CHAPTER 0

Highlights of Calculus

0.1 Distance and Speed // Height and Slope

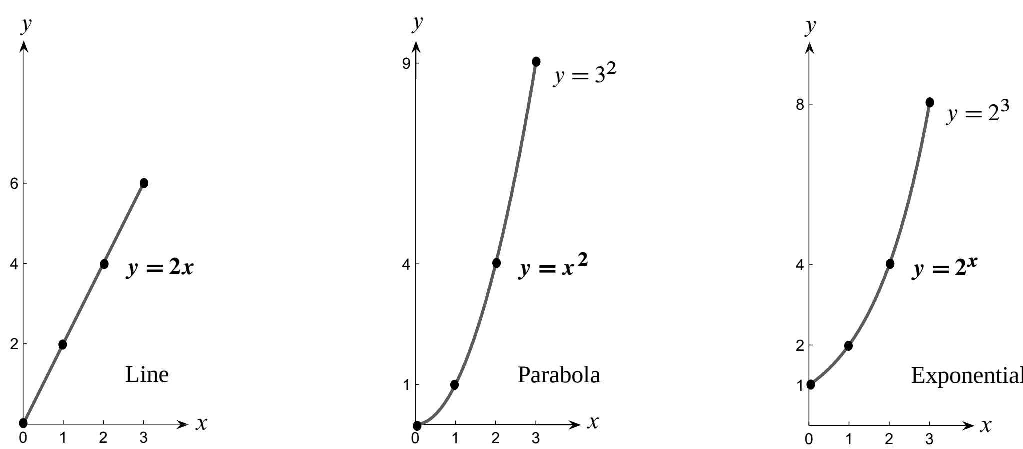
Calculus is about functions. I use that word "functions" in the first sentence, because we can't go forward without it. Like all other words, we learn this one in two different ways: We *define* the word and we *use* the word.

I believe that seeing examples of functions, and using the word to explain those examples, is a fast and powerful way to learn. I will start with three examples:

Linear function	$y(x) = \mathbf{2x}$
Squaring function	$y(x) = \mathbf{x^2}$
Exponential function	$y(x) = \mathbf{2^x}$

The first point is that those are not the same! Their formulas involve 2 and x in very different ways. When I draw their graphs (this is a good way to understand functions) you see that all three are increasing when x is positive. The slopes are positive.

When the input x increases (moving to the right), the output y also increases (the graph goes upward). The three functions increase at different *rates*.



Near the start at $x = 0$, the first function increases the fastest. But the others soon catch up. All three graphs reach the same height $y = 4$ when $x = 2$. Beyond that point the second graph $y = x^2$ pulls ahead. At $x = 3$ the squaring function reaches $y = 3^2 = 9$, while the height of the third graph is only $y = 2^3 = 8$.

Don't be deceived, *the exponential will win*. It pulls even at $x = 4$, because 4^2 and 2^4 are both 16. Then $y = 2^x$ moves ahead of $y = x^2$ and it stays ahead. When you reach $x = 10$, the third graph will have $y = 2^{10} = 1024$ compared to $y = 10^2 = 100$.

The graphs themselves are a *straight line* and a *parabola* and an *exponential*. The straight line has constant growth rate. The parabola has increasing growth rate. The exponential curve has exponentially increasing growth rate. I emphasize these because calculus is all about growth rates.

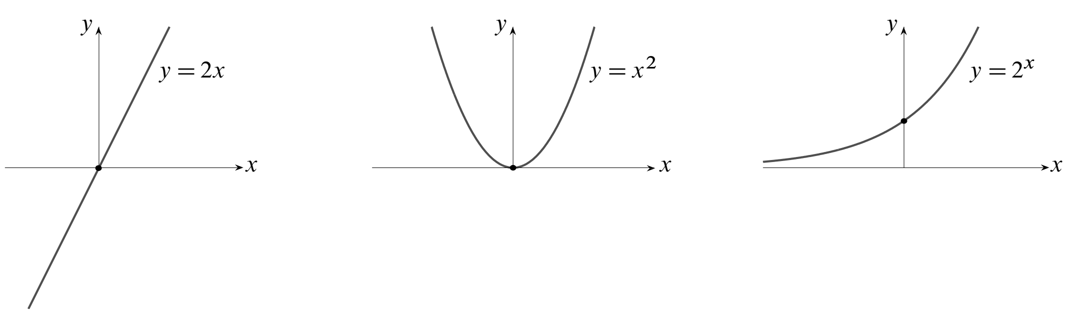
The whole point of differential calculus is to discover the growth rate of a function, and to use that information. So there are actually **two functions** in play at the same time—*the original function and its growth rate*. Before I go further down this all-important road, let me give a working definition of a function $y(x)$:

A function has inputs x and outputs $y(x)$. To each x it assigns one y.

The inputs x come from the "domain" of the function. In our graphs the domain contained all numbers $x \geqslant 0$. The outputs y form the "range" of the function. The ranges for the first two functions $y = 2x$ and $y = x^2$ contained all numbers $y \geqslant 0$. But the range for $y = 2^x$ is limited to $y \geqslant 1$ when the domain is $x \geqslant 0$.

Since these examples are so important, let me also allow x to be *negative*. The three graphs are shown below. Strictly speaking, these are new functions! Their domains have been extended to *all real numbers* x. Notice that the three ranges are also different:

The range of $y = 2x$ is all real numbers y
The range of $y = x^2$ is all nonnegative numbers $y \geqslant 0$
The range of $y = 2^x$ is all positive numbers $y > 0$



One more note about the idea of a function, and then calculus can begin. We have seen the three most popular ways to describe a function:

1. Give a *formula* to find y from x. Example: $y(x) = 2x$.
2. Give a *graph* that shows x (distance across) and y (distance up).
3. Give the *input-output pairs* (x in the domain and y in the range).

In a high-level definition, the "function" is the set of all the input-output pairs. We could also say: The function is the rule that assigns an output y in the range to every input x in the domain.

This shows something that we see for other words too. Logically, the definition should come first. Practically, we understand the definition better after we know examples that use the word. Probably that is the way we learn other words and also the way we will learn calculus. Examples show the general idea, and the definition is more precise. Together, we get it right.

The first words in this book were *Calculus is about functions*. Now I have to update that.

PAIRS OF FUNCTIONS

Calculus is about pairs of functions. Call them Function (1) and Function (2). Our graphs of $y=2x$ and $y=x^2$ and $y=2^x$ were intended to be examples of Function (1). Then we discussed the growth rates of those three examples. **The growth rate of Function (1) is Function (2)**. This is our first task—to find the growth rate of a function. Differential calculus starts with a formula for Function (1) and aims to produce a formula for Function (2).

Let me say right away how calculus operates. There are two ways to compute how quickly y changes when x changes:

Method 1 (*Limits*): Write $\dfrac{\text{Change in } y}{\text{Change in } x} = \dfrac{\Delta y}{\Delta x}$. Take the limit of this ratio as $\Delta x \to 0$.

Method 2 (*Rules*): Follow a rule to produce new growth rates from known rates.

For each new function $y(x)$, look to see if it can be produced from known functions—obeying one of the rules. An important part of learning calculus is to see different ways of producing new functions from old. Then we follow the rules for the growth rate.

Suppose the new function is *not* produced from known functions (2^x is not produced from $2x$ or x^2). Then we have to find its growth rate directly. By "directly" I mean that we compute a limit which is Function (2). This book will explain what a "limit" means and how to compute it.

Here we begin with examples—almost always the best way. I will state the growth rates "dy/dx" for the three functions we are working with:

Function (1)	$y=\mathbf{2x}$	$y=\mathbf{x^2}$	$y=\mathbf{2^x}$
Function (2)	$\dfrac{dy}{dx}=\mathbf{2}$	$\dfrac{dy}{dx}=\mathbf{2x}$	$\dfrac{dy}{dx}=\mathbf{2^x\,(\ln 2)}$

The linear function $y=2x$ has constant growth rate $dy/dx=2$. This section will take that first and easiest step. It is our opportunity to connect the growth rate to the **slope of the graph**. The ratio of *up* to *across* is $2x/x$ which is 2.

Section 0.2 takes the next step. The squaring function $y=x^2$ has linear growth rate $dy/dx=2x$. (This requires the idea of a limit—so fundamental to calculus.) Then we can introduce our first two rules:

Constant factor	The growth rate of $Cy(x)$ is C times the growth rate of $y(x)$.
Sum of functions	The growth rate of y_1+y_2 is the sum of the two growth rates.

The first rule says that $y = 5x^2$ has growth rate $10x$. The factor $C = 5$ multiplies the growth rate $2x$. The second rule says that $y_1 + y_2 = 5x^2 + 2x$ has growth rate $10x + 2$. Notice how we immediately took $5x^2$ as a function y_1 with a known growth rate. Together, the two rules give the growth rate for any "linear combination" of y_1 and y_2:

$$\textbf{The growth rate of } C_1 y_1 + C_2 y_2 \textbf{ is that same combination } C_1 \frac{dy_1}{dx} + C_2 \frac{dy_2}{dx}.$$

In words, the step from Function (1) to Function (2) is *linear*. The slope of $y = x^2 - x$ is $dy/dx = 2x - 1$. This rule is simple but so important.

Finally, Section 0.3 will present the exponential functions $y = 2^x$ and $y = e^x$. Our first job is their meaning—what is "2 *to the power* π"? We understand $2^3 = 8$ and $2^4 = 16$, but how can we multiply 2 by itself π times?

When we meet e^x, we are seeing the great creation of calculus. This is a function with the remarkable property that $dy/dx = y$. **The slope equals the function**. This requires the amazing number e, which was never seen in algebra—because it only appears when you take the right limit.

So these first sections compute growth rates for three essential functions. We are ready for $y = 2x$.

THE SLOPE OF A GRAPH

The slope is distance up divided by distance across. I am thinking now about the graph of a function $y(x)$. The "distance across" is the change $x_2 - x_1$ in the inputs, from x_1 to x_2. The "distance up" is the change $y_2 - y_1$ in the outputs, from y_1 to y_2. The slope is large and the graph is steep when $y_2 - y_1$ is much larger than $x_2 - x_1$. *Change in* y *divided by change in* x matches our ordinary meaning of the word slope:

$$\textbf{Average slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}. \tag{1}$$

I introduced the very useful Greek letter Δ (delta), as a symbol for *change*. We take a step of length Δx to go from x_1 to x_2. For the height $y(x)$ on the graph, that produces a step $\Delta y = y_2 - y_1$. The ratio of Δy to Δx, up divided by across, is the average slope between x_1 and x_2. The slope is the steepness.

Important point: I had to say "average" because the slope could be changing as we go. The graph of $y = x^2$ shows an increasing slope. Between $x_1 = 1$ and $x_2 = 2$, what is the average slope for $y = x^2$? *Divide* Δy *by* Δx:

$$\begin{array}{l} y_1 = 1 \text{ at } x_1 = 1 \\ y_2 = 4 \text{ at } x_2 = 2 \end{array} \qquad \textbf{Average slope} = \frac{4-1}{2-1} = \frac{\Delta y}{\Delta x} = 3.$$

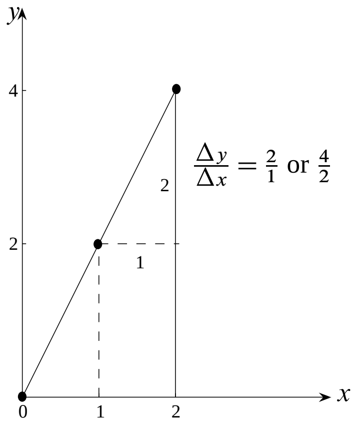
Between $x_1 = 0$ and $x_2 = 2$, we get a different answer (not 3). This graph of x^2 shows the problem of calculus, to deal with changes in slope and changes in speed.

The graph of $y = 2x$ has constant slope. The ratio of Δy to Δx, distance up to distance across, is always 2:

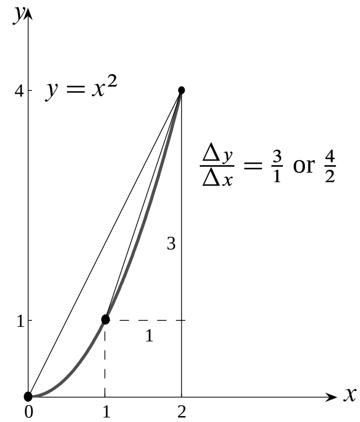
$$\textbf{Constant slope} \qquad \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2x_2 - 2x_1}{x_2 - x_1} = 2.$$

The mathematics is easy, which gives me a chance to emphasize the words and the ideas:

Function (1) = **Height** of the graph   Function (2) = **Slope** of the graph



Constant slope



Changing slope

When Function (1) is $y = Cx$, the ratio $\Delta y / \Delta x$ is always C. A linear function has a constant slope. And those same functions can come from driving a car at constant speed:

Function (1) = **Distance** traveled = $\boldsymbol{Ct}$ Function (2) = **Speed** of the car = $\boldsymbol{C}$

For a graph of Function (1), its rate of change is the **slope**. When Function (1) measures distance traveled, its rate of change is the **speed** (or **velocity**). When Function (1) measures our height, its rate of change is our **growth rate**.

The first point is that *functions are everywhere*. For calculus, those functions come in pairs. *Function* (2) *is the rate of change of Function* (1).

The second point is that Function (1) and Function (2) are measured in different units. That is natural:

$$\left(\text{Speed in } \frac{\textbf{miles}}{\textbf{hour}}\right) \text{ multiplies } \left(\text{Time in } \textbf{hours}\right) \text{ to give } \left(\text{Distance in } \textbf{miles}\right)$$

$$\left(\text{Growth rate in } \frac{\textbf{inches}}{\textbf{year}}\right) \text{ multiplies } \left(\text{Time in } \textbf{years}\right) \text{ to give } \left(\text{Height in } \textbf{inches}\right)$$

When time is in seconds and distance is in meters, then speed is automatically in meters per second. We can choose two units, and they decide the third. Function (2) always involves a division: Δy is divided by Δx or distance is divided by time.

The delicate and tricky part of calculus is coming next. We want the ***slope at one point*** and the ***speed at one instant***. What is the rate of change in *zero time*?

The distance across is $\Delta x = 0$ at a point. The distance up is $\Delta y = 0$. **Formally, their ratio is** $\frac{0}{0}$. It is the inspiration of calculus to give this a useful meaning.

