multiply derivatives*:

4A ***Chain Rule*** Suppose $g(x)$ has a derivative at x and $f(y)$ has a derivative at $y = g(x)$. Then the derivative of $z = f(g(x))$ is

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(g(x))\,g'(x). \tag{3}$$

The slope at x is df/dy (at y) times dg/dx (at x).

Caution The chain rule does *not* say that the derivative of $\sin x^2$ is $(\cos x)(2x)$. True, $\cos y$ is the derivative of $\sin y$. The point is that $\cos y$ *must be evaluated at* y (not at x). We do not want df/dx at x, we want df/dy at $y = x^2$:

$$\textit{The derivative of } \sin x^2 \textit{ is } (\cos x^2) \textit{ times } (2x). \tag{4}$$

EXAMPLE 3 If $z = (\sin x)^2$ then $dz/dx = (2\sin x)(\cos x)$. Here $y = \sin x$ is *inside*.

In this order, $z = y^2$ leads to $dz/dy = 2y$. *It does not lead to* $2x$. The inside function $\sin x$ produces $dy/dx = \cos x$. The answer is $2y\cos x$. We have not yet found the function whose derivative is $2x \cos x$.

EXAMPLE 4 The derivative of $z = \sin 3x$ is $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx} = 3\cos 3x$.

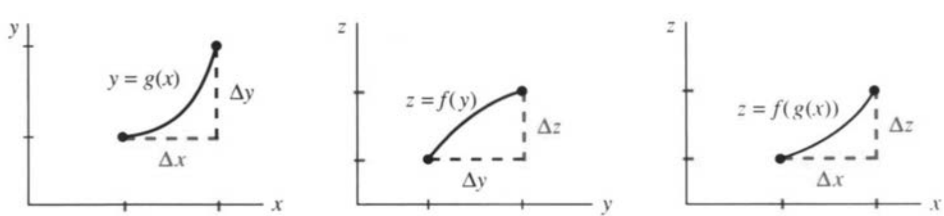


Fig. 4.2 The chain rule: $\dfrac{\Delta z}{\Delta x} = \dfrac{\Delta z}{\Delta y}\dfrac{\Delta y}{\Delta x}$ approaches $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$.

The outside function is $z = \sin y$. The inside function is $y = 3x$. Then $dz/dy = \cos y$—this is $\cos 3x$, not $\cos x$. Remember the other factor $dy/dx = 3$.

I can explain that factor 3, especially if x is switched to t. The distance is $z = \sin 3t$. That oscillates like $\sin t$ except *three times as fast*. The speeded-up function $\sin 3t$ completes a wave at time $2\pi/3$ (instead of 2π). Naturally the velocity contains the extra factor 3 from the chain rule.

EXAMPLE 5 Let $z = f(y) = y^n$. Find the derivative of $f(g(x)) = [g(x)]^n$.

In this case dz/dy is ny^{n-1}. The chain rule multiplies by dy/dx:

$$\frac{dz}{dx} = ny^{n-1}\frac{dy}{dx} \qquad \text{or} \qquad \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{dg}{dx}. \tag{5}$$

This is the ***power rule***! It was already discovered in Section 2.5. Square roots (when $n = 1/2$) are frequent and important. Suppose $y = x^2 - 1$:

$$\frac{d}{dx}\sqrt{x^2-1} = \frac{1}{2}(x^2-1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2-1}}. \tag{6}$$

Question A Buick uses $1/20$ of a gallon of gas per mile. You drive at 60 miles per hour. How many gallons per hour?

Answer (***Gallons/hour***) = (***gallons/mile***)(***miles/hour***). The chain rule is $(dy/dt) = (dy/dx)(dx/dt)$. The answer is $(1/20)(60) = 3$ gallons/hour.

Proof of the chain rule The discussion above was correctly based on

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \qquad \text{and} \qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{7}$$

It was here, over the chain rule, that the "battle of notation" was won by Leibniz. His notation practically tells you what to do: Take the limit of each term. (I have to mention that when Δx is approaching zero, it is theoretically possible that Δy might *hit* zero. If that happens, $\Delta z/\Delta y$ becomes $0/0$. We have to assign it the correct meaning, which is dz/dy.) As $\Delta x \to 0$,

$$\frac{\Delta y}{\Delta x} \to g'(x) \qquad \text{and} \qquad \frac{\Delta z}{\Delta y} \to f'(y) = f'(g(x)).$$

Then $\Delta z/\Delta x$ approaches $f'(y)$ times $g'(x)$, which is the chain rule $(dz/dy)(dy/dx)$. In the table below, the derivative of $(\sin x)^3$ is $3(\sin x)^2\cos x$. That extra factor $\cos x$ is easy to forget. It is even easier to forget the -1 in the last example.

$z = (x^3+1)^5$	$dz/dx = 5(x^3+1)^4$	times $3x^2$
$z = (\sin x)^3$	$dz/dx = 3\sin^2 x$	times $\cos x$
$z = (1-x)^2$	$dz/dx = 2(1-x)$	times -1

Important All kinds of letters are used for the chain rule. We named the output z. Very often it is called y, and the inside function is called u:

$$\textit{The derivative of } y = \sin u(x) \textit{ is } \frac{dy}{dx} = \cos u\frac{du}{dx}.$$

Examples with du/dx are extremely common. I have to ask you to accept whatever letters may come. What never changes is the key idea—*derivative of outside function times derivative of inside function*.

EXAMPLE 6 The chain rule is barely needed for $\sin(x-1)$. Strictly speaking the inside function is $u = x-1$. Then du/dx is just 1 (not -1). ***If*** $y = \sin(x-1)$ ***then*** $dy/dx = \cos(x-1)$. The graph is shifted and the slope shifts too.

Notice especially: The cosine is computed at $x-1$ and not at the unshifted x.

RECOGNIZING $f(y)$ AND $g(x)$

A big part of the chain rule is *recognizing the chain*. The table started with $(x^3+1)^5$. You look at it for a second. Then you see it as u^5. The inside function is $u = x^3+1$. With practice this decomposition (the opposite of composition) gets easy:

$$\cos(2x+1) \text{ is } \cos u \qquad \sqrt{1+\sin t} \text{ is } \sqrt{u} \qquad x\sin x \text{ is } \ldots \text{ (product rule!)}$$

In calculations, the careful way is to write down all the functions:

$$z = \cos u \quad u = 2x+1 \quad dz/dx = (-\sin u)(2) = -2\sin(2x+1).$$

The quick way is to keep in your mind "the derivative of what's inside." The slope of $\cos(2x+1)$ is $-\sin(2x+1)$, *times* 2 *from the chain rule*. The derivative of $2x+1$ is remembered—without z or u or f or g.

EXAMPLE 7 $\sin\sqrt{1-x}$ is a chain of $z = \sin y$, $y = \sqrt{u}$, $u = 1-x$ (*three functions*).

With that triple chain you will have the hang of the chain rule:

$$\text{The derivative of } \sin\sqrt{1-x} \text{ is } (\cos\sqrt{1-x})\left(\frac{1}{2\sqrt{1-x}}\right)(-1).$$

This is $(dz/dy)(dy/du)(du/dx)$. Evaluate them at the right places y, u, x.

Finally there is the question of *second derivatives*. The chain rule gives dz/dx as a product, so d^2z/dx^2 needs the product rule:

$$\underbrace{\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}}_{u \quad v} \qquad \text{leads to} \qquad \frac{d^2z}{dx^2} = \underbrace{\frac{dz}{dy}}_{u}\,\underbrace{\frac{d^2y}{dx^2}}_{v'} + \underbrace{\frac{d}{dx}\left(\frac{dz}{dy}\right)}_{u'}\underbrace{\frac{dy}{dx}}_{v}. \tag{8}$$

That last term needs the chain rule again. It becomes d^2z/dy^2 times $(dy/dx)^2$.

EXAMPLE 8 The derivative of $\sin x^2$ is $2x\cos x^2$. Then the product rule gives $d^2z/dx^2 = 2\cos x^2 - 4x^2\sin x^2$. In this case $y'' = 2$ and $(y')^2 = 4x^2$.

4.1 EXERCISES

Read-through questions

$z = f(g(x))$ comes from $z = f(y)$ and $y =$ __a__. At $x = 2$, the chain $(x^2-1)^3$ equals __b__. Its inside function is $y =$ __c__, its outside function is $z =$ __d__. Then dz/dx equals __e__. The first factor is evaluated at $y =$ __f__ (not at $y = x$). For $z = \sin(x^4-1)$ the derivative is __g__. The triple chain $z = \cos(x+1)^2$ has a shift and a __h__ and a cosine. Then $dz/dx =$ __i__.

The proof of the chain rule begins with $\Delta z/\Delta x =$ (__j__)(__k__) and ends with __l__. Changing letters, $y = \cos u(x)$ has $dy/dx =$ __m__. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx =$ __n__. The slope of $5g(x)$ is __o__ and the slope of $g(5x)$ is __p__. When $f =$ cosine and $g =$ sine and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are __q__.

In 1–10 identify $f(y)$ and $g(x)$. From their derivatives find $\dfrac{dz}{dx}$.

1 $z = (x^2-3)^3$

2 $z = (x^3-3)^2$

3 $z = \cos(x^3)$

4 $z = \tan 2x$

5 $z = \sqrt{\sin x}$ **6** $z = \sin \sqrt{x}$

7 $z = \tan(1/x) + 1/\tan x$ **8** $z = \sin(\cos x)$

9 $z = \cos(x^2 + x + 1)$ **10** $z = \sqrt{x^2}$

In 11–16 write down dz/dx. Don't write down f and g.

11 $z = \sin(17x)$ **12** $z = \tan(x + 1)$

13 $z = \cos(\cos x)$ **14** $z = (x^2)^{3/2}$

15 $z = x^2 \sin x$ **16** $z = (9x + 4)^{3/2}$

Problems 17–22 involve three functions $z(y)$, $y(u)$, and $u(x)$. Find dz/dx from $(dz/dy)(dy/du)(du/dx)$.

17 $z = \sin \sqrt{x + 1}$ **18** $z = \sqrt{\sin(x + 1)}$

19 $z = \sqrt{1 + \sin x}$ **20** $z = \sin(\sqrt{x} + 1)$

21 $z = \sin(1/\sin x)$ **22** $z = (\sin x^2)^2$

In 23–26 find dz/dx by the chain rule and also by rewriting z.