Big Picture

Calculus connects Function (1) with Function (2) = **rate of change** of (1)

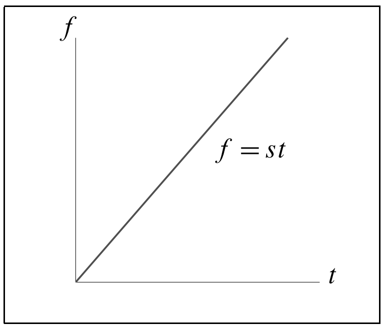
Function (1) Distance traveled $f(t)$ Function (2) Speed $s(t) = df/dt$

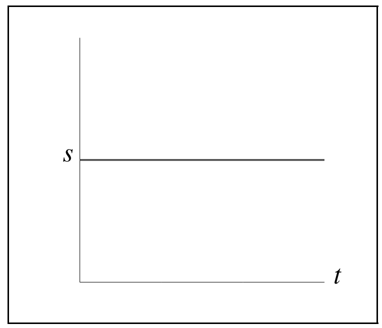
Function (1) Height of graph $y(x)$ Function (2) Slope $s(x) = dy/dx$

Function (2) tells how quickly Function (1) is changing

KEY Constant speed $s = \dfrac{\text{Distance } f}{\text{Time } t}$ Constant slope $s = \dfrac{\text{Distance up}}{\text{Distance across}}$

Graphs of (1) and (2)
$f =$ increasing distance
$s =$ constant speed

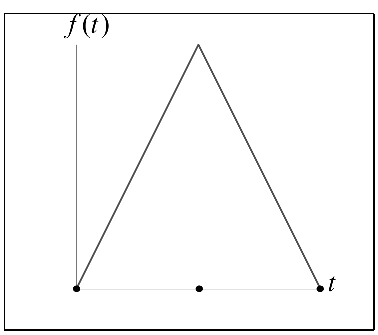


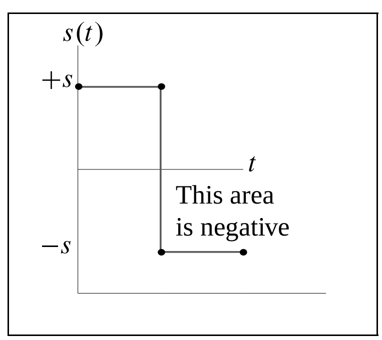


Slope of f-graph $= \dfrac{\text{up}}{\text{across}} = \dfrac{st}{t} = s$

Area under s-graph = area of rectangle $= st = f$

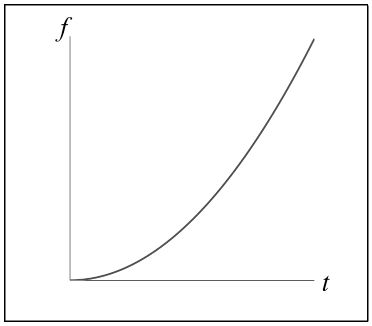
Now run the car backwards.
Speed is negative
Distance goes down to 0
Area "under" $s(t)$ is zero

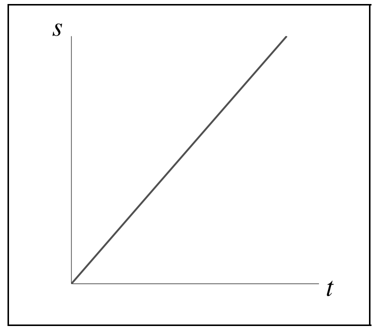




Example with increasing speed   Then distance has steeper slope

$$f = 10t^2$$
$$s = 20t$$





When speed is changing, algebra is not enough $s = \dfrac{f}{t}$ is wrong

Still true that area under s = triangle area $= \frac{1}{2}(t)(20t) = 10t^2 = f$

Still true that s = slope of $f = \dfrac{df}{dt}$ = "derivative" of f

When f is increasing, the slope s is **positive**

When f is decreasing, the slope s is **negative**

When f is at its maximum or minimum, the slope s is **zero**

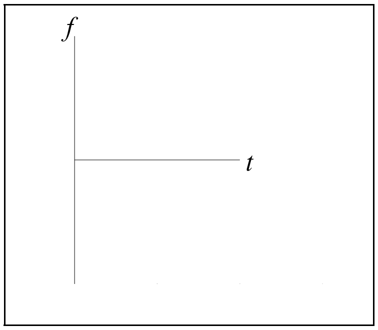
The graphs of any $f(t)$ and $f(t)+10$ have the same slope at every t

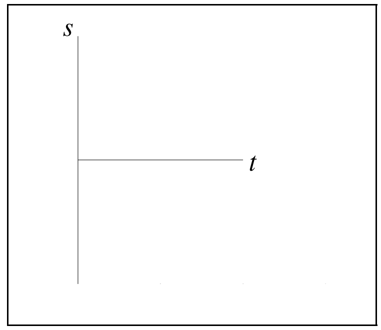
To recover f = Function (1) from $\dfrac{df}{dt}$, good to know a starting height $f(0)$

Practice Questions

1. Draw a graph of $f(t)$ that goes up and down and up again.

Then draw a reasonable graph of its slope.





2. The world population $f(t)$ increased slowly at first, now quickly, then slowly again (we hope and expect). Maybe a limit $\approx$ 12 or 14 billion.

Draw a graph for $f(t)$ and its slope $s(t) = \dfrac{df}{dt}$

3. Suppose $f(t) = 2t$ for $t \le 1$ and then $f(t) = 3t + 2$ for $t \ge 1$

Describe the slope graph $s(t)$. Compare its area out to $t = 3$ with $f(3)$

4. Draw a graph of $f(t) = \cos t$. Then sketch a graph of its slope. At what angles t is the slope zero (slope $= 0$ when $f(t)$ is "flat").

5. Suppose the graph of $f(t)$ is shaped like the capital letter **W**. Describe the graph of its slope $s(t) = \dfrac{df}{dt}$. What is the total area under the graph of s ?

6. A train goes a distance f at constant speed s. Inside the train, a passenger walks forward a distance F at walking speed S. What distance does the passenger go ? At what speed ? (Measure distance from the train station)

0.2 The Changing Slope of $y = x^2$ and $y = x^n$

The second of our three examples is $y = x^2$. Now the slope is changing as we move up the curve. The average slope is still $\Delta y/\Delta x$, but that is not our final goal. We have to answer the crucial questions of differential calculus:

What is the meaning of "slope at a point" and how can we compute it?

My video lecture on *Big Picture*: *Derivatives* also faces those questions. Every student of calculus soon reaches this same problem. What is the meaning of "rate of change" when we are at a single moment in time, and nothing actually changes in that moment? Good question.

The answers will come in two steps. Algebra produces $\Delta y/\Delta x$, and then calculus finds dy/dx. Those steps dy and dx are infinitesimally short, so formally we are looking at $0/0$. Trying to define dy and dx and $0/0$ is not wise, and I won't do it. The successful plan is to realize that the ratio of Δy to Δx is clearly defined, and those two numbers can become very small. ***If that ratio*** $\Delta y/\Delta x$ ***approaches a limit, we have a perfect answer***:

The slope at x is the limit of $$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x}.$$

The distance across, from x to $x + \Delta x$, is just Δx. The distance up is from $y(x)$ to $y(x + \Delta x)$. Let me show how algebra leads directly to $\Delta y/\Delta x$ when $y = x^2$:

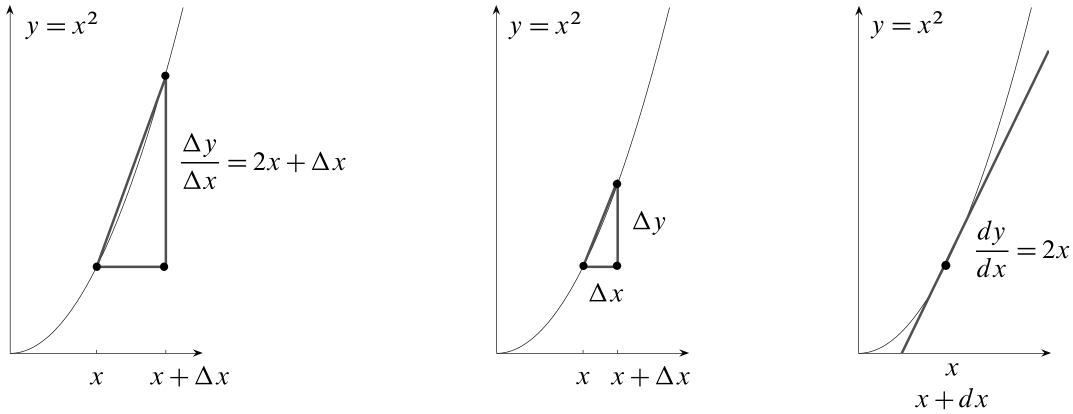
$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x.$$

Notice that calculation! The "leading terms" x^2 and $-x^2$ cancel. The important term here is $2x\Delta x$. This "first-order term" is responsible for most of Δy. The "second-order term" in this example is $(\Delta x)^2$. *After we divide by* Δx, *this term is still small*. That part $(\Delta x)^2/\Delta x$ will disappear as the step size Δx goes to zero.

That limiting process $\Delta x \to 0$ **produces the slope** dy/dx **at a point**. The first-order term survives in dy/dx and higher-order terms disappear.

Slope at a point $$\frac{dy}{dx} = \text{limit of } \frac{\Delta y}{\Delta x} = \text{limit of } 2x + \Delta x = 2x.$$

Algebra produced $\Delta y/\Delta x$. In the limit, calculus gave us dy/dx. Look at the graph, to see the geometry of those steps. The ratio up/across $= \Delta y/\Delta x$ is the slope between two points on the graph. *The two points come together in the limit.* Then $\Delta y/\Delta x$ approaches the slope dy/dx at a ***single point***.



The color lines connecting points on the first two graphs are called "chords." They approach the color line on the third graph, which touches at only *one* point. This is the "**tangent line**" to the curve. Here is the idea of differential calculus:

$$\textbf{Slope of tangent line} = \textbf{Slope of curve} = \quad \text{Function (2)} = \frac{dy}{dx} = \mathbf{2x}.$$

To find the equation for this tangent line, return to algebra. Choose any specific value x_0. Above that position on the x axis, the graph is at height $y_0 = x_0^2$. The slope of the tangent line at that point of the graph is $dy/dx = 2x_0$. **We want the equation for the line through that point with that slope**.

Equation for the tangent line	$y - y_0 = (2x_0)(x - x_0)$	(1)

At the point where $x = x_0$ and $y = y_0$, this equation becomes $0 = 0$. The equation is satisfied and the point is on the line. Furthermore the slope of the line matches the slope $2x_0$ of the curve. You see that directly if you divide both sides by $x - x_0$:

$$\textbf{Tangent line} \qquad \frac{\text{up}}{\text{across}} = \frac{y - y_0}{x - x_0} = 2x_0 \;\; \text{is the correct slope} \;\; \frac{dy}{dx}.$$

Let me say this again. The curve $y = x^2$ is bending, the tangent line is straight. This line stays as close to the curve as possible, near the point where they touch. The tangent line gives a *linear* approximation to the nonlinear function $y = x^2$:

Linear approximation	$y \approx y_0 + (2x_0)(x - x_0) = y_0 + \frac{dy}{dx}(x - x_0)$	(2)

I only moved y_0 to the right side of equation (1). Then I used the symbol $\approx$ for "approximately equal" because the symbol $=$ would be wrong: The curve bends.

Important for the future: This bending comes from the ***second derivative*** of $y = x^2$.

THE SECOND DERIVATIVE

The first derivative is the slope $dy/dx = 2x$. **The second derivative is the slope of the slope**. By good luck we found the slope of $2x$ in the previous section (easy to do, it is just the constant 2). Notice the symbol d^2y/dx^2 for the slope of the slope:

Second derivative $\frac{d^2y}{dx^2}$	The slope of $\frac{dy}{dx} = 2x$ is $\frac{d^2y}{dx^2} = 2$.	(3)

In ordinary language, the first derivative dy/dx tells how fast the function $y(x)$ is changing. The second derivative tells whether we are *speeding up or slowing down*. The example $y = x^2$ is certainly speeding up, since the graph is getting steeper. The curve is bending and the tangent line is steepening.

Think also about $y = x^2$ on the left side (the negative side) of $x = 0$. The graph is coming down to zero. Its slope is certainly negative. But the curve is still bending upwards! The algebra agrees with this picture: The slope $dy/dx = 2x$ is *negative* on the left side of $x = 0$, but the second derivative $d^2y/dx^2 = 2$ is still positive.

An economist or an investor watches all three of those numbers: $y(x)$ tells where the economy is, and dy/dx tells which way it is going (short term, close to the tangent line). But it is d^2y/dx^2 that reveals the longer term prediction. I am writing these words near the end of the economic downturn (I hope). I am sorry that dy/dx has been negative but happy that d^2y/dx^2 has recently been positive.

DISTANCE AND SPEED AND ACCELERATION

An excellent example of $y(x)$ and dy/dx and d^2y/dx^2 comes from driving a car. The function y is the *distance traveled*. Its rate of change (first derivative) is the *speed*. The rate of change of the speed (second derivative) is the *acceleration*. If you are pressing on the gas pedal, all three will be positive. If you are pressing on the brake, the distance and speed are probably still positive but the acceleration is negative: The speed is dropping. If the car is *in reverse* and you are *braking*, what then ?

The speed is negative (going backwards)
The speed is increasing (less negative)
The acceleration is positive (increasing speed).

The video lecture mentions that car makers don't know calculus. The distance meter on the dashboard does not go back toward zero (in reverse gear it should). The speedometer does not go below zero (it should). There is no meter at all (on my car) for acceleration. Spaceships do have accelerometers, and probably aircraft too.

We often hear that an astronaut or a test pilot is subjected to a high number of g's. The ordinary acceleration in free fall is one g, from the gravity of the Earth. An airplane in a dive and a rocket at takeoff will have a high second derivative—the rocket may be hardly moving but it is accelerating like mad.

One more very useful point about this example of motion. *The natural letter to use is not x but t*. The distance is a function of **time**. The slope of a graph is up/across, but now the right ratio is (change of distance) divided by (change in time):

Average speed between t and $t + \Delta t$ $\quad \frac{\Delta y}{\Delta t} = \frac{y(t+\Delta t) - y(t)}{\Delta t}$

Speed at t itself (instant speed) $\quad \frac{dy}{dt} = \text{limit of } \frac{\Delta y}{\Delta t} \text{ as } \Delta t \to 0$

The words “rate of change” and “rate of growth” suggest t. The word “slope” suggests x. But calculus doesn’t worry much about the letters we use. If we graph the distance traveled as a function of time, then the x axis (across) becomes the t axis. And the slope of that graph becomes the speed (velocity is the best word).

Here is something not often seen in calculus books—the **second difference**. We know the first difference $\Delta y = y(t+\Delta t) - y(t)$. It is the change in y. The second difference **$\Delta^2 y$ is the change in Δy**:

$$\textbf{Second difference} \qquad \Delta^2 y = (y(t+\Delta t) - y(t)) - (y(t) - y(t-\Delta t)) \qquad \frac{\boldsymbol{\Delta^2 y}}{\boldsymbol{(\Delta t)^2}} \to \frac{\boldsymbol{d^2 y}}{\boldsymbol{dt^2}} \tag{4}$$

$\Delta^2 y$ simplifies to $y(t+\Delta t) - 2y(t) + y(t-\Delta t)$. We divide by $(\Delta t)^2$ to approximate the acceleration. In the limit as $\Delta t \to 0$, this ratio $\Delta^2 y/(\Delta t)^2$ becomes the **second derivative d^2y/dt^2**.