23 $z = ((x^2)^2)^2$ **24** $z = (3x)^3$

25 $z = (x + 1)^2 + \sin(x + \pi)$ **26** $z = \sqrt{1 - \cos^2 x}$

27 If $f(x) = x^2 + 1$ what is $f(f(x))$? If $U(x)$ is the unit step function (from 0 to 1 at $x = 0$) draw the graphs of $\sin U(x)$ and $U(\sin x)$. If $R(x)$ is the *ramp function* $\frac{1}{2}(x + |x|)$, draw the graphs of $R(x)$ and $R(\sin x)$.

28 (Recommended) If $g(x) = x^3$ find $f(y)$ so that $f(g(x)) = x^3 + 1$. Then find $h(y)$ so that $h(g(x)) = x$. Then find $k(y)$ so that $k(g(x)) = 1$.

29 If $f(y) = y - 2$ find $g(x)$ so that $f(g(x)) = x$. Then find $h(x)$ so that $f(h(x)) = x^2$. Then find $k(x)$ so that $f(k(x)) = 1$.

30 Find two different pairs $f(y)$, $g(x)$ so that $f(g(x)) = \sqrt{1 - x^2}$.

31 The derivative of $f(f(x))$ is ______. Is it $(df/dx)^2$? Test your formula on $f(x) = l/x$.

32 If $f(3) = 3$ and $g(3) = 5$ and $f'(3) = 2$ and $g'(3) = 4$, find the derivative at $x = 3$ if possible for

(a) $f(x)g(x)$ (b) $f(g(x))$ (c) $g(f(x))$ (d) $f(f(x))$

33 For $F(x) = \frac{1}{2}x + 8$, show how iteration gives $F(F(x)) = \frac{1}{4}x + 12$. Find $F(F(F(x)))$—also called $F^{(3)}(x)$. The derivative of $F^{(4)}(x)$ is ______.

34 In Problem 33 the limit of $F^{(n)}(x)$ is a constant $C =$ ______. From any start (try $x = 0$) the iterations $x_{n+1} = F(x_n)$ converge to C.

35 Suppose $g(x) = 3x + 1$ and $f(y) = \frac{1}{3}(y - 1)$. Then $f(g(x)) =$ ______ and $g(f(y)) =$ ______. These are *inverse functions*.

36 Suppose $g(x)$ is continuous at $x = 4$, say $g(4) = 7$. Suppose $f(y)$ is continuous at $y = 7$, say $f(7) = 9$. Then $f(g(x))$ is continuous at $x = 4$ and $f(g(4)) = 9$.

Proof ε is given. Because ______ is continuous, there is a δ such that $|f(g(x)) - 9| < \varepsilon$ whenever $|g(x) - 7| < \delta$. Then because ______ is continuous, there is a θ such that $|g(x) - 7| < \delta$ whenever $|x - 4| < \theta$. Conclusion: *If* $|x - 4| < \theta$ *then* ______. This shows that $f(g(x))$ approaches $f(g(4))$.

37 Only six functions can be constructed by compositions (in any sequence) of $g(x) = 1 - x$ and $f(x) = 1/x$. Starting with g and f, find the other four.

38 If $g(x) = 1 - x$ then $g(g(x)) = 1 - (1 - x) = x$. If $g(x) = 1/x$ then $g(g(x)) = 1/(1/x) = x$. Draw graphs of those g's and explain from the graphs why $g(g(x)) = x$. Find two more g's with this special property.

39 Construct functions so that $f(g(x))$ is always zero, but $f(y)$ is not always zero.

40 **True or false**

(a) If $f(x) = f(-x)$ then $f'(x) = f'(-x)$.

(b) The derivative of the identity function is zero.

(c) The derivative of $f(1/x)$ is $-1/(f(x))^2$.

(d) The derivative of $f(1 + x)$ is $f'(1 + x)$.

(e) The second derivative of $f(g(x))$ is $f''(g(x))g''(x)$.

41 On the same graph draw the parabola $y = x^2$ and the curve $z = \sin y$ (keep y upwards, with x and z across). Starting at $x = 3$ find your way to $z = \sin 9$.

42 On the same graph draw $y = \sin x$ and $z = y^2$ (y upwards for both). Starting at $x = \pi/4$ find $z = (\sin x)^2$ on the graph.

43 Find the second derivative of

(a) $\sin(x^2 + 1)$ (b) $\sqrt{x^2 - 1}$ (c) $\cos \sqrt{x}$

44 Explain why $\dfrac{d}{dx}\left(\dfrac{dz}{dy}\right) = \left(\dfrac{d^2z}{dy^2}\right)\left(\dfrac{dy}{dx}\right)$ in equation (8). Check this when $z = y^2$, $y = x^3$.

Final practice with the chain rule and other rules (and other letters!). Find the x or t derivative of z or y.

45 $z = f(u(t))$ **46** $z = u^3, u = x^3$

47 $y = \sin u(x) \cos u(x)$ **48** $y = \sqrt{u(t)}$

49 $y = x^2 u(x)$ **50** $y = f(x^2) = (f(x))^2$

51 $z = \sqrt{1 - u}, u = \sqrt{1 - x}$ **52** $z = 1/u^n(t)$

53 $z = f(u), u = v^2, v = \sqrt{t}$ **54** $y = u, u = x, x = 1/t$

55 If $f = x^4$ and $g = x^3$ then $f' = 4x^3$ and $g' = 3x^2$. The chain rule multiplies derivatives to get $12x^5$. But $f(g(x)) = x^{12}$ and its derivative is not $12x^5$. Where is the flaw?

56 The derivative of $y = \sin(\sin x)$ is $dy/dx =$

$\cos(\cos x)$ $\sin(\cos x) \cos x$ $\cos(\sin x) \cos x$ $\cos(\cos x) \cos x$.

57 (a) A book has 400 words per page. There are 9 pages per section. So there are ______ words per section.

(b) You read 200 words per minute. So you read ______ pages per minute. How many minutes per section?

58 (a) You walk in a train at 3 miles per hour. The train moves at 50 miles per hour. Your ground speed is ______ miles per hour.

(b) You walk in a train at 3 miles per hour. The train is shown on TV (1 mile train $=$ 20 inches on TV screen). Your speed across the screen is ______ inches per hour.

59 Coke costs $1/3$ dollar per bottle. The buyer gets ______ bottles per dollar. If $dy/dx = 1/3$ then dx/dy= ______.

60 (Computer) Graph $F(x) = \sin x$ and $G(x) = \sin(\sin x)$—not much difference. Do the same for $F'(x)$ and $G'(x)$. Then plot $F''(x)$ and $G''(x)$ to see where the difference shows up.

4.2 Implicit Differentiation and Related Rates

We start with the equations $xy = 2$ and $y^5 + xy = 3$. As x changes, these y's will change—to keep (x, y) on the curve. ***We want to know*** dy/dx ***at a typical point***. For $xy = 2$ that is no trouble, but the slope of $y^5 + xy = 3$ requires a new idea.

In the first case, solve for $y = 2/x$ and take its derivative: $dy/dx = -2/x^2$. The curve is a hyperbola. At $x = 2$ the slope is $-2/4 = -1/2$.

The problem with $y^5 + xy = 3$ is that it can't be solved for y. Galois proved that there is no solution formula for fifth-degree equations.† ***The function*** $y(x)$ ***cannot be given explicitly***. All we have is the *implicit* definition of y, as a solution to $y^5 + xy = 3$. The point $x = 2$, $y = 1$ satisfies the equation and lies on the curve, but how to find dy/dx?

This section answers that question. It is a situation that often occurs. Equations like $\sin y + \sin x = 1$ or $y \sin y = x$ (maybe even $\sin y = x$) are difficult or impossible to solve directly for y. Nevertheless we can find dy/dx at any point.

The way out is ***implicit differentiation***. Work with the equation as it stands. ***Find the*** x ***derivative of every term in*** $y^5 + xy = 3$. That includes the constant term 3, whose derivative is zero.

EXAMPLE 1 The power rule for y^5 and the product rule for xy yield

$$5y^4 \frac{dy}{dx} + x\frac{dy}{dx} + y = 0. \tag{1}$$

Now substitute the typical point $x = 2$ and $y = 1$, and solve for dy/dx:

$$5\frac{dy}{dx} + 2\frac{dy}{dx} + 1 = 0 \quad \text{produces} \quad \frac{dy}{dx} = -\frac{1}{7}. \tag{2}$$

This is implicit differentiation **(ID)**, and you see the idea: Include dy/dx from the chain rule, even if y is not known explicitly as a function of x.

EXAMPLE 2 $\sin y + \sin x = 1$ leads to $\cos y \dfrac{dy}{dx} + \cos x = 0$

EXAMPLE 3 $y \sin y = x$ leads to $y \cos y \dfrac{dy}{dx} = \sin y \dfrac{dy}{dx} = 1$

Knowing the slope makes it easier to draw the curve. We still need points (x, y) that satisfy the equation. Sometimes we can solve for x. Dividing $y^5 + xy = 3$ by y gives $x = 3/y - y^4$. Now the derivative (the x derivative!) is

$$1 = \left(-\frac{3}{y^2} - 4y^3\right)\frac{dy}{dx} = -7\frac{dy}{dx} \text{ at } y = 1. \tag{3}$$

Again $dy/dx = -1/7$. All these examples confirm the main point of the section:

> **4B** (***Implicit differentiation***) An equation $F(x, y) = 0$ can be differentiated directly by the chain rule, without solving for y in terms of x.

The example $xy = 2$, done implicitly, gives $x\,dy/dx + y = 0$. The slope dy/dx is $-y/x$. That agrees with the explicit slope $-2/x^2$.

ID is explained better by examples than theory (maybe everything is). The essential theory can be boiled down to one idea: "***Go ahead and differentiate***."

† That was before he went to the famous duel, and met his end. Fourth-degree equations do have a solution formula, but it is practically never used.

EXAMPLE 4 Find the tangent direction to the circle $x^2+y^2=25$.

We can solve for $y=\pm\sqrt{25-x^2}$, or operate directly on $x^2+y^2=25$:

$$2x+2y\frac{dy}{dx}=0 \qquad \text{or} \qquad \frac{dy}{dx}=-\frac{x}{y}. \tag{4}$$

Compare with the radius, which has slope y/x. The radius goes across x and up y. The tangent goes across $-y$ and up x. The slopes multiply to give $(-x/y)(y/x)=-1$.

To emphasize implicit differentiation, go on to the *second derivative*. The top of the circle is concave down, so d^2y/dx^2 is negative. Use the quotient rule on $-x/y$:

$$\frac{dy}{dx}=-\frac{x}{y} \quad \text{so} \quad \frac{d^2y}{dx^2}=-\frac{y\,dx/dx-x\,dy/dx}{y^2}=-\frac{y+(x^2/y)}{y^2}=-\frac{y^2+x^2}{y^3}. \tag{5}$$

RELATED RATES

There is a group of problems that has never found a perfect place in calculus. They seem to fit here—as applications of the chain rule. The problem is to compute df/dt, but the odd thing is that *we are given another derivative* dg/dt. To find df/dt, we need a relation between f and g.

The chain rule is $df/dt=(df/dg)(dg/dt)$. Here the variable is t because that is typical in applications. From the rate of change of g we find *the rate of change of* f. This is the problem of ***related rates***, and examples will make the point.

EXAMPLE 5 The radius of a circle is growing by $dr/dt=7$. How fast is the circumference growing? Remember that $C=2\pi r$ (this relates C to r).

Solution
$$\frac{dC}{dt}=\frac{dC}{dr}\frac{dr}{dt}=(2\pi)(7)=14\pi.$$

That is pretty basic, but its implications are amazing. Suppose you want to put a rope around the earth that any 7-footer can walk under. If the distance is $24{,}000$ miles, what is the additional length of the rope? Answer: Only 14π feet.

More realistically, if two lanes on a circular track are separated by 5 feet, how much head start should the outside runner get? Only 10π feet. If your speed around a turn is 55 and the car in the next lane goes 56, who wins? See Problem 14.

Examples 6–8 *are from the* 1988 *Advanced Placement Exams* (copyright 1989 by the College Entrance Examination Board). Their questions are carefully prepared.

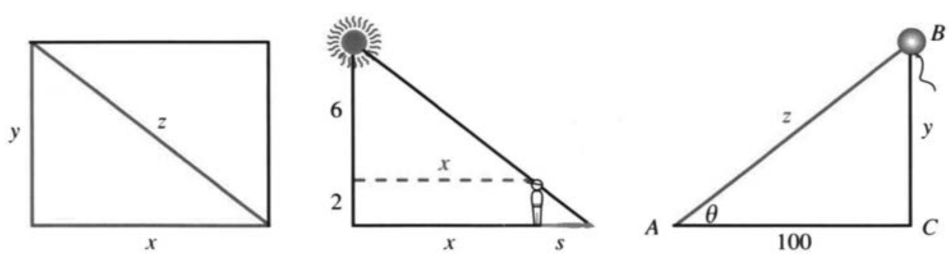


Fig. 4.3 Rectangle for Example 6, shadow for Example 7, balloon for Example 8.

EXAMPLE 6 The sides of the rectangle increase in such a way that $dz/dt=1$ and $dx/dt=3dy/dt$. At the instant when $x=4$ and $y=3$, what is the value of dx/dt?

Solution The key relation is $x^2+y^2=z^2$. Take its derivative (*implicitly*):

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt} \quad \text{produces} \quad 8\frac{dx}{dt}+6\frac{dy}{dt}=10.$$

We used all information, including $z=5$, except for $dx/dt=3dy/dt$. The term $6dy/dt$ equals $2dx/dt$, so we have $10dx/dt=10$. Answer: $dx/dt=1$.

EXAMPLE 7 A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person's shadow is lengthening at the rate of $4/9$ meters per second, at what rate in meters per second is the person walking?

Solution Draw a figure! You must relate the shadow length s to the distance x from the streetlight. The problem gives $ds/dt=4/9$ and asks for dx/dt:

$$\text{By similar triangles } \frac{x}{6}=\frac{s}{2} \quad \text{so} \quad \frac{dx}{dt}=\frac{6}{2}\frac{ds}{dt}=(3)\left(\frac{4}{9}\right)=\frac{4}{3}.$$

Note This problem was hard. I drew three figures before catching on to x and s. It is interesting that *we never knew x or s or the angle*.

EXAMPLE 8 An observer at point A is watching balloon B as it rises from point C. (*The figure is given*.) The balloon is rising at a constant rate of 3 meters per second (*this means $dy/dt=3$*) and the observer is 100 meters from point C.

(a) Find the rate of change in z at the instant when $y=50$. (*They want dz/dt*.)

$$z^2=y^2+100^2 \Rightarrow 2z\frac{dz}{dt}=2y\frac{dy}{dt}$$

$$z=\sqrt{50^2+100^2}=50\sqrt{5} \Rightarrow \frac{dz}{dt}=\frac{2\cdot 50\cdot 3}{2\cdot 50\sqrt{5}}=\frac{3\sqrt{5}}{5}.$$

(b) Find the rate of change in the area of right triangle BCA when $y=50$.

$$A=\frac{1}{2}(100)(y)=50y \qquad \frac{dA}{dt}=50\frac{dy}{dt}=50\cdot 3=150.$$

(c) Find the rate of change in θ when $y=50$. (*They want $d\theta/dt$*.)

$$y=50 \Rightarrow \cos\theta=\frac{100}{50\sqrt{5}}=\frac{2}{\sqrt{5}}$$

$$\tan\theta=\frac{y}{100} \Rightarrow \sec^2\theta\frac{d\theta}{dt}=\frac{1}{100}\frac{dy}{dt} \Rightarrow \frac{d\theta}{dt}=\left(\frac{2}{\sqrt{5}}\right)^2\frac{3}{100}=\frac{3}{125}$$

***In all problems I first wrote down a relation from the figure*. *Then I took its derivative*. *Then I substituted known information*.** (The substitution is *after* taking the derivative of $\tan\theta=y/100$. If we substitute $y=50$ too soon, the derivative of $50/100$ is useless.)

"Candidates are advised to show their work in order to minimize the risk of not receiving credit for it." 50% solved Example 6 and 21% solved Example 7. From $12,000$ candidates, the average on Example 8 (free response) was 6.1 out of 9.

EXAMPLE 9 A is a lighthouse and BC is the shoreline (same figure as the balloon). The light at A turns once a second ($d\theta/dt = 2\pi$ radians /second). How quickly does the receiving point B move up the shoreline?

Solution The figure shows $y = 100\tan\theta$. The speed dy/dt is $100\sec^2\theta\, d\theta/dt$. This is $200\pi\sec^2\theta$, so B speeds up as $\sec\theta$ increases.

Paradox When θ approaches a right angle, $\sec\theta$ approaches infinity. So does dy/dt. B ***moves faster than light***! This contradicts Einstein's theory of relativity. The paradox is resolved (I hope) in Problem 18.

If you walk around a light at A, your shadow at B seems to go faster than light. Same problem. This speed is impossible—something has been forgotten.

Smaller paradox (not destroying the theory of relativity). The figure shows $y = z\sin\theta$. Apparently $dy/dt = (dz/dt)\sin\theta$. ***This is totally wrong***. Not only is it wrong, the exact opposite is true: $dz/dt = (dy/dt)\sin\theta$. If you can explain that (Problem 15), then **ID** and related rates hold no terrors.

4.2 EXERCISES

Read-through questions

For $x^3 + y^3 = 2$ the derivative dy/dx comes from __a__ differentiation. We don't have to solve for __b__. Term by term the derivative is $3x^2+$ __c__ $= 0$. Solving for dy/dx gives __d__. At $x = y = 1$ this slope __e__. The equation of the tangent line is $y - 1 =$ __f__.

A second example is $y^2 = x$. The x derivative of this equation is __g__. Therefore $dy/dx =$ __h__. Replacing y by $\sqrt{x}$, this is $dy/dx =$ __i__.

In related rates, we are given dg/dt and we want df/dt. We need a relation between f and __j__. If $f = g^2$, then $(df/dt) =$ __k__ (dg/dt). If $f^2 + g^2 = 1$, then $df/dt =$ __l__. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt =$ __m__. To find a number (8 is wrong), you also need to know __n__.

By implicit differentiation find dy/dx **in** $1 - 10$.

1 $y^n + x^n = 1$

2 $x^2y + y^2x = 1$

3 $(x-y)^2 = 4$

4 $\sqrt{x} + \sqrt{y} = 3$ at $x = 4$

5 $x = F(y)$

6 $f(x) + F(y) = xy$

7 $x^2y = y^2x$

8 $x = \sin y$

9 $x = \tan y$

10 $y^n = x$ at $x = 1$

11 Show that the hyperbolas $xy = C$ are perpendicular to the hyperbolas $x^2 - y^2 = D$. (Perpendicular means that the product of slopes is -1.)

12 Show that the circles $(x-2)^2 + y^2 = 2$ and $x^2 + (y-2)^2 = 2$ are tangent at the point $(1,1)$.

13 At 25 meters/second, does your car turn faster or slower than a car traveling 5 meters further out at 26 meters/second? Your radius is (a) 50 meters (b) 100 meters.

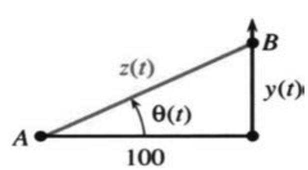
14 Equation (4) is $2x + 2y\, dy/dx = 0$ (on a circle). Directly by **ID** reach d^2y/dx^2 in equation (5).

Problems 15–18 resolve the speed of light paradox in Example 9.

15 (Small paradox first) The right triangle has $z^2 = y^2 + 100^2$. Take the t derivative to show that $z' = y'\sin\theta$.

16 (Even smaller paradox) As B moves up the line, why is dy/dt larger than dz/dt? Certainly z is larger than y, But as θ increases they become ______ .

17 (Faster than light) The derivative of $y = 100\tan\theta$ in Example 9 is $y' = 100\sec^2\theta\,\theta' = 200\pi\sec^2\theta$. Therefore y' passes c (the speed of light) when $\sec^2\theta$ passes ______. Such a speed is impossible—we forget that light takes time to reach B.



θ increases by 2π in 1 second

t is arrival time of light

θ is different from $2\pi t$

18 (Explanation by **ID**) Light travels from A to B in time z/c, distance over speed. Its arrival time is $t = \theta/2\pi + z/c$ so $\theta'/2\pi = 1 - z'/c$. Then $z' = y'\sin\theta$ and $y' = 100\sec^2\theta\,\theta'$ (all these are **ID**) lead to

$$y' = 200\pi c/(c\cos^2\theta + 200\pi\sin\theta)$$

As θ approaches $\pi/2$, this speed approaches ______.

Note: y' still exceeds c for some negative angle. That is for Einstein to explain. See the 1985 *College Math Journal*, page 186, and the 1960 *Scientific American*, "Things that go faster than light."

19 If a plane follows the curve $y = f(x)$, and its ground speed is $dx/dt = 500$ mph, how fast is the plane going up? How fast is the plane going?