THE SLOPE OF $y = x^n$

The slope of $y = x^2$ is $dy/dx = 2x$. Now I want to compute the slopes of $y = x^3$ and $y = x^4$ and all succeeding powers $y = x^n$. The rate of increase of x^n will be found again in Section 2.2. But there are two reasons to discover these special derivatives early:

1. Their pattern is simple: **The slope of each power $y = x^n$ is $\dfrac{dy}{dx} = nx^{n-1}$**.
2. The next section can then introduce $y = e^x$. This amazing function has $\dfrac{dy}{dx} = y$.

Of course $y = x^2$ fits into this pattern for x^n. The exponent drops by 1 from $n = 2$ to $n - 1 = 1$. Also $n = 2$ multiplies that lower power to give $nx^{n-1} = 2x$.

The slope of $y = x^3$ is $dy/dx = 3x^2$. Watch how $3x^2$ appears in $\Delta y/\Delta x$:

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}. \tag{5}$$

Cancel x^3 with $-x^3$. Then divide by Δx:

$$\textbf{Average slope} \qquad \frac{\Delta y}{\Delta x} = \mathbf{3x^2} + 3x\Delta x + (\Delta x)^2.$$

When the step length Δx goes to zero, the limit value dy/dx is $3x^2$. This is nx^{n-1}.

To establish this pattern for $n = 4, 5, 6, \ldots$ the only hard part is $(x + \Delta x)^n$. When n was 3, we multiplied this out in equation (5) above. The result will always start with x^n. We claim that the next term (the “first-order term” in Δy) will be $nx^{n-1}\Delta x$. When we divide this part of Δy by Δx, we have the answer we want—the correct derivative nx^{n-1} of $y(x) = x^n$.

How to see that term $nx^{n-1}\Delta x$? Our multiplications showed that $2x\Delta x$ and $3x^2\Delta x$ are correct for $n = 2$ and 3. Then we can reach $n = 4$ from $n = 3$:

$$\begin{aligned}(x+\Delta x)^4 &= (x+\Delta x)^3 \text{ times } (x+\Delta x)\\ &= (x^3 + 3x^2\Delta x + \cdots) \text{ times } (x+\Delta x)\end{aligned}$$

That multiplication produces x^4 and $4x^3\Delta x$, exactly what we want. We can go from each n to the next one in the same way (this is called “induction”). The derivatives of all the powers $x^4, x^5, \ldots, x^n$ are $4x^3, 5x^4, \ldots, nx^{n-1}$.

Section 2.2 of the book shows you a slightly different proof of this formula. And the video lecture on the *Product Rule* explains one more way. Look at x^{n+1} as the product of x^n times x, and use the rule for the slope of y_1 times y_2:

Product Rule	**Slope of $y_1 y_2 = y_2$ (slope of y_1) + y_1 (slope of y_2)**	(6)

With $y_1 = x^n$ and $y_2 = x$, the slope of $y_1 y_2 = x^{n+1}$ comes out right:

$$x(\text{slope of } x^n) + x^n(\text{slope of } x) = x(nx^{n-1}) + x^n(1) = \boldsymbol{(n+1)x^n}. \qquad (7)$$

Again we can increase n one step at a time. Soon comes the truly valuable fact that this derivative formula is correct for *all powers* $y = x^n$. The exponent n can be negative, or a fraction, or any number at all. The slope dy/dx is always nx^{n-1}.

By combining different powers of x, you know the slope of every "polynomial." An example is $y = x + x^2/2 + x^3/3$. Compute dy/dx one term at a time, as the Sum Rule allows:

$$\frac{d}{dx}\left(x + \frac{x^2}{2} + \frac{x^3}{3}\right) = 1 + x + x^2.$$

The slope of the slope is $d^2y/dx^2 = 1 + 2x$. The fourth derivative is zero!

Function (1) tells us the height y above each point x

The problem is to find the "instant slope" at x

This slope $s(x)$ is written $\dfrac{dy}{dx}$ It is **Function (2)**

KEY: $\dfrac{\boldsymbol{\Delta y}}{\boldsymbol{\Delta x}} = \dfrac{y(x+\Delta x) - y(x)}{\Delta x} = \dfrac{\text{up}}{\text{across}}$ approaches $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}}$ as $\Delta x \to 0$

Compute the **instant slope** $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}}$ for the function $y = x^3$

First find the average slope between x and $x + \Delta x$

Average slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{(x+\Delta x)^3 - x^3}{\Delta x}$

Write $(x+\Delta x)^3 = x^3 + \boldsymbol{3x^2\Delta x} + 3x(\Delta x)^2 + (\Delta x)^3$

Subtract x^3 and divide by Δx

$$\frac{\Delta y}{\Delta x} = \frac{\boldsymbol{3x^2\Delta x} + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \boldsymbol{3x^2} + 3x\Delta x + (\Delta x)^2$$

When $\Delta x \to 0$, this becomes $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}} \boldsymbol{= 3x^2}$ $\qquad \dfrac{d}{dx}(x^n) = nx^{n-1}$

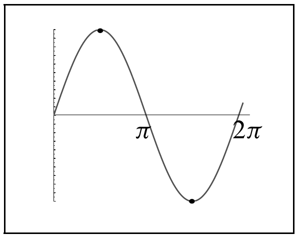
$y = Cx^n$ has slope Cnx^{n-1} The slope of $y = 7x^2$ is $\dfrac{dy}{dx} = 14x$

Multiply y by C → Multiply Δy by C → Multiply $\dfrac{dy}{dx}$ by C

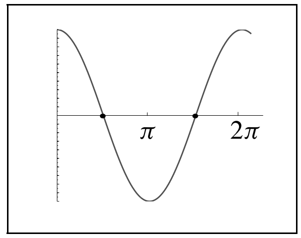
Neat Fact: The slope of $\boldsymbol{y = \sin x}$ is $\boldsymbol{\dfrac{dy}{dx} = \cos x}$

The graphs show this is reasonable

Slope at the start is 1 (to find later)



$y = \sin x$
slope $= \cos x$



Sine curve climbing → Cosine curve > 0

Top of sine curve (flat) → Cosine is zero at the first bullet

Sine curve falling → Cosine curve < 0 between bullets

Bottom of sine curve (flat) → Cosine back to zero at the second bullet

Practice Questions

1. For $y = 2x^3$, what is the average slope $= \dfrac{\Delta y}{\Delta x}$ from $x = 1$ to $x = 2$?

2. What is the instant slope of $y = 2x^3$ at $x = 1$? What is $\dfrac{d^2y}{dx^2}$?

3. $y = x^n$ has $\dfrac{dy}{dx} = nx^{n-1}$. What is $\dfrac{dy}{dx}$ when $y(x) = \dfrac{1}{x} = x^{-1}$?

4. For $y = x^{-1}$, what is the average slope $\dfrac{\Delta y}{\Delta x}$ from $x = \dfrac{1}{2}$ to $x = 1$?

5. What is the instant slope of $y = x^{-1}$ at $x = \dfrac{1}{2}$?

6. Suppose the graph of $y(x)$ climbs up to its maximum at $x = 1$

Then it goes downward for $x > 1$

6A. What is the sign of $\dfrac{dy}{dx}$ for $x < 1$ and then for $x > 1$?

6B. What is the instant slope at $x = 1$?

7 If $y = \sin x$, write an expression for $\dfrac{\Delta y}{\Delta x}$ at any point x.

We see later that this $\dfrac{\Delta y}{\Delta x}$ approaches $\cos x$

0.3 The Exponential $y = e^x$

The great function that calculus creates is the exponential $y = e^x$. There are different ways to reach this function, and Section 6.2 of this textbook mentions five ways. Here I will describe the approach to e^x that I now like best. It uses the derivative of x^n, the first thing we learn.

In all approaches, a "limiting step" will be involved. So the amazing number $e = 2.7\ldots$ is not seen in algebra (e is not a fraction). The question is where to take that limiting step, and my answer starts with this truly remarkable fact: When $y = e^x$ is **Function** (**1**), **it is also Function** (**2**).

$$\textit{The exponential function } y = e^x \textit{ solves the equation } \frac{dy}{dx} = y.$$

The function equals its slope. This is a first example of a **differential equation**—connecting an unknown function y with its own derivatives. Fortunately $dy/dx = y$ is the most important differential equation—a model that other equations try to follow.

I will add one more requirement, to eliminate solutions like $y = 2e^x$ and $y = 8e^x$. When $y = e^x$ solves our equation, all other functions Ce^x solve it too. ($C = 2$ and $C = 8$ will appear on both sides of $dy/dx = y$, and they cancel.) At $x = 0$, e^0 will be the "zeroth power" of the positive number e. *All zeroth powers are* 1. So we want $y = e^x$ to equal 1 when $x = 0$:

$$y = e^x \textit{ is the solution of } \frac{dy}{dx} = y \textit{ that starts from } y = 1 \textit{ at } x = 0.$$

Before solving $dy/dx = y$, look at what this equation means. When y starts from 1 at $x = 0$, its slope is also 1. So y increases. Therefore dy/dx also increases, staying equal to y. So y increases faster. The graph gets steeper as the function climbs higher. This is what "growing exponentially" means.

INTRODUCING e^x

Exponential growth is quite ordinary and reasonable. When a bank pays interest on your money, the interest is proportional to the amount you have. After the interest is added, you have more. The new interest is based on the higher amount. Your wealth is growing "geometrically," one step at a time.

At the end of this section on e^x, I will come back to *continuous* compounding—interest is added at every instant instead of every year. That word "continuous" signals that we need calculus. There is a limiting step, from every year or month or day or second to every instant. You don't get infinite interest, you do get exponentially increasing interest.

I will also describe other ways to introduce e^x. This is an important question with many answers! I like equation (1) below, because we know the derivative of each power x^n. If you take their derivatives in equation (1), you get back the same e^x: *amazing*. So that sum solves $dy/dx = y$, starting from $y = 1$ as we wanted.

The difficulty is that the sum involves every power x^n: an *infinite series*. When I go step by step, you will see that those powers are all needed. For this infinite series, I am asking you to believe that everything works. *We can add the series to get* e^x, and we can add all derivatives to see that the slope of e^x is e^x.

For me, the advantage of using only the powers x^n is overwhelming.

CONSTRUCTING $y = e^x$

I will solve $dy/dx = y$ a step at a time. At the start, $y = 1$ means that $dy/dx = 1$:

Start $\begin{array}{r} y=1 \\ dy/dx=1 \end{array}$ **Change y** $\begin{array}{r} y=1+x \\ dy/dx=1 \end{array}$ **Change $\frac{dy}{dx}$** $\begin{array}{r} y=1+x \\ dy/dx=1+x \end{array}$

After the first change, $y = 1 + x$ has the correct derivative $dy/dx = 1$. But then I had to change dy/dx to keep it equal to y. And I can't stop there:

$\begin{array}{r} y=1+x \\ dy/dx=1+x \end{array}$ **Update y to** $1+x+\frac{1}{2}x^2$ **Then update** $\frac{dy}{dx}$ **to** $1+x+\frac{1}{2}x^2$

The extra $\frac{1}{2}x^2$ gave the correct x in the slope. Then $\frac{1}{2}x^2$ also had to go into dy/dx, to keep it equal to y. Now we need a new term with this derivative $\frac{1}{2}x^2$.

The term that gives $\frac{1}{2}x^2$ has x^3 divided by 6. The derivative of x^n is nx^{n-1}, so I *must divide by n* (to cancel correctly). Then the derivative of $x^3/6$ is $3x^2/6 = \frac{1}{2}x^2$ as we wanted. After that comes x^4 divided by 24:

$$\frac{x^3}{6} = \frac{x^3}{(3)(2)(1)} \quad \text{has slope} \quad \frac{x^2}{(2)(1)}$$
$$\frac{x^4}{24} = \frac{x^4}{(4)(3)(2)(1)} \quad \text{has slope} \quad \frac{4x^3}{(4)(3)(2)(1)} = \frac{x^3}{6}.$$

The pattern becomes more clear. The x^n term is divided by n *factorial*, which is $n! = (n)(n-1)\ldots(1)$. The first five factorials are $1, 2, 6, 24, 120$. **The derivative of that term $x^n/n!$ is the previous term $x^{n-1}/(n-1)!$** (because the n's cancel). As long as we don't stop, this sum of infinitely many terms does achieve $dy/dx = y$:

$$\boldsymbol{y(x) = e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots + \tfrac{1}{n!}x^n + \cdots} \qquad (1)$$

If we substitute $x = 10$ into this series, do the infinitely many terms add to a finite number e^{10}? *Yes*. The numbers $n!$ grow much faster than 10^n (or any other x^n). So the terms $x^n/n!$ in this "exponential series" become extremely small as $n \to \infty$. Analysis shows that the sum of the series (which is $y = e^x$) does achieve $dy/dx = y$.

Note 1 Let me just remember a series that you know, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. If I replace $\frac{1}{2}$ by x, this becomes the *geometric series* $1 + x + x^2 + x^3 + \cdots$ and it adds up to $1/(1-x)$. This is the most important series in mathematics, but it runs into a problem at $x = 1$: the infinite sum $1 + 1 + 1 + 1 + \cdots$ doesn't "converge."

I emphasize that the series for e^x is always safe, because the powers x^n are divided by the rapidly growing numbers $n! = n$ *factorial*. This is a great example to meet, long before you learn more about convergence and divergence.

Note 2 Here is another way to look at that series for e^x. Start with x^n and take its derivative n times. First get nx^{n-1} and then $n(n-1)x^{n-2}$. Finally the nth derivative is $n(n-1)(n-2)\ldots(1)x^0$, which is n *factorial*. When we divide by that number, **the nth derivative of $x^n/n!$ is equal to 1**.

Now look at e^x. All its derivatives are still e^x. They all equal 1 at $x = 0$. *The series is matching every derivative of the function e^x at the starting point $x = 0$.*

Set $x = 1$ in the exponential series. This tells us the amazing number $e^1 = e$:

The number e	$\boldsymbol{e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots}$	(2)

The first three terms add to 2.5. The first five terms almost reach 2.71. *We never reach* 2.72. With quite a few terms (how many ?) you can pass 2.71828. It is certain that e is not a fraction. It never appears in algebra, but it is the key number for calculus.

MULTIPLYING BY ADDING EXPONENTS

We write e^2 in the same way that we write 3^2. *Is it true that e times e equals e^2 ?* Up to now, e and e^2 come from setting $x = 1$ and $x = 2$ in the infinite series. The wonderful fact is that for every x, the series produces the "xth power of the number e." When $x = -1$, we get e^{-1} which is $1/e$:

Set $x = -1$
$$e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots$$

If we multiply that series for $1/e$ by the series for e, we get 1.

The best way is to go straight for all multiplications of e^x times any power e^X. The rule of adding exponents says that the answer is e^{x+X}. The series must say this too! When $x = 1$ and $X = -1$, this rule produces e^0 from e^1 times e^{-1}.