20 Why can't we differentiate $x = 7$ and reach $1 = 0$?

Problems 21–29 are applications of related rates.

21 (Calculus classic) The bottom of a 10-foot ladder is going away from the wall at $dx/dt = 2$ feet per second. How fast is the top going down the wall? Draw the right triangle to find dy/dt when the height y is (a) 6 feet (b) 5 feet (c) zero.

22 The top of the 10-foot ladder can go faster than light. At what height y does $dy/dt = -c$?

23 How fast does the level of a Coke go down if you drink a cubic inch a second? The cup is a cylinder of radius 2 inches—first write down the volume.

24 A jet flies at 8 miles up and 560 miles per hour. How fast is it approaching you when (a) it is 16 miles from you; (b) its shadow is 8 miles from you (the sun is overhead); (c) the plane is 8 miles from you (exactly above)?

25 Starting from a $3-4-5$ right triangle, the short sides increase by 2 meters/second but the angle between them decreases by 1 radian/second. How fast does the area increase or decrease?

26 A pass receiver is at $x = 4$, $y = 8t$. The ball thrown at $t = 3$ is at $x = c(t-3)$, $y = 10c(t-3)$.

(a) Choose c so the ball meets the receiver.

*(b) At that instant the distance D between them is changing at what rate?

27 A thief is 10 meters away (8 meters ahead of you, across a street 6 meters wide). The thief runs on that side at 7 meters/second, you run at 9 meters/second. How fast are you approaching if (a) you follow on your side; (b) you run toward the thief; (c) you run away on your side?

28 A spherical raindrop evaporates at a rate equal to twice its surface area. Find dr/dt.

29 Starting from $P = V = 5$ and maintaining $PV = T$, find dV/dt if $dP/dt = 2$ and $dT/dt = 3$.

30 (a) The crankshaft AB turns twice a second so $d\theta/dt =$ ______.

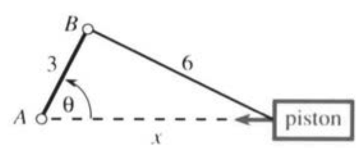
(b) Differentiate the cosine law $6^2 = 3^2 + x^2 - 2(3x\cos\theta)$ to find the piston speed dx/dt when $\theta = \pi/2$ and $\theta = \pi$.

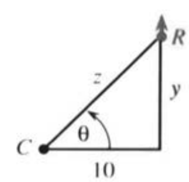
31 A camera turns at C to follow a rocket at R.

(a) Relate dz/dt to dy/dt when $y = 10$.

(b) Relate $d\theta/dt$ to dy/dt based on $y = 10\tan\theta$.

(c) Relate $d^2\theta/dt^2$ to d^2y/dt^2 and dy/dt.





4.3 Inverse Functions and Their Derivatives

There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "***inverse functions***." That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.

Inverse functions: Start with any input, say $x = 5$. *Compute* $y = g(x)$, say $y = 3$. Then compute $f(y)$, and *the answer must be* 5. What one function does, the inverse function undoes. If $g(5) = 3$ then $f(3) = 5$. ***The inverse function*** f ***takes the output*** y ***back to the input*** x.

EXAMPLE 1 $g(x) = x - 2$ and $f(y) = y + 2$ are inverse functions. Starting with $x = 5$, the function g subtracts 2. That produces $y = 3$. Then the function f adds 2. *That brings back* $x = 5$. To say it directly: ***The inverse of*** $y = x - 2$ ***is*** $x = y + 2$.

EXAMPLE 2 $y = g(x) = \frac{5}{9}(x - 32)$ and $x = f(y) = \frac{9}{5}y + 32$ are inverse functions (for temperature). Here x is degrees Fahrenheit and y is degrees Celsius. From $x = 32$ (freezing in Fahrenheit) you find $y = 0$ (freezing in Celsius). The inverse function takes $y = 0$ back to $x = 32$. Figure 4.4 shows how $x = 50°$F matches $y = 10°$C.

Notice that $\frac{5}{9}(x - 32)$ subtracts 32 *first*. The inverse $\frac{9}{5}y + 32$ adds 32 *last*. In the same way g multiplies last by $\frac{5}{9}$ while f multiplies first by $\frac{9}{5}$.

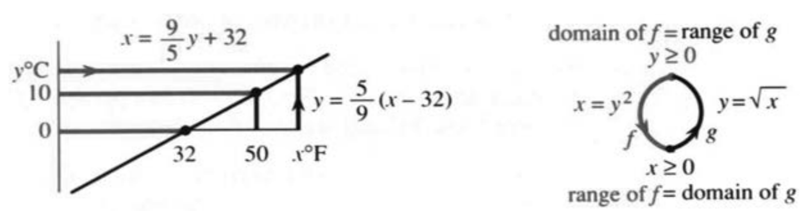


Fig. 4.4 °F to °C to °F. Always $g^{-1}(g(x)) = x$ and $g(g^{-1} = (y)) = y$. If $f = g^{-1}$ then $g = f^{-1}$.

The inverse function is written $f = g^{-1}$ ***and pronounced*** "g inverse." ***It is not*** $1/g(x)$.

If the demand y is a function of the price x, then the price is a function of the demand. Those are inverse functions. ***Their derivatives obey a fundamental rule***: dy/dx ***times*** dx/dy ***equals*** 1. In Example 2, dy/dx is $5/9$ and dx/dy is $9/5$.

There is another important point. When f and g are applied in the *opposite order*, they still come back to the start. First f adds 2, then g subtracts 2. The chain $g(f(y)) = (y + 2) - 2$ brings back y. ***If*** f ***is the inverse of*** g ***then*** g ***is the inverse of*** f. The relation is completely symmetric, and so is the definition:

Inverse function: ***If*** $y = g(x)$ ***then*** $x = g^{-1}(y)$. ***If*** $x = g^{-1}(y)$ ***then*** $y = g(x)$.

The loop in the figure goes from x to y to x. The composition $g^{-1}(g(x))$ is the "identity function." Instead of a new point z it returns to the original x. This will make the chain rule particularly easy—leading to $(dy/dx)(dx/dy) = 1$.

EXAMPLE 3 $y = g(x) = \sqrt{x}$ and $x = f(y) = y^2$ are inverse functions.

Starting from $x = 9$ we find $y = 3$. The inverse gives $3^2 = 9$. The square of $\sqrt{x}$ is $f(g(x)) = x$. In the opposite direction, the square root of y^2 is $g(f(y)) = y$.

Caution That example does not allow x to be negative. The domain of g—the set of numbers with square roots—is restricted to $x \geqslant 0$. This matches the range of g^{-1}. The outputs y^2 are nonnegative. With *domain of* $g =$ *range of* g^{-1}, the equation $x = (\sqrt{x})^2$ is possible and true. The nonnegative x goes into g and comes out of g^{-1}.

In this example y is also nonnegative. You might think we could square anything, but y must come back as the square root of y^2. So $y \geqslant 0$.

To summarize: ***The domain of a function matches the range of its inverse***. The inputs to g^{-1} are the outputs from g. The inputs to g are the outputs from g^{-1}.

If $g(x) = y$ ***then solving that equation for*** x ***gives*** $x = g^{-1}(y)$:

$$\text{if } y = 3x - 6 \quad \text{then } x = \tfrac{1}{3}(y+6) \quad (\text{this is } g^{-1}(y))$$
$$\text{if } y = x^3 + 1 \quad \text{then } x = \sqrt[3]{y-1} \quad (\text{this is } g^{-1}(y))$$

In practice that is how g^{-1} is computed: *Solve* $g(x) = y$. This is the reason inverses are important. Every time we solve an equation we are computing a value of g^{-1}.

Not all equations have one solution. ***Not all functions have inverses***. For each y, the equation $g(x) = y$ is only allowed to produce one x. That solution is $x = g^{-1}(y)$. If there is a second solution, then g^{-1} will not be a function—because a function cannot produce two x's from the same y.

EXAMPLE 4 There is more than one solution to $\sin x = \frac{1}{2}$. Many angles have the same sine. On the interval $0 \leqslant x \leqslant \pi$, the inverse of $y = \sin x$ is not a function. Figure 4.5 shows how two x's give the same y.

Prevent x from passing $\pi/2$ and the sine has an inverse. Write $x = \sin^{-1} y$.

The function g *has no inverse if two points* x_1 *and* x_2 *give* $g(x_1) = g(x_2)$. Its inverse would have to bring the same y back to x_1 and x_2. No function can do that; $g^{-1}(y)$ cannot equal both x_l and x_2. There must be only one x for each y.

To be invertible over an interval, g ***must be steadily increasing or steadily decreasing***.

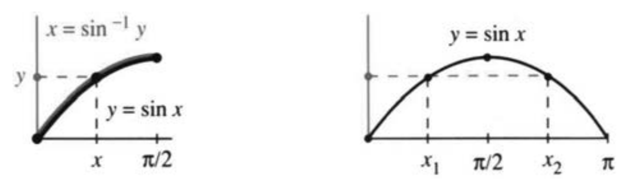


Fig. 4.5 Inverse exists (one x for each y). No inverse function (two x's for one y).

THE DERIVATIVE OF g^{-1}

It is time for calculus. Forgive me for this very humble example.

EXAMPLE 5 (ordinary multiplication) The inverse of $y = g(x) = 3x$ is $x = f(y) = \frac{1}{3}y$.

This shows with special clarity the rule for derivatives: ***The slopes*** $dy/dx = 3$ ***and*** $dx/dy = \frac{1}{3}$ ***multiply to give*** 1. This rule holds for all inverse functions, even if their slopes are not constant. It is a crucial application of the chain rule to the derivative of $f(g(x)) = x$.

4C (***Derivative of inverse function***) From $f(g(x)) = x$ the chain rule gives $f'(g(x))g'(x) = 1$. Writing $y = g(x)$ and $x = f(y)$, this rule looks better:

$$\frac{dx}{dy}\frac{dy}{dx} = 1 \qquad \textit{or} \qquad \frac{dx}{dy} = \frac{1}{dy/dx}. \tag{1}$$

The slope of $x = g^{-1}(y)$ times the slope of $y = g(x)$ equals one.

This is the chain rule with a special feature. Since $f(g(x)) = x$, *the derivative of both sides is* 1. If we know g' we now know f'. That rule will be tested on a familiar example. In the next section it leads to totally new derivatives.

EXAMPLE 6 The inverse of $y = x^3$ is $x = y^{1/3}$. We can find dx/dy two ways:

$$\text{directly}: \frac{dx}{dy} = \frac{1}{3}y^{-2/3} \qquad \text{indirectly}: \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3x^2} = \frac{1}{3y^{2/3}}.$$

The equation $(dx/dy)(dy/dx) = 1$ is not ordinary algebra, but it is true. Those derivatives are limits of fractions. The fractions are $(\Delta x/\Delta y)(\Delta y/\Delta x) = 1$ and we let $\Delta x \to 0$.

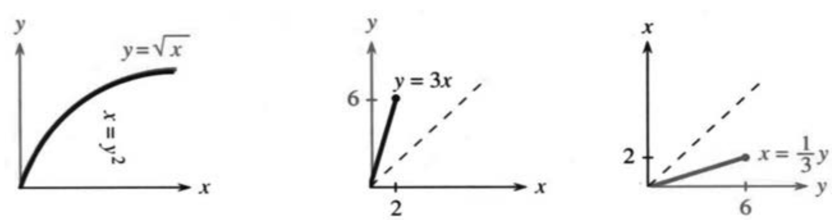


Fig. 4.6 Graphs of inverse functions: $x = \frac{1}{3}y$ is the mirror image of $y = 3x$.

Before going to new functions, I want to draw graphs. Figure 4.6 shows $y = \sqrt{x}$ and $y = 3x$. What is special is that *the same graphs also show the inverse functions*. The inverse of $y = \sqrt{x}$ is $x = y^2$. The pair $x = 4$, $y = 2$ is the same for both. That is the whole point of inverse functions—if $2 = g(4)$ then $4 = g^{-1}(2)$. Notice that the graphs go steadily up.

The only problem is, the graph of $x = g^{-1}(y)$ is on its side. To change the slope from 3 to $\frac{1}{3}$, you would have to turn the figure. After that turn there is another problem—the axes don't point to the right and up. You also have to look in a mirror! (The typesetter refused to print the letters backward. He thinks it's crazy but it's not.) To keep the book in position, and the typesetter in position, we need a better idea.

The graph of $x = \frac{1}{3}y$ comes from ***turning the picture across the*** $45°$ ***line***. The y axis becomes horizontal and x goes upward. The point $(2, 6)$ on the line $y = 3x$ goes into the point $(6, 2)$ on the line $x = \frac{1}{3}y$. The eyes see a reflection across the $45°$ line (Figure 4.6c). The mathematics sees the same pairs x and y. The special properties of g and g^{-1} allow us to know two functions—and draw two graphs—at the same time.† ***The graph of*** $x = g^{-1}(y)$ ***is the mirror image of the graph of*** $y = g(x)$.

†I have seen graphs with $y = g(x)$ and also $y = g^{-1}(x)$. For me that is wrong: it has to be $x = g^{-1}(y)$. If $y = \sin x$ then $x = \sin^{-1} y$.

EXPONENTIALS AND LOGARITHMS

I would like to add two more examples of inverse functions, because they are so important. Both examples involve the *exponential* and the *logarithm*. One is made up of linear pieces that imitate 2^x; it appeared in Chapter 1. The other is the true function 2^x, which is not yet defined—and it is not going to be defined here. The functions b^x and $\log_b y$ are so overwhelmingly important that they deserve and will get a whole chapter of the book (at least). But you have to see the graphs.

The slopes in the linear model are powers of 2. So are the heights y at the start of each piece. ***The slopes*** $1, 2, 4, \ldots$ ***equal the heights*** $1, 2, 4, \ldots$ ***at those special points***.

The inverse is a discrete model for the logarithm (to base 2). The logarithm of 1 is 0, because $2^0 = 1$. The logarithm of 2 is 1, because $2^1 = 2$. The logarithm of 2^j is the exponent j. Thus the model gives the correct $x = \log_2 y$ at the breakpoints $y = 1, 2, 4, 8, \ldots$. The slopes are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ because $dx/dy = 1/(dy/dx)$.

The model is good, but the real thing is better. The figure on the right shows the true exponential $y = 2^x$. At $x = 0, 1, 2, \ldots$ the heights y are the same as before. But now the height at $x = \frac{1}{2}$ is the number $2^{1/2}$, which is $\sqrt{2}$. The height at $x = .10$ is the tenth root $2^{1/10} = 1.07\ldots$. *The slope at* $x = 0$ *is no longer* 1—*it is closer to* $\Delta y/\Delta x = .07/.10$. The exact slope is a number c (near .7) that we are not yet prepared to reveal.

The special property of $y = 2^x$ is that the slope at all points is cy. ***The slope is proportional to the function***. The exponential solves $dy/dx = cy$.

Now look at the inverse function—*the logarithm*. Its graph is the mirror image:

$$\textit{If } y = 2^x \textit{ then } x = \log_2 y. \textit{ If } 2^{1/10} \approx 1.07 \textit{ then } \log_2 1.07 \approx 1/10.$$

What the exponential does, the logarithm undoes—and vice versa. ***The logarithm of*** 2^x ***is the exponent*** x. Since the exponential starts with slope c, the logarithm must start with slope $1/c$. Check that numerically. The logarithm of 1.07 is near $1/10$. The slope is near $.10/.07$. The beautiful property is that $dx/dy = 1/cy$.

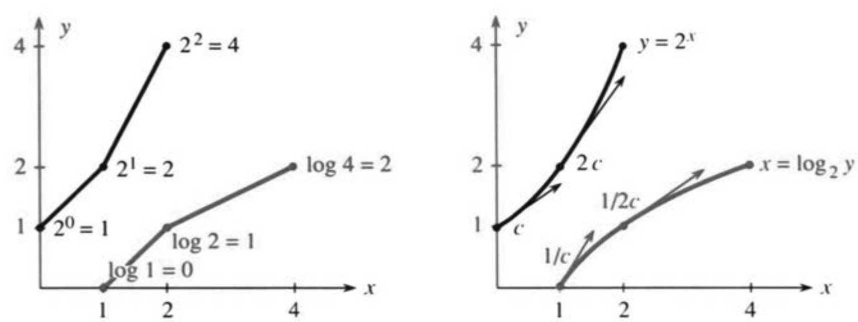


Fig. 4.7 Piecewise linear models and smooth curves: $y = 2^x$ and $x = \log_2 y$. Base $b = 2$.

I have to mention that calculus avoids logarithms to base 2. The reason lies in that mysterious number c. It is the "natural logarithm" of 2, which is $.693147\ldots$ —and who wants that? Also $1/.693147\ldots$ enters the slope of $\log_2 y$. Then $(dx/dy)(dy/dx) = 1$. The right choice is to use "natural logarithms" throughout. In place of 2, they are based on the special number e:

$$y = e^x \textbf{\textit{ is the inverse of }} x = \ln y. \tag{2}$$

The derivatives of those functions are sensational—they are saved for Chapter 6. Together with x^n and $\sin x$ and $\cos x$, they are the backbone of calculus.

Note It is almost possible to go directly to Chapter 6. The inverse functions $x = \sin^{-1} y$ and $x = \tan^{-1} y$ can be done quickly. The reason for including integrals first (Chapter 5) is that they solve differential equations with no guesswork:

$$\frac{dy}{dx} = y \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{y} \quad \text{leads to} \quad \int dx = \int \frac{dy}{y} \quad \text{or} \quad x = \ln y + C.$$

Integrals have applications of all kinds, spread through the rest of the book. But do not lose sight of 2^x and e^x. They solve $dy/dx = cy$—the key to applied calculus.

THE INVERSE OF A CHAIN $h(g(x))$

The functions $g(x) = x - 2$ and $h(y) = 3y$ were easy to invert. For g^{-1} we added 2, and for h^{-1} we divided by 3. Now the question is: If we create the composite function $z = h(g(x))$, or $z = 3(x-2)$, what is its inverse?

Virtually all known functions are created in this way, from chains of simpler functions. *The problem is to invert a chain using the inverse of each piece*. The answer is one of the fundamental rules of mathematics:

> **4D** The inverse of $z = h(g(x))$ is a chain of inverses ***in the opposite order***:
>
> $$x = g^{-1}(h^{-1}(z)). \tag{3}$$
>
> h^{-1} is applied first because h was applied last: $g^{-1}(h^{-1}(h(g(x)))) = x$.

That last equation looks like a mess, but it holds the key. In the middle you see h^{-1} and h. That part of the chain does nothing! The inverse functions cancel, to leave $g^{-1}(g(x))$. *But that is* x. The whole chain collapses, when g^{-1} and h^{-1} are in the correct order—which is opposite to the order of $h(g(x))$.

EXAMPLE 7 $z = h(g(x)) = 3(x-2)$ and $x = g^{-1}(h^{-1}(z)) = \frac{1}{3}z + 2$.

First h^{-1} divides by 3. Then g^{-1} adds 2. The inverse of $h \circ g$ is $g^{-1} \circ h^{-1}$. *It can be found directly by solving* $z = 3(x-2)$. A chain of inverses is like writing in prose—we do it without knowing it.

EXAMPLE 8 Invert $z = \sqrt{x-2}$ by writing $z^2 = x - 2$ and then $x = z^2 + 2$.

The inverse adds 2 and takes the square—*but not in that order*. That would give $(z+2)^2$, which is wrong. The correct order is $z^2 + 2$.

The domains and ranges are explained by Figure 4.8. We start with $x \geqslant 2$. Subtracting 2 gives $y \geqslant 0$. Taking the square root gives $z \geqslant 0$. Taking the square brings back $y \geqslant 0$. Adding 2 brings back $x \geqslant 2$—which is in the original domain of g.

Fig. 4.8 The chain $g^{-1}(h^{-1}(h(g(x)))) = x$ is one-to-one at every step.

EXAMPLE 9 Inverse matrices $(AB)^{-1} = B^{-1}A^{-1}$ (this linear algebra is optional).

Suppose a vector x is multiplied by a square matrix B: $y = g(x) = Bx$. The inverse function multiplies by the ***inverse matrix***: $x = g^{-1}(y) = B^{-1}y$. It is like multiplication by $B = 3$ and $B^{-1} = 1/3$, except that x and y are vectors.

Now suppose a second function multiplies by another matrix A: $z = h(g(x)) = ABx$. The problem is to recover x from z. The first step is to invert A, because that came last: $Bx = A^{-1}z$. Then the second step multiplies by B^{-1} and brings back $x = B^{-1}A^{-1}z$. ***The product*** $B^{-1}A^{-1}$ ***inverts the product*** AB. The rule for matrix inverses is like the rule for function inverses—in fact it is a special case.

I had better not wander too far from calculus. The next section introduces the inverses of the sine and cosine and tangent, and finds their derivatives. Remember that the ultimate source is the chain rule.