Add the exponents	$\boldsymbol{(e^x)(e^X) = e^{x+X}}$	(3)

We only know e^x and e^X from the infinite series. For this all-important rule, we can multiply those series and recognize the answer as the series for e^{x+X}. Make a start:

Multiply each term
e^x times e^X
Hoping for
e^{x+X}

$$e^x = \mathbf{1} + \boldsymbol{x} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$
$$e^X = \mathbf{1} + \boldsymbol{X} + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots$$
$$(e^x)(e^X) = \mathbf{1} + \boldsymbol{x} + \boldsymbol{X} + \frac{1}{2}x^2 + \boldsymbol{xX} + \frac{1}{2}X^2 + \cdots \qquad (4)$$

Certainly you see $x + X$. Do you see $\frac{1}{2}(x+X)^2$ in equation (4) ? No problem:

$$\frac{1}{2}(x+X)^2 = \frac{1}{2}(x^2 + 2xX + X^2) \quad \text{matches the "second degree" terms.}$$

The step to third degree takes a little longer, but it also succeeds:

$$\frac{1}{6}(x+X)^3 = \frac{1}{6}x^3 + \frac{3}{6}x^2X + \frac{3}{6}xX^2 + \frac{1}{6}X^3 \quad \text{matches the next terms in (4).}$$

For high powers of $x + X$ we need the *binomial theorem* (or a healthy trust that mathematics comes out right). When e^x multiplies e^X, the coefficient of $x^n X^m$ will be $1/n!$ times $1/m!$. Now look for that same term in the series for e^{x+X}:

$$\frac{(x+X)^{n+m}}{(n+m)!} \text{ includes } \frac{x^n X^m}{(n+m)!} \text{ times } \boldsymbol{\frac{(n+m)!}{n!\,m!}} \text{ which gives } \frac{x^n X^m}{n!\,m!}. \qquad (5)$$

That binomial number $(n+m)!/n!\,m!$ is known to successful gamblers. It counts the number of ways to choose n aces out of $n+m$ aces. Out of 4 aces, you could choose 2 aces in $4!/2!2! = 6$ ways. To a mathematician, there are 6 ways to choose 2 x's out of $xxxx$. This number 6 will be the coefficient of x^2X^2 in $(x+X)^4$.

That 6 shows up in the fourth degree term. It is divided by 4! (to produce $1/4$). When e^x multiplies e^X, $\frac{1}{2}x^2$ multiplies $\frac{1}{2}X^2$ (which also produces $1/4$). All terms are good, but we are not going there—we accept $(e^x)(e^X) = e^{x+X}$ as now confirmed.

Note A different way to see this rule for $(e^x)(e^X)$ is based on $dy/dx = y$. Starting from $y = 1$ at $x = 0$, follow this equation. At the point x, you reach $y = e^x$. Now go an additional distance X to arrive at e^{x+X}.

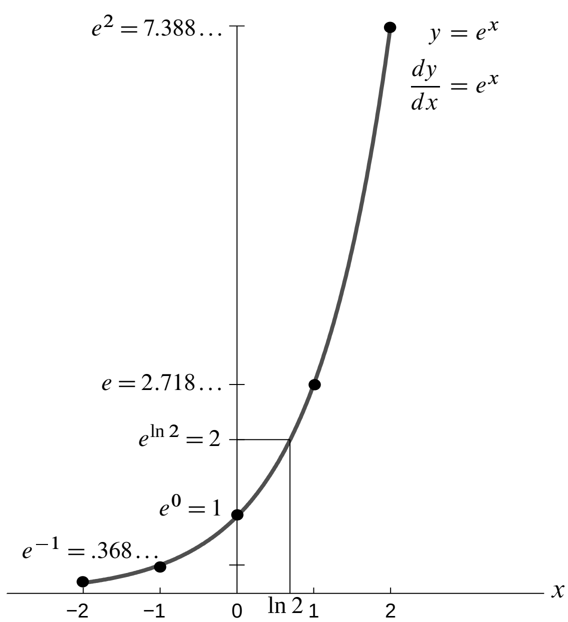
Notice that the additional part starts from e^x (instead of starting from 1). That starting value e^x will multiply e^X in the additional part. So e^x times e^X must be the same as e^{x+X}. (This is a "differential equations proof" that the exponents are added. Personally, I was happy to multiply the series and match the terms.)

The rule immediately gives e^x times e^x. The answer is $e^{x+x} = e^{2x}$. If we multiply again by e^x, we find $(e^x)^3$. This is equal to $e^{2x+x} = e^{3x}$. We are finding a new rule for all powers $(e^x)^n = (e^x)(e^x)\cdots(e^x)$:

Multiply exponents $\qquad \boldsymbol{(e^x)^n = e^{nx}} \qquad$ (6)

This is easy to see for $n = 1, 2, 3, \ldots$ and then $n = -1, -2, -3, \ldots$ *It remains true for all numbers x and n.*

That last sentence about "all numbers" is important! Calculus cannot develop properly without working with all exponents (not just whole numbers or fractions). The infinite series (1) defines e^x for every x and we are on our way. Here is the graph: **Function (1) = Function (2) = e^x = exp(x).**



$$(e^x)(e^X) = e^{x+X}$$
$$(e^x)^n = e^{nx}$$
$$e^{\ln y} = y$$

THE EXPONENTIALS 2^x AND b^x

We know that $2^3 = 8$ and $2^4 = 16$. But what is the meaning of 2^π? One way to get close to that number is to replace π by 3.14 which is $314/100$. As long as we have a fraction in the exponent, we can live without calculus:

Fractional power $\quad 2^{314/100} = 314\text{th power of the 100th root } 2^{1/100}.$

But this is only "close" to 2^π. And in calculus, we will want the slope of the curve $y = 2^x$. The good way is to connect 2^x with e^x, whose slope we know (it is e^x again). So we need to connect 2 with e.

The key number is the **logarithm of 2**. This is written "ln 2" and it is the power of e that produces 2. It is specially marked on the graph of e^x:

Natural logarithm of 2 $\quad e^{\ln 2} = 2$

This number ln 2 is about $7/10$. A calculator knows it with much higher accuracy. In the graph of $y = e^x$, the number ln 2 on the x axis produces $y = 2$ on the y axis.

This is an example where we want the output $y = 2$ and we ask for the input $x = \ln 2$. That is the opposite of knowing x and asking for y. "The logarithm $x = \ln y$ is the *inverse* of the exponential $y = e^x$." This idea will be explained in Section 4.3 and in two video lectures—inverse functions are not always simple.

Now 2^x has a meaning for every x. When we have the number ln 2, meeting the requirement $2 = e^{\ln 2}$, we can take the xth power of both sides:

Powers of 2 from powers of e $\quad 2 = e^{\ln 2} \quad \text{and} \quad \mathbf{2^x = e^{x \ln 2}}. \quad (7)$

All powers of e are defined by the infinite series. The new function 2^x also grows exponentially, but not as fast as e^x (because 2 is smaller than e). Probably $y = 2^x$ could have the same graph as e^x, if I stretched out the x axis. That stretching multiplies the slope by the constant factor ln 2. Here is the algebra:

Slope of $y = 2^x$ $\quad \frac{d}{dx} 2^x = \frac{d}{dx} e^{x \ln 2} = (\ln 2)\, e^{x \ln 2} = (\mathbf{\ln 2) 2^x}.$

For any positive number b, the same approach leads to the function $y = b^x$. First, find the natural logarithm $\ln b$. This is the number (positive or negative) so that $b = e^{\ln b}$. Then take the xth power of both sides:

Connect b to e $\quad b = e^{\ln b} \quad \text{and} \quad b^x = e^{x \ln b} \quad \text{and} \quad \frac{d}{dx} b^x = (\ln b) b^x \quad (8)$

When b is e (the perfect choice), $\ln b = \ln e = 1$. When b is e^n, then $\ln b = \ln e^n = n$. "***The logarithm is the exponent***." Thanks to the series that defines e^x for every x, that exponent can be any number at all.

Allow me to mention Euler's Great Formula $\boldsymbol{e^{ix} = \cos x + i \sin x}$. The exponent ix has become an **imaginary number**. (You know that $i^2 = -1$.) If we faithfully use $\cos x + i \sin x$ at 90° and 180° (where $x = \pi/2$ and $x = \pi$), we arrive at these amazing facts:

Imaginary exponents $\quad \boldsymbol{e^{i\pi/2} = i} \quad \text{and} \quad \boldsymbol{e^{i\pi} = -1}. \quad (9)$

Those equations are not imaginary, they come from the great series for e^x.

CONTINUOUS COMPOUNDING OF INTEREST

There is a different and important way to reach e and e^x (not by an infinite series). We solve the key equation $dy/dx = y$ in small steps. As these steps approach zero (a limit is always involved !) the small-step solution becomes the exact $y = e^x$.

I can explain this idea in two different languages. Each step multiplies by $1 + \Delta x$:

1. *Compound interest*. After each step Δx, the interest is added to y. Then the next step begins with a larger amount, and y increases exponentially.
2. *Finite differences*. The continuous dy/dx is replaced by small steps $\Delta Y/\Delta x$:

$$\frac{dy}{dx} = y \text{ changes to } \frac{Y(x+\Delta x) - Y(x)}{\Delta x} = Y(x) \text{ with } Y(0) = 1. \qquad (10)$$

This is Euler's method of approximation. $Y(x)$ approaches $y(x)$ as $\Delta x \to 0$.

Let me compute compound interest when 1 year is divided into 12 months, and then 365 days. The interest rate is 100% and you start with $Y(0) = \$1$. If you only get interest once, at the end of the year, then you have $Y(1) = \$2$.

If interest is added every month, you now get $\frac{1}{12}$ of 100% each time (12 times). So Y is multiplied each month by $1+\frac{1}{12}$. (The bank adds $\frac{1}{12}$ for every 1 you have.) Do this 12 times and the final value \$2 is improved to \$2.61:

$$\textbf{After 12 months} \qquad Y(1) = \left(1 + \frac{1}{12}\right)^{12} = \$2.61$$

Now add interest every day. $Y(0) = \$1$ is multiplied 365 times by $1+\frac{1}{365}$:

$$\textbf{After 365 days} \qquad Y(1) = \left(1 + \frac{1}{365}\right)^{365} = \$2.71 \ (\textit{close to } e)$$

Very few banks use minutes, and nobody divides the year into $N{=}31,536,000$ seconds. It would add less than a penny to \$2.71. But many banks are willing to use *continuous compounding*, the limit as $N \to \infty$. After one year you have \$$e$:

Another limit gives e	$\left(1+\frac{1}{N}\right)^N \to e = 2.718\ldots$ as $N \to \infty$	(11)

You could invest at the 100% rate for x years. Now each of the N steps is for x/N years. Again the bank multiplies at every step by $1+\frac{x}{N}$. The 1 keeps what you have, the x/N adds the interest in that step. After N steps you are close to e^x:

A formula for e^x	$\left(1+\frac{x}{N}\right)^N \to e^x$ as $N \to \infty$	(12)

Finally, I will change the interest rate to a. Go for x years at the interest rate a. The differential equation changes from $dy/dx = y$ to $\boldsymbol{dy/dx = ay}$. The exponential function still solves it, but now that solution is $y = e^{ax}$:

Change the rate to a	$\frac{dy}{dx} = ay$ ***is solved by*** $y(x) = e^{ax}$	(13)

You can write down the series $e^{ax} = 1 + ax + \frac{1}{2}(ax)^2 + \cdots$ and take its derivative:

$$\frac{d}{dx}(e^{ax}) = a + a^2x + \cdots = a(1 + ax + \cdots) = \boldsymbol{ae^{ax}} \tag{14}$$

The derivative of e^{ax} brings down the extra factor a. So $y = e^{ax}$ solves $dy/dx = ay$.

The Exponential $y = e^x$

Looking for a function $y(x)$ that equals its own derivative $\dfrac{dy}{dx}$

A differential equation! We start at $x = 0$ with $y = 1$

Infinite Series $y(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \left(\dfrac{x^n}{n!}\right) + \cdots$

Take derivative $\dfrac{dy}{dx} = 0 + 1 + x + \dfrac{x^2}{2!} + \cdots + \left(\dfrac{x^{n-1}}{(n-1)!}\right) + \cdots$

Term by term $\dfrac{dy}{dx}$ agrees with y Limit step = add up this series

$n! = (n)(n-1)\cdots(1)$ grows much faster than x^n so the terms get very small

At $x = 1$ the number $y(1)$ is called e. Set $x = 1$ in the series to find e

$\boldsymbol{e} = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \cdots = \mathbf{2.71828\ldots}$

GOAL Show that $y(x)$ agrees with e^x for all x Series gives powers of e

Check that the series follows the rule to add exponents as in $e^2e^3 = e^5$

Directly multiply series e^x times e^X to get e^{x+X}

$\left(1 + x + \frac{1}{2}x^2\right)$ times $\left(1 + X + \frac{1}{2}X^2\right)$ produces the right start for e^{x+X}

$1 + (x + X) + \frac{1}{2}(x + X)^2 + \cdots$ HIGHER TERMS ALSO WORK

The series gives us e^x for EVERY x, not just whole numbers

CHECK $\dfrac{de^x}{dx} = \lim \dfrac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \left(\lim \dfrac{e^{\Delta x} - 1}{\Delta x}\right) = e^x$ YES!

SECOND KEY RULE $\boldsymbol{(e^x)^n = e^{nx}}$ for every x and n

---

Another approach to e^x uses multiplication instead of an infinite sum

Start with \$1. Earn interest every day at yearly rate x

Multiply 365 times by $\left(1 + \dfrac{x}{365}\right)$. End the year with \$ $\left(1 + \dfrac{x}{365}\right)^{365}$

Now pay n times in the year. End the year with $\left(1 + \dfrac{x}{n}\right)^n \to \$\, e^x$ as $n \to \infty$

We are solving $\dfrac{\Delta Y}{\Delta x} = Y$ in n short steps Δx. The limit solves $\dfrac{dy}{dx} = y$.

Practice Questions

1. What is the derivative of $\dfrac{x^{10}}{10!}$? What is the derivative of $\dfrac{x^9}{9!}$?

2. How to see that $\dfrac{x^n}{n!}$ gets small as $n \to \infty$?

Start with $\dfrac{x}{1}$ and $\dfrac{x^2}{2}$, possibly big. But we multiply by $\dfrac{x}{3}, \dfrac{x}{4}, \cdots$ which gets small.

3. Why is $\dfrac{1}{e^x}$ the same as e^{-x} ? Use equation (3) and also use (6).

4. Why is $e^{-1} = 1 - 1 + \dfrac{1}{2} - \dfrac{1}{6} + \cdots$ between $\dfrac{1}{3}$ and $\dfrac{1}{2}$? Then $2 < e < 3$.

5. Can you solve $\dfrac{dy}{dx} = y$ starting from $y = 3$ at $x = 0$?

Why is $y = 3e^x$ the right answer? Notice how 3, multiplies e^x.

6. Can you solve $\dfrac{dy}{dx} = 5y$ starting from $y = 1$ at $x = 0$?

Why is $y = e^{5x}$ the right answer? Notice 5 in the exponent!

7. Why does $\dfrac{e^{\Delta x} - 1}{\Delta x}$ approach 1 as Δx gets smaller? Use the $e^{\Delta x}$ series.

8. Draw the graph of $x = \ln y$, just by flipping the graph of $y = e^x$ across the 45° line $y = x$. Remember that y stays positive but $x = \ln y$ can be negative.

9. What is the exact sum of $1 + \ln 2 + \dfrac{1}{2}(\ln 2)^2 + \dfrac{1}{3!}(\ln 2)^3 + \cdots$?

10. If you replace ln 2 by 0.7, what is the sum of those four terms?

11. From Euler's Great Formula $e^{ix} = \cos x + i \sin x$, what number is $e^{2\pi i}$?

12. How close is $\left(1 + \dfrac{1}{10}\right)^{10}$ to e ?

13. What is the limit of $\left(1 + \dfrac{1}{N}\right)^{2N}$ as $N \to \infty$?

0.4 Video Summaries and Practice Problems

This section is to help readers who also look at the **Highlights of Calculus** video lectures. The first five videos are just released on `ocw.mit.edu` as I write these words. Sections 0.1–0.2–0.3 discussed the content of three lectures in full detail. The summaries and practice problems for the other two will come first in this section:

4. Maximum and Minimum and Second Derivative

5. Big Picture of Integrals

That Lecture 5 is a taste of *Integral Calculus*. A second set of video lectures goes deeper into *Differential Calculus*—the rules for computing and using derivatives.

This second set is right now with the video editors, to zoom in when I write on the blackboard and zoom out for the big picture. I just borrowed a video camera from MIT's OpenCourseWare and set it up in an empty room. I am not good at looking at the audience anyway, so it was easier with nobody watching !

I hope it will be helpful to print here the summaries and practice problems that are planned to accompany those videos. Here are the topics:

6. Derivative of the Sine and Cosine

7. Product and Quotient Rules

8. Chain Rule for the Slope of $f(g(x))$

9. Inverse Functions and Logarithms

10. Growth Rates and Log Graphs

11. Linear Approximation and Newton's Method

12. Differential Equations of Growth

13. Differential Equations of Motion

14. Power Series and Euler's Formula

15. Six Functions, Six Rules, Six Theorems

That last lecture summarizes the theory of differential calculus. The other lectures explain the steps. Here are the first lines written for the max-min video.

Maximum and Minimum and Second Derivative

To find the maximum and minimum values of a function $y(x)$

Solve $\dfrac{dy}{dx} = 0$ to find points x^* where **slope = zero**

Test each x^* for a possible minimum or maximum

Example $y(x) = x^3 - 12x$ $\quad \dfrac{dy}{dx} = 3x^2 - 12$ $\quad$ Solve $3x^2 = 12$

The slope is $\dfrac{dy}{dx} = 0$ at $\boldsymbol{x^* = 2}$ and $\boldsymbol{x^* = -2}$

At those points $y(2) = 8 - 24 = -16 =$ **min** and $y(-2) = -8 + 24 = 16 =$ **max**

$x^* = 2$ is a minimum Look at $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) =$ **second derivative**

$\dfrac{d^2y}{dx^2} =$ derivative of $3x^2 - 12$. This second derivative is $6x$.

$\dfrac{d^2y}{dx^2} > 0$ $\dfrac{dy}{dx}$ increases Slope goes from down to up at $x^* = 2$

The bending is upwards and this x^* is a **minimum**

$\dfrac{d^2y}{dx^2} < 0$ $\dfrac{dy}{dx}$ decreases Slope goes from up to down at $x^* = -2$

The bending is downwards and x^* is a **maximum**

Find the maximum of $y(x) = \sin x + \cos x$ using $\dfrac{dy}{dx} = \cos x - \sin x$

The slope is zero when $\cos x = \sin x$ at $x^* = 45$ degrees $= \dfrac{\pi}{4}$ radians

That point x^* has $y = \sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2}$

The second derivative is $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

At $x^* = \dfrac{\pi}{4}$ this is < 0 y is bending down x^* is a **maximum**

$\dfrac{d^2y}{dx^2} > 0$ when the curve bends up $\dfrac{d^2y}{dx^2} < 0$ when the curve bends down

Direction of bending changes at a **point of inflection** where $\boldsymbol{\dfrac{d^2y}{dx^2} = 0}$

Which x^* gives the minimum of $y = (x-1)^2 + (x-2)^2 + (x-6)^2$?

You can write $y = (x^2 - 2x + 1) + (x^2 - 4x + 4) + (x^2 - 12x + 36)$

The slope is $\dfrac{dy}{dx} = 2x - 2 + 2x - 4 + 2x - 12 = 0$ at the minimum point x^*

Then $6x^* = 18$ and $x^* = 3$ Minimum point is the average of 1, 2, 6

Key for **max / min** word problems is to choose a suitable meaning for x

Practice Questions

1. Which x^* gives the minimum of $y(x) = x^2 + 2x$? Solve $\frac{dy}{dx} = 0$.

2. Find $\frac{d^2y}{dx^2}$ for $y(x) = x^2 + 2x$. This is > 0 so the parabola bends up.

3. Find the maximum height of $y(x) = 2 + 6x - x^2$. Solve $\frac{dy}{dx} = 0$.

4. Find $\frac{d^2y}{dx^2}$ to show that this parabola bends down.

5. For $y(x) = x^4 - 2x^2$ show that $\frac{dy}{dx} = 0$ at $x = -1, 0, 1$.

Find $y(-1)$, $y(0)$, $y(1)$. Check max versus min by the sign of d^2y/dx^2.

6. At a minimum point explain why $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

7. Bending down $\left(\frac{d^2y}{dx^2} < 0\right)$ changes to bending up $\left(\frac{d^2y}{dx^2} > 0\right)$ at a point of _____: At this point $\frac{d^2y}{dx^2} = 0$ Does $y = \sin x$ have such a point ?

8. Suppose $x + X = 12$. What is the maximum of x times X ?

This question asks for the maximum of $y = x(12 - x) = 12x - x^2$.

Find where the slope $\frac{dy}{dx} = 12 - 2x$ is zero. What is x times X ?

The Big Picture of Integrals

Key problem Recover the integral $y(x)$ from its derivative $\frac{dy}{dx}$

Find the total distance traveled from a record of the speed

Find Function (1) = total height knowing Function (2) = slope since the start

Simplest way Recognize $\frac{dy}{dx}$ as derivative of a known $y(x)$

If $\frac{dy}{dx} = x^3$ then its **integral** $y(x)$ was $\frac{1}{4}x^4 + C$ = **Function (1)**

If $\frac{dy}{dx} = e^{2x}$ then $y = \frac{1}{2}e^{2x} + C$

Integral Calculus is the reverse of Differential Calculus

$y(x) = \int \frac{dy}{dx} dx$ adds up the whole history of slopes $\frac{dy}{dx}$ to find $y(x)$

Integral is like sum Derivative is like difference

Sums y_0 y_1 y_2 y_3 y_4

Differences $y_1 - y_0$ $y_2 - y_1$ $y_3 - y_2$ $y_4 - y_3$

Notice cancellation $(y_1 - y_0) + (y_2 - y_1) = y_2 - y_0 =$ change in height

Divide and multiply the differences by the step size Δx

Sum of $\frac{\Delta y}{\Delta x}\Delta x = \frac{y_1 - y_0}{\Delta x}\Delta x + \frac{y_2 - y_1}{\Delta x}\Delta x$ is still $y_2 - y_0$

Now let $\Delta x \to 0$ Sum changes to integral $\boldsymbol{\int \frac{dy}{dx} dx = y_{\text{end}} - y_{\text{start}}}$

Fundamental Theorem of Calculus $\int \frac{dy}{dx} dx = y(x) + C$

The integral reverses the derivative and brings back $y(x)$

Integration and Differentiation are inverse operations

Fundamental Theorem in the opposite order $\frac{d}{dx}\int_0^x s(t)\,dt = s(x)$

KEY What is the meaning of an integral $\int_0^x s(t)\,dt$? Add up short _____ .

Example $s(t) = 6t$ shows increasing speed and slope. Find $y(t)$.

Method 1 $y = 3t^2$ has the required derivative $6t$ (this is the simplest way !)

Method 2 The triangle under the graph of $s(t) = 6t$ has area $3t^2$

From 0 to t, base $= t$ and height $= 6t$ and area $= \frac{1}{2}t(6t)$.

[Most shapes are more difficult! Area comes from integrating $s(t)$ or $s(x)$]

Method 3 (fundamental) **Add up short time steps each at constant speed**

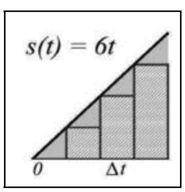
In a step Δt, the distance is close to $s(t^*)\Delta t$

t^* is the starting time for that step and $s(t^*)$ is the starting speed

This is not exact because the speed changes a little within time Δt

The total distance becomes exact as $\Delta t \to 0$ and **sum → integral**

Picture of each step shows a tall thin rectangle



$s(t^*)\Delta t$ = height times base
= area of rectangle
t^* = start point of the time step

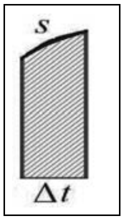
Sum of $s(t^*)\Delta t =$ total area of all rectangles

Now $\Delta t \to 0$ The rectangles fill up the triangle

Integral of $s(t)\,dt =$ exact area $y(t)$ under the graph

<u>Fundamental Theorem</u> Area $y(t)$ has the desired derivative $s(t)$

Reason: Δy is the thin area under $s(t)$ between t and $t+\Delta t$



Δt is the base of that thin "rectangle"

$\dfrac{\Delta y}{\Delta t}$ is the height of that thin "rectangle"

This height $\Delta y/\Delta t$ approaches $s(t)$ as the base $\Delta t \to 0$

Practice Questions

1. What functions $y(t)$ have the constant derivative $s(t)=7$?
2. What is the area from 0 to t under the graph of $s(t)=7$?
3. From $t=0$ to 2, find the integral $\int_0^2 7\,dt =$ _____ .
4. What function $y(t)$ has the derivative $s(t)=7+6t$?
5. From $t=0$ to 2, find area $=$ integral $\int_0^2 (7+6t)\,dt$.
6. At this instant $t=2$, what is $\dfrac{d(\text{area})}{dt}$?

7. From 0 to t, the area under the curve $s=e^t$ IS NOT $y=e^t$.

If t is small, the area must be small. The wrong answer e^t is not small!

8. From 0 to t, the correct area under $s=e^t$ is $\boldsymbol{y=e^t-1}$**.**

The slope $\dfrac{dy}{dt}$ is _____ and now the starting area $y(0)$ is _____

9. Same for sums. Notice y_0 in $(y_1-y_0)+(y_2-y_1)+(y_3-y_2)=$ _____ .

The sum of $\Delta y = \dfrac{\Delta y}{\Delta t}\Delta t$ becomes the integral of $\dfrac{dy}{dt}dt$

The area under $\boldsymbol{s(t)}$ from 0 to $\boldsymbol{t}$ becomes $\boldsymbol{y(t)-y(0)}$.

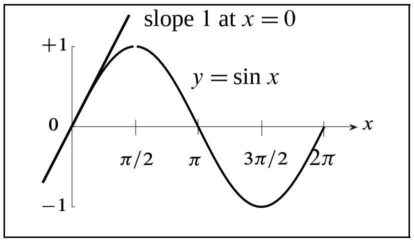
Derivative of the Sine and Cosine

This lecture shows that $\boldsymbol{\frac{d}{dx}(\sin x) = \cos x}$ and $\boldsymbol{\frac{d}{dx}(\cos x) = -\sin x}$

We have to measure the angle x in **radians** 2π radians = full 360 degrees

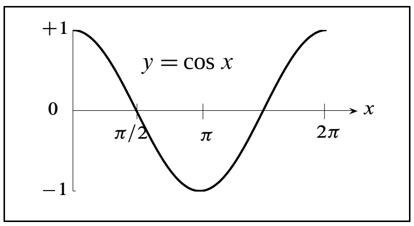
All the way around the circle (2π radians) **Length** $= \boldsymbol{2\pi}$ when the radius is 1

Part way around the circle (x radians) **Length** $= \boldsymbol{x}$ when the radius is 1



Slope $\cos x$

at $x = 0$	slope $1 = \cos 0$
at $x = \pi/2$	slope $0 = \cos \pi/2$
at $x = \pi$	slope $-1 = \cos \pi$



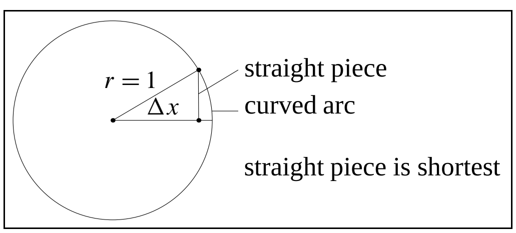
Slope $-\sin x$

at $x = 0$	slope $= 0 = -\sin 0$
at $x = \pi/2$	slope $-1 = -\sin \pi/2$
at $x = \pi$	slope $= 0 = -\sin \pi$

Problem: $\frac{\Delta y}{\Delta x} = \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$ is not as simple as $\frac{(x + \Delta x)^2 - x^2}{\Delta x}$

Good idea to start at $x = 0$ Show $\boldsymbol{\frac{\Delta y}{\Delta x} = \frac{\sin \Delta x}{\Delta x}}$ **approaches 1**

Draw a right triangle with angle Δx to see $\boldsymbol{\sin \Delta x \le \Delta x}$

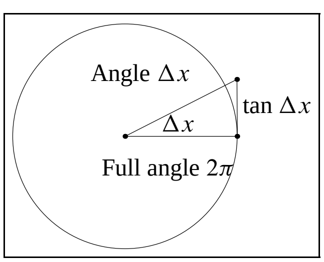


straight length $= \sin \Delta x$

curved length $= \Delta x$

IDEA $\frac{\sin \Delta x}{\Delta x} < 1$ and $\frac{\sin \Delta x}{\Delta x} > \cos \Delta x$ will **squeeze** $\boldsymbol{\frac{\sin \Delta x}{\Delta x} \to 1}$ as $\Delta x \to 0$

To prove $\frac{\sin \Delta x}{\Delta x} > \cos \Delta x$ which is $\tan \Delta x > \Delta x$ **Go to a bigger triangle**



Triangle area $= \frac{1}{2}(1)(\tan \Delta x)$ greater than

Circular area $= \left(\frac{\Delta x}{2\pi}\right)$ (whole circle) $= \frac{1}{2}(\Delta x)$

The squeeze $\cos\Delta x < \dfrac{\sin\Delta x}{\Delta x} < 1$ tells us that $\dfrac{\sin\Delta x}{\Delta x}$ **approaches 1**

$\dfrac{(\sin\Delta x)^2}{(\Delta x)^2} < 1$ means $\dfrac{(1-\cos\Delta x)}{\Delta x}(1+\cos\Delta x) < \Delta x$

So $\dfrac{1-\cos\Delta x}{\Delta x} \to 0$ **Cosine curve has slope = 0**

For the slope at other x remember a formula from trigonometry: $\sin(x+\Delta x) = \sin x\,\cos\Delta x + \cos x\,\sin\Delta x$

We want $\Delta y = \sin(x+\Delta x) - \sin x$ Divide that by Δx

$$\frac{\Delta y}{\Delta x} = (\sin x)\left(\frac{\cos\Delta x - 1}{\Delta x}\right) + (\cos x)\left(\frac{\sin\Delta x}{\Delta x}\right) \quad \text{Now let } \Delta x \to 0$$

In the limit $\dfrac{dy}{dx} = (\sin x)(0) + (\cos x)(1) = \cos x =$ Derivative of $\sin x$

For $y = \cos x$ the formula for $\cos(x+\Delta x)$ leads similarly to $\dfrac{dy}{dx} = -\sin x$

Practice Questions

1. What is the slope of $y = \sin x$ at $x = \pi$ and at $x = 2\pi$?
2. What is the slope of $y = \cos x$ at $x = \pi/2$ and $x = 3\pi/2$?
3. The slope of $(\sin x)^2$ is $2\sin x\cos x$. The slope of $(\cos x)^2$ is $-2\cos x\sin x$. Combined, the slope of $(\sin x)^2 + (\cos x)^2$ is **zero**. Why is this true ?
4. What is the **second derivative** of $y = \sin x$ (derivative of the derivative) ?
5. At what angle x does $y = \sin x + \cos x$ have zero slope ?