4.3 EXERCISES

Read-through questions

The functions $g(x) = x - 4$ and $f(y) = y + 4$ are __a__ functions, because $f(g(x)) =$ __b__. Also $g(f(y)) =$ __c__. The notation is $f = g^{-1}$ and $g =$ __d__. The composition __e__ is the identity function. By definition $x = g^{-1}(y)$ if and only if $y =$ __f__. When y is in the range of g, it is in the __g__ of g^{-1}. Similarly x is in the __h__ of g when it is in the __i__ of g^{-1}. If g has an inverse then $g(x_1)$ __j__ $g(x_2)$ at any two points. The function g must be steadily __k__ or steadily __l__.

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)($__m__$)$ = __n__. The slope of g^{-1} times the slope of g equals __o__. More directly $dx/dy = 1/$__p__. For $y = 2x + 1$ and $x = \frac{1}{2}(y-1)$, the slopes are $dy/dx =$ __q__ and $dx/dy =$ __r__. For $y = x^2$ and $x =$ __s__, the slopes are $dy/dx =$ __t__ and $dx/dy =$ __u__. Substituting x^2 for y gives $dx/dy =$ __v__. Then $(dx/dy)(dy/dx) =$ __w__.

The graph of $y = g(x)$ is also the graph of $x =$ __x__, but with x across and y up. For an ordinary graph of g^{-1}, take the reflection in the line __y__. If $(3, 8)$ is on the graph of g, then its mirror image (__z__) is on the graph of g^{-1}. Those particular points satisfy $8 = 2^3$ and $3 =$ __A__.

The inverse of the chain $z = h(g(x))$ is the chain $x =$ __B__. If $g(x) = 3x$ and $h(y) = y^3$ then $z =$ __C__. Its inverse is $x =$ __D__, which is the composition of __E__ and __F__.

Solve equations 1–10 for x, to find the inverse function $x = g^{-1}(y)$. When more than one x gives the same y, write "no inverse."

1 $y = 3x - 6$ **2** $y = Ax + B$

3 $y = x^2 - 1$ **4** $y = x/(x-1)$ [solve $xy - y = x$]

5 $y = 1 + x^{-1}$ **6** $y = |x|$

7 $y = x^3 - 1$ **8** $y = 2x + |x|$

9 $y = \sin x$ **10** $y = x^{1/5}$ [draw graph]

11 Solving $y = \dfrac{1}{x-a}$ gives $xy - ay = 1$ *or* $x = \dfrac{1+ay}{y}$. Now solve that equation for y.

12 Solving $y = \dfrac{x+1}{x-1}$ gives $xy - y = x + 1$ or $x = \dfrac{y+1}{y-1}$. Draw the graph to see why f and f^{-1} are the same. Compute dy/dx and dx/dy.

13 Suppose f is increasing and $f(2) = 3$ and $f(3) = 5$. What can you say about $f^{-1}(4)$?

14 Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 5$. What can you say about f^{-1}?

15 Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 0$. How do you know that there is no function f^{-1}?

16 ***Vertical line test***: If no vertical line touches its graph twice then $f(x)$ is a ***function*** (one y for each x). ***Horizontal line test***: If no horizontal line touches its graph twice then $f(x)$ is ***invertible*** because ______ .

17 If $f(x)$ and $g(x)$ are increasing, which two of these might not be increasing?

$f(x) + g(x)$ $f(x)g(x)$ $f(g(x))$ $f^{-1}(x)$ $1/f(x)$

18 If $y = 1/x$ then $x = 1/y$. If $y = 1 - x$ then $x = 1 - y$. The graphs are their own mirror images in the 45° line. Construct two more functions with this property $f = f^{-1}$ or $f(f(x)) = x$.

19 For which numbers m are these functions invertible?
(a) $y = mx + b$ (b) $y = mx + x^3$ (c) $y = mx + \sin x$

20 From its graph show that $y = |x| + cx$ is invertible if $c > 1$ and also if $c < -1$. The inverse of a piecewise linear function is piecewise ______ .

In 21–26 find dy/dx in terms of x and dx/dy in terms of y.

21 $y = x^5$

22 $y = 1/(x-1)$

23 $y = x^3 - 1$

24 $y = 1/x^3$

25 $y = \dfrac{x}{x-1}$

26 $y = \dfrac{ax+b}{cx+d}$

27 If $dy/dx = 1/y$ then $dx/dy =$ ______ and $x =$ ______.

28 If $dx/dy = 1/y$ then $dy/dx =$ ______ (these functions are $y = e^x$ and $x = \ln y$, soon to be honored properly).

29 The slopes of $f(x) = \frac{1}{3}x^3$ and $g(x) = -1/x$ are x^2 and $1/x^2$. Why isn't $f = g^{-1}$? What is g^{-1}? Show that $g'(g^{-1})' = 1$.

30 At the points x_1, x_2, x_3 a piecewise constant function jumps to y_1, y_2, y_3. Draw its graph starting from $y(0) = 0$. The mirror image is piecewise constant with jumps at the points ______ to the heights ______. Why isn't this the inverse function?

In 31–38 draw the graph of $y = g(x)$. Separately draw its mirror image $x = g^{-1}(y)$.

31 $y = 5x - 10$

32 $y = \cos x$, $0 \leqslant x \leqslant \pi$

33 $y = 1/(x+1)$

34 $y = |x| - 2x$

35 $y = 10^x$

36 $y = \sqrt{1-x^2}$, $0 \leqslant x \leqslant 1$

37 $y = 2^{-x}$

38 $y = 1/\sqrt{1-x^2}$, $0 \leqslant x < 1$

In 39–42 find dx/dy at the given point.

39 $y = \sin x$ at $x = \pi/6$

40 $y = \tan x$ at $x = \pi/4$

41 $y = \sin x^2$ at $x = 3$

42 $y = x - \sin x$ at $x = 0$

43 If y is a decreasing function of x, then x is a ______ function of y. Prove by graphs and by the chain rule.

44 If $f(x) > x$ for all x, show that $f^{-1}(y) < y$.

45 ***True or false***, with example:

(a) If $f(x)$ is invertible so is $h(x) = (f(x))^2$.

(b) If $f(x)$ is invertible so is $h(x) = f(f(x))$.

(c) $f^{-1}(y)$ has a derivative at every y.

In the ehains 46–51 write down $g(x)$ and $f(y)$ and their inverses. Then find $x = g^{-1}(f^{-1}(z))$.

46 $z = 5(x-4)$

47 $z = (x^m)^n$

48 $z = (6+x)^3$

49 $z = 6 + x^3$

50 $z = \frac{1}{2}\left(\frac{1}{2}x + 4\right) + 4$

51 $z = \log(10^x)$

52 Solving $f(x) = 0$ is a large part of applied mathematics. Express the solution x^* in terms of f^{-1}: $x^* =$ ______.

53 (a) Show by example that d^2x/dy^2 is not $1/(d^2y/dx^2)$.

(b) If y is in meters and x is in seconds, then d^2y/dx^2 is in ______ and d^2x/dy^2 is in ______.

54 Newton's method solves $f(x^*) = 0$ by applying a linear approximation to f^{-1}:

$$f^{-1}(0) \approx f^{-1}(y) + (df^{-1}/dy)(0-y).$$

For $y = f(x)$ this is Newton's equation $x^* \approx x +$ ______.

55 If the demand is $1/(p+1)^2$ when the price is p, then the demand is y when the price is ______. If the range of prices is $p \geqslant 0$, what is the range of demands?

56 If $dF/dx = f(x)$ show that the derivative of $G(y) = yf^{-1}(y) - F(f^{-1}(y))$ is $f^{-1}(y)$.

57 For each number y find the maximum value of $yx - 2x^4$. This maximum is a function $G(y)$. Verify that the derivatives of $G(y)$ and $2x^4$ are inverse functions.

58 (for professors only) If $G(y)$ is the maximum value of $yx - F(x)$, prove that $F(x)$ is the maximum value of $xy - G(y)$. Assume that $f(x) = dF/dx$ is increasing, like $8x^3$ in Problem 57.

59 Suppose the richest x percent of people in the world have $10\sqrt{x}$ percent of the wealth. Then y percent of the wealth is held by ______ percent of the people.

4.4 Inverses of Trigonometric Functions

Mathematics is built on basic functions like the sine, and on basic ideas like the inverse. Therefore ***it is totally natural to invert the sine function***. The graph of $x = \sin^{-1} y$ is a mirror image of $y = \sin x$. This is a case where we pay close attention to the domains, since the sine goes up and down infinitely often. We only want *one piece* of that curve, in Figure 4.9.

For the bold line the domain is restricted. *The angle x lies between $-\pi/2$ and $+\pi/2$*. On that interval the sine is increasing, so ***each y comes from exactly one angle x***. If the whole sine curve is allowed, infinitely many angles would have $\sin x = 0$. The sine function could not have an inverse. By restricting to an interval where $\sin x$ is increasing, we make the function invertible.

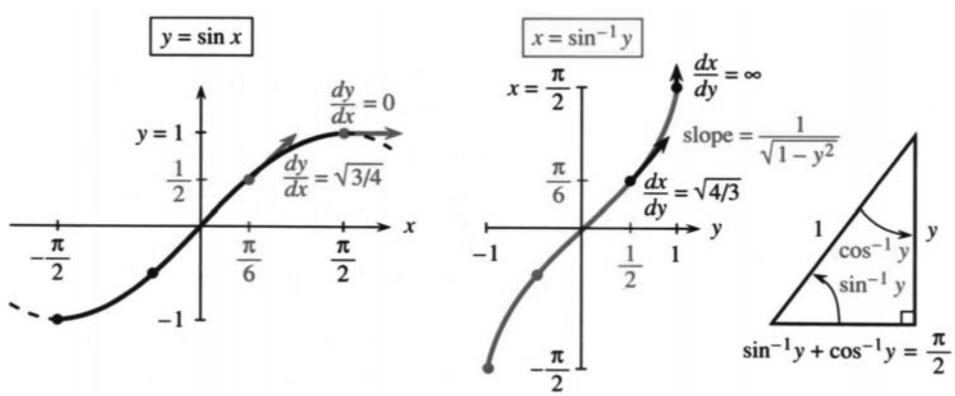


Fig. 4.9 Graphs of $\sin x$ and $\sin^{-1} y$. Their slopes are $\cos x$ and $1/\sqrt{1-y^2}$.

The inverse function brings y back to x. It is $x = \sin^{-1} y$ (the ***inverse sine***):

$$x = \sin^{-1} y \text{ when } y = \sin x \text{ and } |x| \leqslant \pi/2. \tag{1}$$

The inverse starts with a number y between -1 and 1. It produces an angle $x = \sin^{-1} y$—***the angle whose sine is*** y. The angle x is between $-\pi/2$ and $\pi/2$, with the required sine. Historically x was called the "arc sine" of y, and *arcsin* is used in computing. The mathematical notation is $\sin^{-1}$. *This has nothing to do with* $1/\sin x$.

The figure shows the 30° angle $x = \pi/6$. Its sine is $y = \frac{1}{2}$. ***The inverse sine of*** $\frac{1}{2}$ ***is*** $\pi/6$. Again: The symbol $\sin^{-1}(1)$ stands for the angle whose sine is 1 (this angle is $x = \pi/2$). We are seeing $g^{-1}(g(x)) = x$:

$$\sin^{-1}(\sin x) = x \ \text{ for } \ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \qquad \sin(\sin^{-1} y) = y \ \text{ for } \ -1 \leqslant y \leqslant 1.$$

EXAMPLE 1 (important) If $\sin x = y$ find a formula for $\cos x$.

Solution We are given the sine, we want the cosine. The key to this problem must be $\cos^2 x = 1 - \sin^2 x$. When the sine is y, the cosine is the square root of $1 - y^2$:

$$\cos x = \cos(\sin^{-1} y) = \sqrt{1 - y^2}. \tag{2}$$

This formula is crucial for computing derivatives. We use it immediately.

THE DERIVATIVE OF THE INVERSE SINE

The calculus problem is to find the slope of the inverse function $f(y) = \sin^{-1} y$. The chain rule gives (*slope of inverse function*) $= 1/($*slope of original function*$)$. Certainly the slope of $\sin x$ is $\cos x$. To switch from x to y, use equation (2):

$$y = \sin x \text{ gives } \frac{dy}{dx} = \cos x \text{ so that } \frac{dx}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}. \qquad (3)$$

This derivative $1/\sqrt{1-y^2}$ gives a new *v–f* pair that is extremely valuable in calculus:

$$\textit{velocity} \quad v(t) = 1/\sqrt{1-t^2} \qquad \textit{distance} \quad f(t) = \sin^{-1} t.$$

Inverse functions will soon produce two more pairs, from the derivatives of $\tan^{-1} y$ and $\sec^{-1} y$. ***The table at the end lists all the essential facts***.

EXAMPLE 2 The slope of $\sin^{-1} y$ at $y = 1$ is *infinite*: $1/\sqrt{1-y^2} = 1/0$. Explain.

At $y = 1$ the graph of $y = \sin x$ is horizontal. The slope is zero. So its mirror image is vertical. The slope $1/0$ is an extreme case of the chain rule.

Question What is $d/dx(\sin^{-1} x)$? Answer $1/\sqrt{1-x^2}$. I just changed letters.

THE INVERSE COSINE AND ITS DERIVATIVE

Whatever is done for the sine can be done for the cosine. But the domain and range have to be watched. The graph cannot be allowed to go up and down. Each y from -1 to 1 should be the cosine of *only one angle* x. That puts x between 0 and π. Then the cosine is steadily decreasing and $y = \cos x$ has an inverse:

$$\cos^{-1}(\cos x) = x \text{ and } \cos(\cos^{-1} y) = y. \qquad (4)$$

The cosine of the angle $x = 0$ is the number $y = 1$. The inverse cosine of $y = 1$ is the angle $x = 0$. Those both express the same fact, that $\cos 0 = 1$.

For the slope of $\cos^{-1} y$, we could copy the calculation that succeeded for $\sin^{-1} y$. The chain rule could be applied as in (3). But there is a faster way, because of a special relation between $\cos^{-1} y$ and $\sin^{-1} y$. ***Those angles always add to a right angle***:

$$\cos^{-1} y + \sin^{-1} y = \pi/2. \qquad (5)$$

Figure 4.9c shows the angles and Figure 4.10c shows the graphs. The sum is $\pi/2$ (the dotted line), and its derivative is zero. So the derivatives of $\cos^{-1} y$ and $\sin^{-1} y$ must add to zero. *Those derivatives have opposite sign*. There is a ***minus*** for the inverse cosine, and its graph goes downward:

$$\textbf{\textit{The derivative of}} \;\; x = \cos^{-1} y \quad \textbf{\textit{is}} \quad dx/dy = -1/\sqrt{1-y^2}. \qquad (6)$$

Question How can two functions $x = \sin^{-1} y$ and $x = -\cos^{-1} y$ have the *same derivative*?

Answer $\sin^{-1} y$ must be the same as $-\cos^{-1} y + C$. Equation (5) gives $C = \pi/2$.

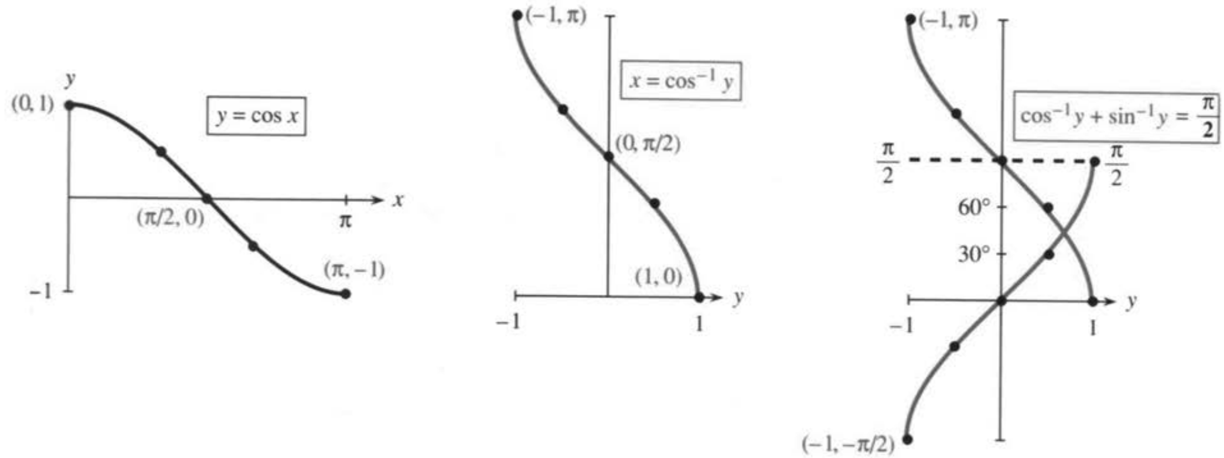


Fig. 4.10 The graphs of $y = \cos x$ and $x = \cos^{-1} y$. Notice the domain $0 \leqslant x \leqslant \pi$.

THE INVERSE TANGENT AND ITS DERIVATIVE

The tangent is $\sin x/\cos x$. The inverse tangent is *not* $\sin^{-1} y/\cos^{-1} y$. The inverse function produces ***the angle whose tangent is*** y. Figure 4.11 shows that angle, which is between $-\pi/2$ and $\pi/2$. The tangent can be any number, but the inverse tangent is in the *open interval* $-\pi/2 < x < \pi/2$. (The interval is "open" because its endpoints are not included.) The tangents of $\pi/2$ and $-\pi/2$ are not defined.

The slope of $y = \tan x$ is $dy/dx = \sec^2 x$. What is the slope of $x = \tan^{-1} y$?

$$\text{By the chain rule } \frac{dx}{dy} = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}. \tag{7}$$

4E *The derivative of* $f(y) = \tan^{-1} y$ *is* $\dfrac{df}{dy} = \dfrac{1}{1+y^2}$. (8)

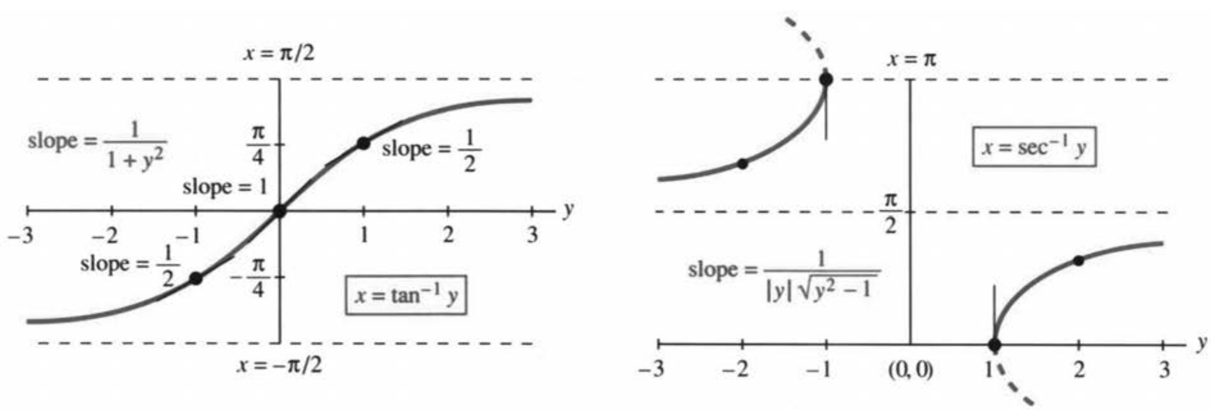


Fig. 4.11 $x = \tan^{-1} y$ has slope $1/(1+y^2)$. $x = \sec^{-1} y$ has slope $1/|y|\sqrt{y^2-1}$.

EXAMPLE 3 The tangent of $x = \pi/4$ is $y = 1$. We check slopes. On the inverse tangent curve, $dx/dy = 1/(1+y^2) = \frac{1}{2}$. On the tangent curve, $dy/dx = \sec^2 x$. At

$\pi/4$ the secant squared equals 2. The slopes $dx/dy = \frac{1}{2}$ and $dy/dx = 2$ multiply to give 1.

Important Soon will come the following question. *What function has the derivative* $1/(1+x^2)$? One reason for reading this section is to learn the answer. The function is in equation (8)—*if we change letters*. ***It is*** $f(x) = \tan^{-1} x$ ***that has slope*** $1/(1+x^2)$.

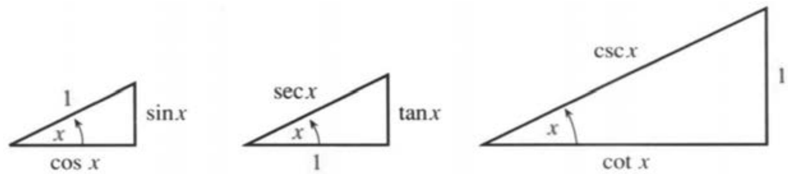


Fig. 4.12 $\cos^2 x + \sin^2 x = 1$ and $1+\tan^2 x = \sec^2 x$ and $1+\cot^2 x = \csc^2 x$.