6. Here are amazing infinite series for $\sin x$ and $\cos x$. $e^{ix} = \cos x + i\sin x$

$$\sin x = \frac{x}{1} - \frac{x^3}{3\cdot2\cdot1} + \frac{x^5}{5\cdot4\cdot3\cdot2\cdot1} - \cdots \quad (\textit{odd powers of } x)$$

$$\cos x = 1 - \frac{x^2}{2\cdot1} + \frac{x^4}{4\cdot3\cdot2\cdot1} - \cdots \quad (\textit{even powers of } x)$$

7. Take the derivative of the sine series to see the cosine series.
8. Take the derivative of the cosine series to see **minus** the sine series.
9. Those series tell us that for small angles $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$. With these approximations check that $(\sin x)^2 + (\cos x)^2$ is close to 1.

Product and Quotient Rules

Goal To find the derivative of $\boldsymbol{y = f(x)g(x)}$ from $\dfrac{df}{dx}$ and $\dfrac{dg}{dx}$

Idea Write $\Delta y = f(x+\Delta x)\,g(x+\Delta x) - f(x)g(x)$ by separating Δf and Δg

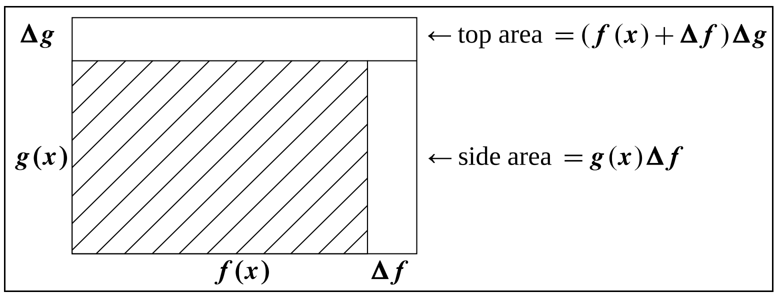
That same Δy is $f(x+\Delta x)\,[g(x+\Delta x)-g(x)] + g(x)\,[f(x+\Delta x)-f(x)]$

$$\frac{\Delta y}{\Delta x} = f(x+\Delta x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x}$$

Product Rule $\boldsymbol{\dfrac{dy}{dx} = f(x)\dfrac{dg}{dx} + g(x)\dfrac{df}{dx}}$

Example $\quad y = x^2 \sin x \quad$ Product Rule $\dfrac{dy}{dx} = x^2 \cos x + 2x \sin x$

A picture shows the two unshaded pieces of $\Delta y = f(x+\Delta x)\Delta g + g(x)\Delta f$



Example $\quad f(x) = x^n \quad g(x) = x \quad \boldsymbol{y = f(x)g(x) = x^{n+1}}$

Product Rule $\dfrac{dy}{dx} = x^n \dfrac{dx}{dx} + x\dfrac{dx^n}{dx} = x^n + x\,n x^{n-1} = \boldsymbol{(n+1)x^n}$

The correct derivative of x^n leads to the correct derivative of x^{n+1}

Quotient Rule If $y = \dfrac{f(x)}{g(x)}$ then $\boldsymbol{\dfrac{dy}{dx} = \left(g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}\right)\Big/ g^2}$

EXAMPLE $\dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = (\cos x(\cos x) - \sin x(-\sin x))\Big/\cos^2 x$

This says that $\boldsymbol{\dfrac{d}{dx}\tan x = \dfrac{1}{\cos^2 x} = \sec^2 x}$ $\quad$ (Notice $(\cos x)^2 + (\sin x)^2 = 1$)

EXAMPLE $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{x^4 \text{ times } 0 - 1 \text{ times } 4x^3}{x^8} = \boldsymbol{\dfrac{-4}{x^5}}$ $\quad$ This is nx^{n-1}

Prove the Quotient Rule $\quad \Delta y = \dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)} = \boldsymbol{\dfrac{f+\Delta f}{g+\Delta g} - \dfrac{f}{g}}$

Write this Δy as $\dfrac{g(f+\Delta f) - f(g+\Delta g)}{g(g+\Delta g)} = \boldsymbol{\dfrac{g\Delta f - f\Delta g}{g(g+\Delta g)}}$

Now divide that Δy by Δx $\quad$ As $\Delta x \to 0$ we have the Quotient Rule

Practice Questions

1. Product Rule: Find the derivative of $y = (x^3)(x^4)$. Simplify and explain.
2. Product Rule: Find the derivative of $y = (x^2)(x^{-2})$. Simplify and explain.
3. Quotient Rule: Find the derivative of $y = \dfrac{\cos x}{\sin x}$.
4. Quotient Rule: Show that $y = \dfrac{\sin x}{x}$ has a maximum (zero slope) at $x = 0$.
5. Product and Quotient! Find the derivative of $y = \dfrac{x \sin x}{\cos x}$.
6. $g(x)$ has a minimum when $\dfrac{dg}{dx} = 0$ and $\dfrac{d^2 g}{dx^2} > 0$ The graph is bending up

   $y = \dfrac{1}{g(x)}$ has a *maximum* at that point: Show that $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2 y}{dx^2} < 0$

Chain Rule for the Slope of $f(g(x))$

$y = g(x) \quad z = f(y) \quad \longrightarrow$ the chain is $z = f(g(x))$

$y = x^5 \quad z = y^4 \quad \longrightarrow$ the chain is $z = (x^5)^4 = x^{20}$

Average slope $\dfrac{\Delta z}{\Delta x} = \left(\dfrac{\Delta z}{\Delta y}\right)\left(\dfrac{\Delta y}{\Delta x}\right)$ Just cancel Δy

Instant slope $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx} =$ **CHAIN RULE** (like cancelling dy)

You MUST change y to $g(x)$ in the final answer

Example of chain $z = y^4 = (x^5)^4 \quad \dfrac{dz}{dy} = \mathbf{4y^3} \quad \dfrac{dy}{dx} = \mathbf{5x^4}$

Chain rule $\dfrac{dz}{dx} = \left(\dfrac{dz}{dy}\right)\left(\dfrac{dy}{dx}\right) = (4y^3)(5x^4) = \mathbf{20y^3x^4}$

Replace y by x^5 to get only x $\dfrac{dz}{dx} = 20(x^5)^3 x^4 = \mathbf{20\,x^{19}}$

CHECK $z = (x^5)^4 = x^{20}$ does have $\dfrac{dz}{dx} = 20\,x^{19}$

1. Find $\dfrac{dz}{dx}$ for $z = \cos(4x)$ Write $y = 4x$ and $z = \cos y$ so $\dfrac{dz}{dx} =$

2. Find $\dfrac{dz}{dx}$ for $z = (1+4x)^2$ Write $y = 1+4x$ and $z = y^2$ so $\dfrac{dz}{dx} =$

CHECK $(1+4x)^2 = 1 + 8x + 16x^2$ so $\dfrac{dz}{dx} =$

Practice Questions

3. Find $\frac{dh}{dx}$ for $h(x) = (\sin 3x)(\cos 3x)$

Product rule first Then the Chain rule for each factor

$$\frac{dh}{dx} = (\sin 3x)\frac{d}{dx}(\cos 3x) + (\cos 3x)\frac{d}{dx}(\sin 3x)$$
$$= (\sin 3x)(\text{CHAIN}) + (\cos 3x)(\text{CHAIN}) = \quad ?$$

4. Tough challenge: Find the **second derivative** of $z(x) = f(g(x))$

FIRST DERIV $\frac{dz}{dx} = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right)$ Function of $y(x)$ times function of x

PRODUCT RULE $\frac{d^2z}{dx^2} = \left(\frac{dz}{dy}\right)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)\frac{d}{dx}\left(\frac{dz}{dy}\right)$

SECOND DERIV $\left(\frac{dz}{dy}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^2z}{dy^2}\right)\left(\frac{dy}{dx}\right)$ $\frac{dy}{dx}$ *twice*!

Check $y = x^5$ $z = y^4 = x^{20}$ $\frac{dz}{dx} = 20x^{19}$ $\frac{d^2z}{dx^2} = 380x^{18}$

SECOND DERIV $(4y^3)(20x^3) + (5x^4)(12y^2)(5x^4)$ $80 + 300 = 380$ OK

Inverse Functions and Logarithms

A function assigns an **output** $\boldsymbol{y = f(x)}$ to each **input** $\boldsymbol{x}$

A one-to-one function has different outputs y for different inputs x

For the **inverse function** the input is y and the output is $\boldsymbol{x = f^{-1}(y)}$

Example If $y = f(x) = x^5$ then $x = f^{-1}(y) = y^{\frac{1}{5}}$

KEY If $y = ax + b$ then solve for $x = \frac{y-b}{a}$ = inverse function

Notice that $\boldsymbol{x = f^{-1}(f(x))}$ and $\boldsymbol{y = f(f^{-1}(y))}$

The **chain rule** will connect the derivatives of f^{-1} and f

The great function of calculus is $\boldsymbol{y = e^x}$

Its inverse function is the **"natural logarithm"** $\boldsymbol{x = \ln y}$

Remember that x is the exponent in $y = e^x$

The rule $e^x e^X = e^{x+X}$ tells us that $\boldsymbol{\ln(yY) = \ln y + \ln Y}$

Add logarithms because you add exponents: $\boldsymbol{\ln(e^2 e^3) = 5}$

$(e^x)^n = e^{nx}$ (multiply exponent) tells us that $\boldsymbol{\ln(y^n) = n \ln y}$

We can change from base e to base 10: New function $\boldsymbol{y = 10^x}$

The inverse function is the logarithm to base 10   Call it log: $\boldsymbol{x = \log y}$

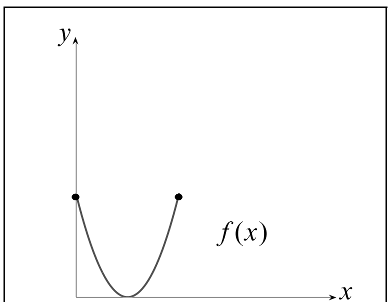
Then $\log 100 = 2$ and $\log \frac{1}{100} = -2$ and $\log 1 = 0$

We will soon find the beautiful derivative of $\ln y$ $\boldsymbol{\frac{d}{dy}(\ln y) = \frac{1}{y}}$

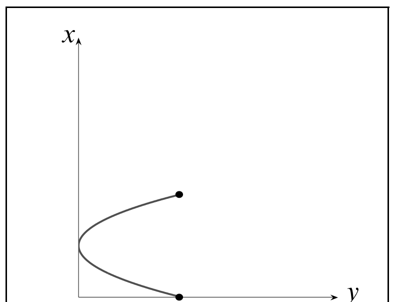
You can change letters to write that as $\boldsymbol{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

Practice Questions

1. What is $x = f^{-1}(y)$ if $y = 50x$?
2. What is $x = f^{-1}(y)$ if $y = x^4$? Why do we keep $x \geq 0$?
3. Draw a graph of an increasing function $y = f(x)$. This has different outputs y for different x. **Flip the graph (switch the axes) to see $\boldsymbol{x = f^{-1}(y)}$**
4. This graph has the same $\boldsymbol{y}$ **from two $\boldsymbol{x}$'s. There is no $\boldsymbol{f^{-1}(y)}$**



$\boldsymbol{f(x)}$ is NOT one-to-one



$\boldsymbol{f^{-1}(y)}$ is NOT a function

5. The natural logarithm of $y = 1/e$ is $\ln(e^{-1}) = ?$ What is $\ln(\sqrt{e})$?
6. The natural logarithm of $y = 1$ is $\ln 1 = ?$ and also base 10 has $\log 1 = ?$
7. The natural logarithm of $(e^2)^{50}$ is ? The base 10 logarithm of $(10^2)^{50}$ is ?
8. I believe that $\boldsymbol{\ln y = (\ln 10)(\log y)}$ because we can write y in two ways $y = e^{\ln y}$ and also $y = 10^{\log y} = e^{(\ln 10)(\log y)}$. Explain those last steps.
9. Change from base e and base 10 to **base 2**. Now $y = 2^x$ means $\boldsymbol{x = \log_2 y}$. What are $\log_2 32$ and $\log_2 2$? Why is $\log_2(e) > 1$?

Growth Rates and Log Graphs

In order of fast growth as x gets large

$\log x$	x, x^2, x^3	$2^x, e^x, 10^x$	$x!, x^x$
logarithmic	polynomial	exponential	factorial

Choose $x = 1000 = 10^3$ so that $\log x = 3$ OK to use $x! \approx \dfrac{x^x}{e^x}$

$\log 1000 = 3$ $10^3, 10^6, 10^9$ $10^{300}, 10^{434}, 10^{1000}$ $10^{2566}, 10^{3000}$

Why is $1000^{1000} = 10^{3000}$? Logarithms are best for big numbers

Logarithms are exponents! $\log 10^9 = 9$ $\log \log x$ is VERY slow

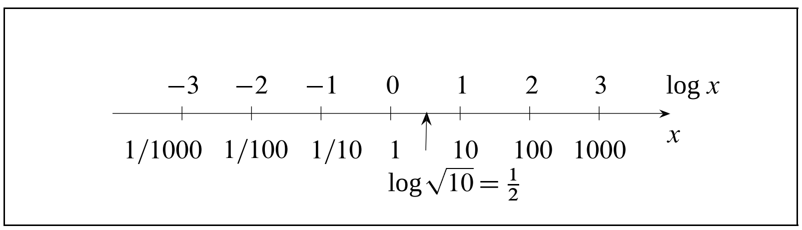
Logarithms **3, 6, 9** **300, 434, 1000** **2566, 3000**

Polynomial growth $\ll$ Exponential growth $\ll$ Factorial growth

Decay to zero for NEGATIVE powers and exponents

$\dfrac{1}{x^2} = x^{-2}$ decays much more slowly than the exponential $\dfrac{1}{e^x} = e^{-x}$

Logarithmic scale shows $x = 1, 10, 100$ equally spaced. NO ZERO!