INVERSE COTANGENT, INVERSE SECANT, INVERSE COSECANT

There is no way we can avoid completing this miserable list! But it can be painless. The idea is to use $1/(dy/dx)$ for $y = \cot x$ and $y = \sec x$ and $y = \csc x$:

$$\frac{dx}{dy} = \frac{-1}{\csc^2 x} \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{\sec x \tan x} \quad \text{and} \quad \frac{dx}{dy} = \frac{-1}{\csc x \cot x}. \tag{9}$$

In the middle equation, replace $\sec x$ by y and $\tan x$ by $\pm\sqrt{y^2-1}$. Choose the sign for positive slope (compare Figure 4.11). That gives the middle equation in (10):

The derivatives of $\cot^{-1} y$ ***and*** $\sec^{-1} y$ ***and*** $\csc^{-1} y$ ***are***

$$\frac{d}{dy}(\cot^{-1} y) = \frac{-1}{1+y^2} \quad \frac{d}{dy}(\sec^{-1} y) = \frac{1}{|y|\sqrt{y^2-1}} \quad \frac{d}{dy}(\csc^{-1} y) = \frac{-1}{|y|\sqrt{y^2-1}}. \tag{10}$$

Note about the inverse secant When y is negative there is a choice for $x = \sec^{-1} y$. We selected the angle in the second quadrant (between $\pi/2$ and π). Its cosine is negative, so its secant is negative. This choice makes $\sec^{-1} y = \cos^{-1}(1/y)$, which matches $\sec x = 1/\cos x$. It also makes $\sec^{-1} y$ an increasing function, where $\cos^{-1} y$ is a decreasing function. So we needed the absolute value $|y|$ in the derivative.

Some mathematical tables make a different choice. The angle x could be in the third quadrant (between $-\pi$ and $-\pi/2$). Then the slope omits the absolute value.

Summary For the six inverse functions it is only necessary to learn three derivatives. The other three just have minus signs, as we saw for $\sin^{-1} y$ and $\cos^{-1} y$. Each inverse function and its "cofunction" add to $\pi/2$, so their derivatives add to zero. Here are the six functions for quick reference, with the three new derivatives.

function $f(y)$	***inputs*** y	***outputs*** x	***slope*** dx/dy
$\sin^{-1} y$, $\cos^{-1} y$	$\lvert y\rvert \leqslant 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0,\pi]$	$\pm\frac{1}{\sqrt{1-y^2}}$
$\tan^{-1} y$, $\cot^{-1} y$	all y	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $(0,\pi)$	$\pm\frac{1}{1+y^2}$
$\sec^{-1} y$, $\csc^{-1} y$	$\lvert y\rvert \geqslant 1$	$[0,\pi]^*$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^*$	$\pm\frac{1}{\lvert y\rvert\sqrt{y^2-1}}$

If $y = \cos x$ or $y = \sin x$ then $|y| \leqslant 1$. For $y = \sec x$ and $y = \csc x$ the opposite is true; we must have $|y| \geqslant 1$. The graph of $\sec^{-1} y$ misses all the points $-1 < y < 1$.

Also, that graph misses $x = \pi/2$—where the cosine is zero. The secant of $\pi/2$ would be $1/0$ (impossible). Similarly $\csc^{-1} y$ misses $x = 0$, because $y = \csc 0$ cannot be $1/\sin 0$. The asterisks in the table are to remove those points $x = \pi/2$ and $x = 0$.

The column of derivatives is what we need and use in calculus.

4.4 EXERCISES

Read-through questions

The relation $x - \sin^{-1} y$ means that ____a____ is the sine of ____b____. Thus x is the angle whose sine is ____c____. The number y lies between ____d____ and ____e____. The angle x lies between ____f____ and ____g____. (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle $\sin^{-1} y$ is $\sqrt{\text{____h____}}$. The derivative of $x = \sin^{-1} y$ is dx/dy=____i____

The relation $x = \cos^{-1} y$ means that y equals ____j____. Again the number y lies between ____k____ and ____l____. This time the angle x lies between ____m____ and ____n____ (so that each y comes from only one angle x). The sum $\sin^{-1} y + \cos^{-1} y =$____o____. (The angles are called ____p____, and they add to a ____q____ angle.) Therefore the derivative of $x = \cos^{-1} y$ is $dx/dy =$____r____, the same as for $\sin^{-1} y$ except for a ____s____ sign.

The relation $x = \tan^{-1} y$ means that $y =$____t____. The number y lies between ____u____ and ____v____. The angle x lies between ____w____ and ____x____. The derivative is $dx/dy =$____y____. Since $\tan^{-1} y + \cot^{-1} y =$____z____ , the derivative of $\cot^{-1} y$ is the same except for a ____A____ sign.

The relation $x = \sec^{-1} y$ means that ____B____. The number y *never* lies between ____C____ and ____D____. The angle x lies between ____E____ and ____F____, but never at $x =$____G____. The derivative of $x = \sec^{-1} y$ is $dx/dy =$____H____.

In 1–4, find the angles $\sin^{-1} y$ **and** $\cos^{-1} y$ **and** $\tan^{-1} y$ **in radians.**

1 $y = 0$ **2** $y = -1$ **3** $y = 1$ **4** $y = \sqrt{3}$

5 We know that $\sin \pi = 0$. Why isn't $\pi = \sin^{-1} 0$?

6 Suppose $\sin x = y$. Under what restriction is $x = \sin^{-1} y$?

7 Sketch the graph of $x = \sin^{-1} y$ and locate the points with slope $dx/dy = 2$.

8 Find dx/dy if $x = \sin^{-1} \frac{1}{2} y$. Draw the graph.

9 If $y = \cos x$ find a formula for $\sin x$. First draw a right triangle with angle x and near side y—what are the other two sides?

10 If $y = \sin x$ find a formula for $\tan x$. First draw a right triangle with angle x and far side y—what are the other sides?

11 Take the x derivative of $\sin^{-1}(\sin x) = x$ by the chain rule. Check that $d(\sin^{-1} y)/dy = -1/\sqrt{1-y^2}$ gives a correct result.

12 Take the y derivative of $\cos(\cos^{-1} y) = y$ by the chain rule. Check that $d(\cos^{-1} y)/dy = -1/\sqrt{1-y^2}$ gives a correct result.

13 At $y = 0$ and $y = 1$, find the slope dx/dy of $x = \sin^{-1} y$ and $x = \cos^{-1} y$ and $x = \tan^{-1} y$.

14 At $x = 0$ and $x = 1$, find the slope dx/dy of $x = \sin^{-1} y$ and $x = \cos^{-1} y$ and $x = \tan^{-1} y$.

15 **True or false**, with reason:

(a) $(\sin^{-1} y)^2 + (\cos^{-1} y)^2 = 1$

(b) $\sin^{-1} y = \cos^{-1} y$ has no solution

(c) $\sin^{-1} y$ is an increasing function

(d) $\sin^{-1} y$ is an odd function

(e) $\sin^{-1} y$ and $-\cos^{-1} y$ have the same slope—so they are the same.

(f) $\sin(\cos x) = \cos(\sin x)$

16 Find $\tan(\cos^{-1}(\sin x))$ by drawing a triangle with sides $\sin x$, $\cos x$, 1.

Compute the derivatives in 17–28 (using the letters as given).

17 $u = \sin^{-1} x$

18 $u = \tan^{-1} 2x$

19 $z = \sin^{-1}(\sin 3x)$

20 $z = \sin^{-1}(\cos x)$

21 $z = (\sin^{-1} x)^2$

22 $z = (\sin^{-1} x)^{-1}$

23 $z = \sqrt{1-y^2}\sin^{-1} y$

24 $z = (1+x^2)\tan^{-1} x$

25 $x = \sec^{-1}(y+1)$

26 $u = \sec^{-1}(\sec x^2)$

27 $u = \sin^{-1} y / \cos^{-1} \sqrt{1-y^2}$

28 $u = \sin^{-1} y + \cos^{-1} y + \tan^{-1} y$

29 Draw a right triangle to show why $\tan^{-1} y + \cot^{-1} y = \pi/2$.

30 Draw a right triangle to show why $\tan^{-1} y = \cot^{-1}(1/y)$.

31 If $y = \tan x$ find $\sec x$ in terms of y.

32 Draw the graphs of $y = \cot x$ and $x = \cot^{-1} y$.

33 Find the slope dx/dy of $x = \tan^{-1} y$ at

(a) $y = -3$ (b) $x = 0$ (c) $x = -\pi/4$

34 Find a function $u(t)$ whose slope satisfies $u' + t^2 u' = 1$.

35 What is the second derivative d^2x/dy^2 of $x = \sin^{-1} y$?

36 What is d^2u/dy^2 for $u = \tan^{-1} y$?

Find the derivatives in 37–44.

37 $y = \sec \frac{1}{2}x$

38 $x = \sec^{-1} 2y$

39 $u = \sec^{-1}(x^n)$

40 $u = \sec^{-1}(\tan x)$

41 $\tan y = (x-1)/(x+1)$

42 $z = (\sin x)(\sin^{-1} x)$

43 $y = \sec^{-1}\sqrt{x^2+1}$

44 $z = \sin(\cos^{-1} x) - \cos(\sin^{-1} x)$

45 Differentiate $\cos^{-1}(1/y)$ to find the slope of $\sec^{-1} y$ in a new way.

46 The domain and range of $x = \csc^{-1} y$ are ______ .

47 Find a function $u(y)$ such that $du/dy = 4/\sqrt{1-y^2}$.

48 Solve the differential equation $du/dx = 1/(1+4x^2)$.

49 If $du/dx = 2/\sqrt{1-x^2}$ find $u(1) - u(0)$.

50 (recommended) With $u(x) = (x-1)/(x+1)$, find the derivative of $\tan^{-1} u(x)$. This is also the derivative of ______. So the difference between the two functions is a ______ .

51 Find $u(x)$ and $\tan^{-1} u(x)$ and $\tan^{-1} x$ at $x = 0$ and $x = \infty$. Conclusion based on Problem 50 : $\tan^{-1} u(x) - \tan^{-1} x$ equals the number

52 Find $u(x)$ and $\tan^{-1} u(x)$ and $\tan^{-1} x$ as $x \to -\infty$. Now $\tan^{-1} u(x) - \tan^{-1} x$ equals ______. Something has happened to $\tan^{-1} u(x)$. At what x do $u(x)$ and $\tan^{-1} u(x)$ change instantly?