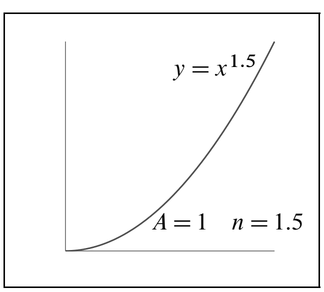
Question If $x = 1, 2, 4, 8$ are plotted, what would you see ?
Answer THEY ARE EQUALLY SPACED TOO!

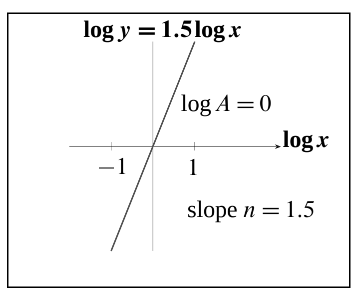
log-log graphs (log scale up and also across)

If $y = Ax^n$, how to see A and n on the graph ?

Plot $\log y$ versus $\log x$ to get a straight line

$\boldsymbol{\log y = \log A + n \log x}$ **Slope on a log-log graph is the exponent n**





For $y = Ab^x$ use **semilog** (x versus $\log y$ is now a line) $\log y = \log A + x \log b$

New type of question How quickly does $\dfrac{\Delta f}{\Delta x}$ approach $\dfrac{df}{dx}$ as $\Delta x \to 0$?

The error $E = \dfrac{\Delta f}{\Delta x} - \dfrac{df}{dx}$ will be $E \approx A(\Delta x)^n$ What is n ?

Usual one-sided $\dfrac{\Delta f}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ only has $n = 1$

Centered difference $\dfrac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$ has $n = 2$

Centered is much better than one-sided $E \approx (\Delta x)^2$ vs $E \approx \Delta x$

[IDEA FOR $f(x) = e^x$ / PROJECT at $x = 0$] One-sided E vs centered E
Graph $\log E$ vs $\log \Delta x$ Should see slope 1 or 2

Practice Questions

1. Does x^{100} grow faster or slower than e^x as x gets large?
2. Does $100 \ln x$ grow faster or slower than x as x gets large?
3. Put these in increasing order for large n:

   $\dfrac{1}{n}, \quad n \log n, \quad n^{1.1}, \quad \dfrac{10^n}{n!}$
4. Put these in increasing order for large x:

   $2^{-x}, \quad e^{-x}, \quad \dfrac{1}{x^2}, \quad \dfrac{1}{x^{10}}$

5. Describe the log-log graph of $y = 10x^5$ (graph $\log y$ vs $\log x$)

Why don't we see $y = 0$ at $x = 0$ on this graph?

What is the slope of the straight line on the log-log graph?

The line crosses the vertical axis when $x =$ _____ and $y =$ _____

Then $\log x = 0$ and $\log y =$ _____

The line crosses the horizontal axis when $x =$ _____ and $y = 1$

Then $\log x =$ _____ and $\log y = 0$

6. Draw the semilog graph (a line) of $y = 10e^x$ (graph $\log y$ versus x)
7. That line cross the $x = 0$ axis at which $\log y$? What is the slope?

Linear Approximation and Newton's Method

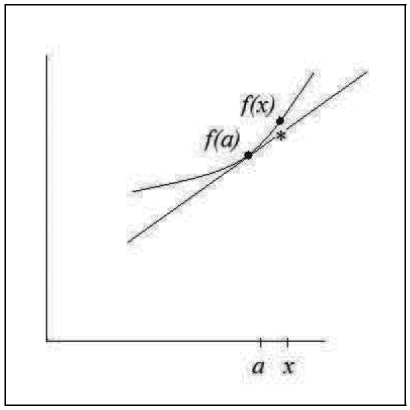
Start at $x = a$ with known $f(a) =$ height and $f'(a) =$ slope

KEY IDEA $\boldsymbol{f'(a) \approx \dfrac{f(x) - f(a)}{x - a}}$ when x is near a

Tangent line has slope $f'(a)$

Solve for $f(x)$

$\boldsymbol{f(x) \approx f(a) + (x - a)f'(a)}$



$\approx$ means "approximately"
curve $\approx$ line near $x = a$

Examples of linear approximation to $f(x)$

1. $f(x) = e^x$ $f(0) = e^0 = 1$ and $f'(0) = e^0 = 1$ are known at $a = 0$

Follow the tangent line $e^x \approx 1 + (x - 0)1 = \mathbf{1 + x}$

$1 + x$ is the linear part of the series for e^x

2. $f(x) = x^{10}$ and $f'(x) = 10x^9$ $f(1) = 1$ and $f'(1) = 10$ known at $a = 1$

Follow the tangent line $x^{10} \approx \mathbf{1 + (x - 1)10}$ near $x = 1$

Take $x = 1.1$ $(1.1)^{10}$ is approximately $1 + 1 = 2$

Newton's Method (looking for x to nearly solve $f(x) = 0$)

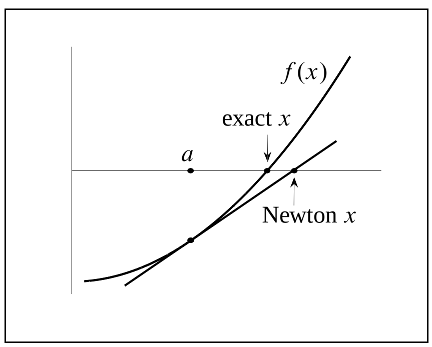
Go back to $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$

$f(a)$ and $f'(a)$ are again known

Solve for x when $f(x) = 0$

$\boldsymbol{x - a \approx -\dfrac{f(a)}{f'(a)}}$ **Newton x**

Line crossing near curve crossing



Examples of Newton's Method Solve $f(x) = x^2 - 1.2 = 0$

1. $a = 1$ gives $f(a) = 1 - 1.2 = \mathbf{-.2}$ and $f'(a) = 2a = \mathbf{2}$

Tangent line hits 0 at $x - 1 = -\dfrac{(-.2)}{2}$ Newton's x will be 1.1

2. For a better x, Newton starts again from that point $a = 1.1$

Now $f(a) = 1.1^2 - 1.2 = \mathbf{.01}$ and $f'(a) = 2a = \mathbf{2.2}$

The new tangent line has $x - 1.1 = -\dfrac{.01}{2.2}$ For this x, x^2 is very close to 1.2

Practice Questions

1. The graph of $y = f(a) + (x - a)f'(a)$ is a straight _____

At $x = a$ the height is $y =$ _____

At $x = a$ the slope is $dy/dx =$ _____

This graph is t_____t to the graph of $f(x)$ at $x = a$

For $f(x) = x^2$ at $a = 3$ this linear approximation is $y =$ _____

2. $y = f(a) + (x - a)f'(a)$ has $y = 0$ when $x - a =$ _____

Instead of the curve $f(x)$ crossing 0, Newton has tangent line y crossing 0

$f(x) = x^3 - 8.12$ at $a = 2$ has $f(a) =$ _____ and $f'(a) = 3a^2 =$ _____

Newton's method gives $x - 2 = -\dfrac{f(a)}{f'(a)} =$ _____

This Newton $x = 2.01$ nearly has $x^3 = 8.12$. It actually has $(2.01)^3 =$ _____ .

Differential Equations of Growth

$\boldsymbol{\dfrac{dy}{dt} = cy}$ Complete solution $y(t) = Ae^{ct}$ for any A

Starting from $y(0)$ $y(t) = y(0)e^{ct}$ $A = y(0)$

Now include a constant source term s This gives a new equation

$\boldsymbol{\dfrac{dy}{dt} = cy} + s$ $s > 0$ is saving, $s < 0$ is spending, cy is interest

Complete solution $y(t) = -\dfrac{s}{c} + \boldsymbol{Ae^{ct}}$ (any A gives a solution)

$y = -\dfrac{s}{c}$ is a constant solution with $cy + s = 0$ and $\dfrac{dy}{dt} = 0$ and $A = 0$

For that solution, the spending s exactly balances the income cy

Choose $\boldsymbol{A}$ to start from $y(0)$ at $t = 0$ $y(t) = -\dfrac{s}{c} + \left(\boldsymbol{y(0) + \dfrac{s}{c}}\right)e^{ct}$

Now add a nonlinear term sP^2 coming from competition

$P(t) =$ world population at time t (for example) follows a new equation

$\boldsymbol{\dfrac{dP}{dt} = cP - sP^2}$ $\quad c =$ birth rate minus death rate

"LOGISTIC EQN" $\quad P^2$ since each person competes with each person

To bring back a linear equation set $y = \dfrac{1}{P}$

Then $\dfrac{dy}{dt} = -\dfrac{dP/dt}{P^2} = \dfrac{(-cP + sP^2)}{P^2} = -\dfrac{c}{P} + s = -cy + s$

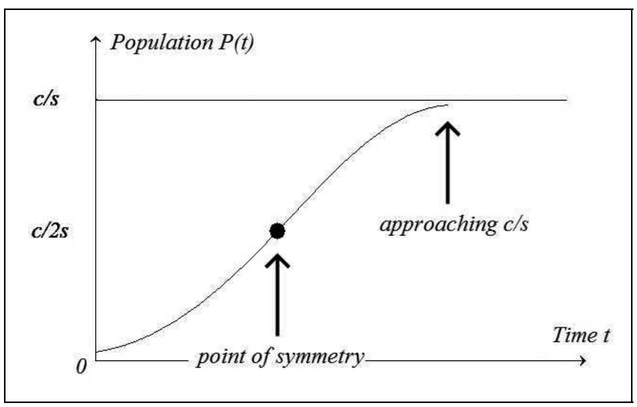
$y = 1/P$ produced our linear equation (no y^2) with $-c$ not $+c$

$y(t) = \dfrac{s}{c} + Ae^{-ct} = \dfrac{s}{c} + \left(y(0) - \dfrac{s}{c}\right)e^{-ct} =$ old solution with change to $-c$

At $t = 0$ we correctly get $y(0)$ $\quad$ CORRECT START

As $t \to \infty$ and $e^{-ct} \to 0$ we get $y(\infty) = \dfrac{s}{c}$ and $P(\infty) = \dfrac{c}{s}$

The population $P(t)$ increases along an $\boldsymbol{S}$**-curve** approaching $\dfrac{c}{s}$



$P = \dfrac{c}{2s}$ has $P'' = 0$ $\quad$ Inflection point $\quad$ Bending changes from up to down

CHECK $\quad \dfrac{d^2P}{dt^2} = \dfrac{d}{dt}\left(cP - sP^2\right) = (c - 2sP)\dfrac{dP}{dt} = 0$ at $P = \dfrac{c}{2s}$

World population approaches the limit $\dfrac{c}{s} \approx 12$ billion (FOR THIS MODEL!)

Population now ≈ 7 billion $\quad$ Try Google for "World population"

Practice Questions

$\boldsymbol{\frac{dy}{dt} = cy} - s$ has $s =$ spending rate not savings rate (with minus sign)

1. The constant solution is $y =$ _____ when $\frac{dy}{dt} = 0$

In that case interest income balances spending: $cy = s$

2. The complete solution is $y(t) = \frac{s}{c} + Ae^{ct}$. Why is $A = y(0) - \frac{s}{c}$?

3. If you start with $y(0) > \frac{s}{c}$ why does wealth approach ∞ ?

   If you start with $y(0) < \frac{s}{c}$ why does wealth approach $-\infty$?

4. The complete solution to $\boldsymbol{\frac{dy}{dt} = s}$ is $y(t) = st + A$

What solution $y(t)$ starts from $y(0)$ at $t = 0$?

5. If $\boldsymbol{\frac{dP}{dt} = -sP^2}$ and $y = \frac{1}{P}$ explain why $\frac{dy}{dt} = s$

Pure competition. Show that $P(t) \to 0$ as $t \to \infty$

6. If $\boldsymbol{\frac{dP}{dt} = cP - sP^4}$ find a linear equation for $y = \frac{1}{P^3}$

Differential Equations of Motion

A differential equation for $y(t)$ can involve dy/dt and also d^2y/dt^2

Here are examples with solutions C and D can be any numbers

$$\frac{d^2y}{dt^2} = -y \text{ and } \boldsymbol{\frac{d^2y}{dt^2} = -\omega^2 y} \quad \text{Solutions} \quad \begin{array}{l} y = C\cos t + D\sin t \\ \boldsymbol{y = C\cos \omega t + D\sin \omega t} \end{array}$$

Now include dy/dt and look for a solution method

$$m\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + ky = 0 \text{ has a damping term } 2r\frac{dy}{dt}. \qquad \textbf{Try } \boldsymbol{y = e^{\lambda t}}$$

Substituting $e^{\lambda t}$ gives $\quad m\lambda^2 e^{\lambda t} + 2r\lambda e^{\lambda t} + ke^{\lambda t} = 0$

Cancel $e^{\lambda t}$ to leave the key equation for λ $\quad \boldsymbol{m\lambda^2 + 2r\lambda + k = 0}$

The quadratic formula gives $\boldsymbol{\lambda} = \frac{-r \pm \sqrt{r^2 - km}}{m}$ Two solutions λ_1 and λ_2

The differential equation is solved by $\boldsymbol{y = Ce^{\lambda_1 t} + De^{\lambda_2 t}}$

Special case $r^2 = km$ has $\lambda_1 = \lambda_2$ Then t enters $y = Ce^{\lambda_1 t} + D\boldsymbol{t}e^{\lambda_1 t}$

EXAMPLE 1 $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+8y=0 \quad m=1$ and $2r=6$ and $k=8$

$\lambda_1,\lambda_2=\frac{-r\pm\sqrt{r^2-km}}{m}$ is $-3\pm\sqrt{9-8}$ Then $\lambda_1=-\mathbf{2}$, $\lambda_2=-\mathbf{4}$

Solution $\boldsymbol{y=Ce^{-2t}+De^{-4t}}$ Overdamping with no oscillation

EXAMPLE 2 Change to $k=10$ $\lambda=-3\pm\sqrt{9-10}$ has $\boldsymbol{\lambda_1=-3+i}$, $\boldsymbol{\lambda_2=-3-i}$

Oscillations from the imaginary part of λ **Decay** from the real part -3

Solution $y=Ce^{\lambda_1 t}+De^{\lambda_2 t}=Ce^{(-3+i)t}+De^{(-3-i)t}$
$e^{it}=\cos t+i\sin t$ leads to $y=(C+D)e^{-3t}\cos t+(C-D)e^{-3t}\sin t$

EXAMPLE 3 Change to $k=9$ Now $\lambda=-3,-3$ (repeated root)

Solution $y=Ce^{-3t}+\boldsymbol{Dte^{-3t}}$ includes the factor $\boldsymbol{t}$

Practice Questions

1. For $\frac{d^2y}{dt^2}=\mathbf{4}y$ find two solutions $y=Ce^{at}+De^{bt}$. What are a and b?
2. For $\frac{d^2y}{dt^2}=-\mathbf{4}y$ find two solutions $y=C\cos\omega t+D\sin\omega t$. What is ω?
3. For $\frac{d^2y}{dt^2}=\mathbf{0}y$ find two solutions $y=Ce^{0t}$ and (???)
4. Put $y=e^{\lambda t}$ into $2\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y=0$ to find λ_1 and λ_2 (**real** numbers)
5. Put $y=e^{\lambda t}$ into $2\frac{d^2y}{dt^2}+5\frac{dy}{dt}+3y=0$ to find λ_1 and λ_2 (**complex** numbers)
6. Put $y=e^{\lambda t}$ into $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$ to find λ_1 and λ_2 (**equal** numbers)

   Now $y=Ce^{\lambda_1 t}+D\boldsymbol{t}e^{\lambda_1 t}$. The factor t appears when $\lambda_1=\lambda_2$

Power Series and Euler's Formula

At $x=0$, the nth derivative of x^n is the number $n!$ Other derivatives are 0.

Multiply the nth derivatives of $f(x)$ by $x^n/n!$ to match function with series

TAYLOR SERIES $$f(x)=f(0)+f'(0)\frac{x}{1}+f''(0)\frac{x^2}{2}+\cdots+f^{(n)}(0)\frac{x^n}{n!}+\cdots$$

EXAMPLE 1 $f(x)=e^x$ All derivatives $=1$ at $x=0$ Match with $x^n/n!$

Taylor Series
Exponential Series $$= \quad e^x=1+1\frac{x}{1}+1\frac{x^2}{2}+\cdots+1\frac{x^n}{n!}+\cdots$$

EXAMPLE 2 $f=\sin x$ $\quad f'=\cos x$ $\quad f''=-\sin x$ $\quad f'''=-\cos x$

At $x=0$ this is $0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1$ REPEAT

$$\boldsymbol{\sin x}=1\cdot\frac{x}{1}-1\frac{x^3}{3!}+1\frac{x^5}{5!}-\cdots$$ ODD POWERS $\sin(-x)=-\sin x$

EXAMPLE 3 $f=\cos x$ produces $1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0$ REPEAT

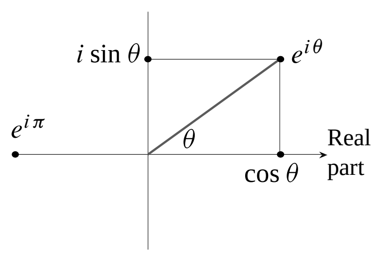
$$\boldsymbol{\cos x}=1-1\frac{x^2}{2!}+1\frac{x^4}{4!}-\cdots$$ EVEN POWERS $\frac{d}{dx}(\cos x)=-\sin x$

Imaginary $i^2=-1$ and then $i^3=-i$ **Find the exponential e^{ix}**

$$e^{ix}=1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\cdots$$
$$=\left(1-\frac{x^2}{2!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\cdots\right)$$

Those are **$\cos x + i \sin x$**

EULER'S GREAT FORMULA $e^{ix}=\cos x+i\sin x$



$e^{i\theta}=\cos\theta+i\sin\theta$

$e^{i\theta}+e^{-i\theta}=2\cos\theta$

$e^{i\pi}=-1$ combines 4 great numbers

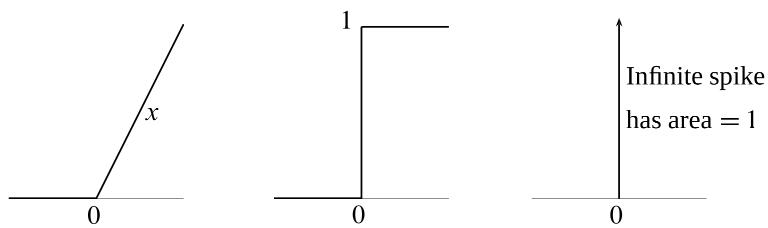
Two more examples of Power Series (Taylor Series for $f(x)$)

$$f(x)=\frac{\mathbf{1}}{\mathbf{1-x}}=1+x+x^2+x^3+\cdots$$ "Geometric series"

$$f(x)=-\mathbf{ln(1-x)}=\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$ "Integral of geometric series"

Summary: Six Functions, Six Rules, Six Theorems

Integrals	**Six Functions**	***Derivatives***
$x^{n+1}/(n+1),\ n \neq -1$	$\boldsymbol{x^n}$	nx^{n-1}
$-\cos x$	$\mathbf{sin}\ \boldsymbol{x}$	$\cos x$
$\sin x$	$\mathbf{cos}\ \boldsymbol{x}$	$-\sin x$
e^{cx}/c	$\boldsymbol{e^{cx}}$	ce^{cx}
$x\ln x - x$	$\mathbf{ln}\,\boldsymbol{x}$	$1/x$
Ramp function	**Step function**	**Delta function**



Six Rules of Differential Calculus

1. The derivative of $\boldsymbol{af(x)+bg(x)}$ is $a\dfrac{df}{dx}+b\dfrac{dg}{dx}$ **Sum**

2. The derivative of $\boldsymbol{f(x)g(x)}$ is $f(x)\dfrac{dg}{dx}+g(x)\dfrac{df}{dx}$ **Product**

3. The derivative of $\dfrac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}$ is $\left(g\dfrac{df}{dx}-f\dfrac{dg}{dx}\right)\Big/ g^2$ **Quotient**

4. The derivative of $\boldsymbol{f(g(x))}$ is $\dfrac{df}{dy}\dfrac{dy}{dx}$ where $y=g(x)$ **Chain**

5. The derivative of $x=\boldsymbol{f^{-1}(y)}$ is $\dfrac{dx}{dy}=\dfrac{1}{dy/dx}$ **Inverse**

6. When $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, what about $f(x)/g(x)$? **l'Hôpital**

$\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{df/dx}{dg/dx}$ if these limits exist. Normally this is $\dfrac{\boldsymbol{f'(a)}}{\boldsymbol{g'(a)}}$

Fundamental Theorem of Calculus

If $f(x)=\displaystyle\int_a^x s(t)\,dt$ then **derivative of integral** $=\dfrac{df}{dx}=\boldsymbol{s(x)}$

If $\dfrac{df}{dx}=s(x)$ then **integral of derivative** $=\displaystyle\int_a^b s(x)\,dx=\boldsymbol{f(b)-f(a)}$

Both parts assume that $s(x)$ is a continuous function.

All Values Theorem Suppose $f(x)$ is a continuous function for $a \leqslant x \leqslant b$. Then on that interval, $f(x)$ reaches its maximum value M and its minimum m. And $f(x)$ takes all values between m and M (there are no jumps).

Mean Value Theorem If $f(x)$ has a derivative for $a \leqslant x \leqslant b$ then

$$\frac{f(b)-f(a)}{b-a}=\frac{df}{dx}(c) \quad \text{at some } c \text{ between } a \text{ and } b$$

"At some moment c, instant speed $=$ average speed"

Taylor Series Match all the derivatives $f^{(n)}=d^n f/dx^n$ at the basepoint $x=a$

$$f(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\,f^{(n)}(a)\,(x-a)^n$$

Stopping at $(x-a)^n$ leaves the error $f^{n+1}(c)(x-a)^{n+1}/(n+1)!$

[c is somewhere between a and x] [$n=0$ is the Mean Value Theorem]

The Taylor series looks best around $a=0$ $\quad f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\,f^{(n)}(0)\,x^n$

Binomial Theorem shows Pascal's triangle

$$\begin{array}{ll}(1+x) & \mathbf{1}+\mathbf{1}x\\ (1+x)^2 & \mathbf{1}+\mathbf{2}x+\mathbf{1}x^2\\ (1+x)^3 & \mathbf{1}+\mathbf{3}x+\mathbf{3}x^2+\mathbf{1}x^3\\ (1+x)^4 & \mathbf{1}+\mathbf{4}x+\mathbf{6}x^2+\mathbf{4}x^3+\mathbf{1}x^4\end{array}$$

Those are just the Taylor series for $f(x)=(1+x)^p$ when $p=1,2,3,4$

$$\begin{array}{lcccc} f^{(n)}(x)= & (1+x)^p & p(1+x)^{p-1} & p(p-1)(1+x)^{p-2} & \cdots\\ f^{(n)}(0)= & \mathbf{1} & \boldsymbol{p} & \boldsymbol{p(p-1)} & \cdots\end{array}$$

Divide by $n!$ to find the Taylor coefficients $=$ **Binomial coefficients**

$$\frac{1}{n!}f^{(n)}(0)=\frac{p(p-1)\cdots(p-n+1)}{n(n-1)\cdots(1)}=\frac{p!}{(p-n)!\,n!}=\binom{\boldsymbol{p}}{\boldsymbol{n}}$$

The series stops at x^n when $p=n$ Infinite series for other p

Every $(1+x)^p=1+px+\frac{p(p-1)}{(2)(1)}x^2+\frac{p(p-1)(p-2)}{(3)(2)(1)}x^3+\cdots$

Practice Questions

1. Check that the derivative of $y = x\ln x - x$ is $dy/dx = \ln x$.

2. The "sign function" is $S(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$

What ramp function $F(x)$ has $\dfrac{dF}{dx} = S(x)$? F is the integral of S.

Why is the derivative $\dfrac{dS}{dx} = 2$ **delta**(x) ? (Infinite spike at $x = 0$ with area 2)

3. (l'Hôpital) What is the limit of $\dfrac{2x+3x^2}{5x+7x^2}$ as $x \to 0$? What about $x \to \infty$?

4. l'Hôpital's Rule says that $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = ??$ when $f(0) = 0$. Here $g(x) = x$.

5. **Derivative is like Difference Integral is like Sum**

   Difference of sums If $f_n = s_1 + s_2 + \cdots + s_n$, what is $f_n - f_{n-1}$?

   Sums of differences What is $(f_1 - f_0) + (f_2 - f_1) + \cdots + (f_n - f_{n-1})$?

   Those are the **Fundamental Theorems** of **"Difference Calculus"**

6. Draw a non-continuous graph for $0 \leq x \leq 1$ where your function does NOT reach its maximum value.

7. For $f(x) = x^2$, which in-between point c gives $\dfrac{f(5) - f(1)}{5-1} = \dfrac{df}{dx}(c)$?

8. If your average speed on the Mass Pike is 75, then at some instant your speedometer will read _____ .

9. Find three Taylor coefficients A, B, C for $\sqrt{1+x}$ (around $x = 0$).

$$(1+x)^{\frac{1}{2}} = A + Bx + Cx^2 + \cdots$$

10. Find the Taylor (= Binomial) series for $f = \dfrac{1}{1+x}$ around $x = 0$ $(p = -1)$.

0.5 Graphs and Graphing Calculators

This book started with the sentence "*Calculus is about functions*." When these functions are given by formulas like $y = x + x^2$, we now know a formula for the slope (and even the slope of the slope). When we only have a rough graph of the function, we can't expect more than a rough graph of the slope. But graphs are very valuable in applications of calculus!

From a graph of $y(x)$, this section extracts the basic information about the growth rate (the slope) and the minimum/maximum and the bending (and area too). A big part of that information is contained in *a plus or minus sign*. Is $y(x)$ increasing? Is its slope increasing? Is the area under its graph increasing? In each case some number is greater than zero. The three numbers are dy/dx and d^2y/dx^2 and $y(x)$ itself.

When one of those numbers is *exactly zero* we always have a special situation. It is a good thing that mathematics invented zero, we need it.

This section is organized by two themes:

(1) Graphs that are drawn without a formula for $y(x)$. From that graph you can draw other graphs—the slope dy/dx, the second derivative d^2y/dx^2, the area $A(x)$ under the graph.

You can also identify where those functions are positive or negative—and especially the points where dy/dx or d^2y/dx^2 or $y(x)$ is *zero*.

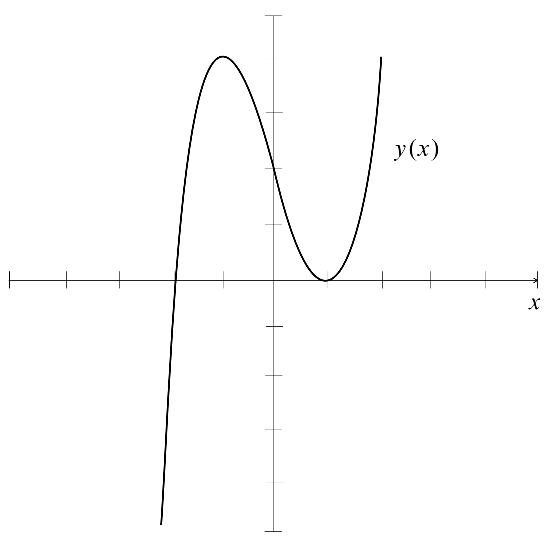
(2) Graphs that are drawn by a calculator or computer. Now there is a formula for $y(x)$. The display allows us to guess rules for derivatives:

Chain Rule **Inverse Rule** **l'Hôpital's Rule**

These rules come into later chapters of the book. They are also explained in *Highlights of Calculus*, the video lectures that are available to everyone. One specific goal is to see how the derivative of 2^x is proportional to 2^x.

This section was much improved by ideas that were offered by Benjamin Goldstein.

GRAPH WITHOUT FORMULAS



1. Suppose this is the graph of some function $y(x)$

a. At what value(s) of x does $y(x)$ have a local minimum?

b. At what value(s) of x does $y(x)$ have a local maximum?

c. At what value(s) of x does $y(x)$ have an inflection point? (Estimate.)

2. Let's change the problem. Suppose this is the graph of dy/dx, the derivative of $y(x)$. Answer the following questions about $y(x)$, the original function.

a. At what value(s) of x does $y(x)$ have a local minimum?

b. At what value(s) of x does $y(x)$ have a local maximum?

c. At what value(s) of x does $y(x)$ have an inflection point?

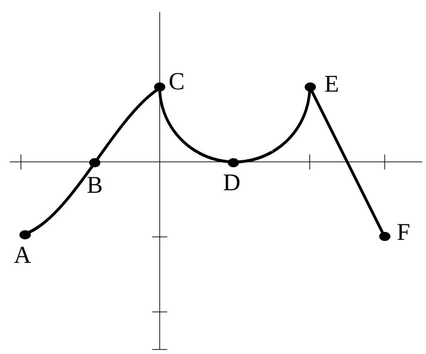
3. One more variation. Suppose this is the graph of the second derivative d^2y/dx^2 (slope of the slope). If any of these questions can't be answered, explain why.

a. At what value(s) of x does $y(x)$ have a local minimum?

b. At what value(s) of x does $y(x)$ have a local maximum?

c. At what value(s) of x does $y(x)$ have an inflection point?

4. Answer the same 9 questions for this second graph.



5. The following table shows the velocity of a car at selected times.

time	0	5	10	15	20	25	30	35
velocity	45	40	30	40	45	40	30	25

a. Was there any time t when the car was moving with acceleration $d^2y/dt^2 = 0$? Justify your answer.

b. If $y(t)$ represents the car's position as a function of time, was there ever a time when $d^3y/dt^3 = 0$? Justify your answer. The third derivative is sometimes referred to as '*jerk*' because it indicates the jerkiness of the motion. This is *important* to roller-coaster designers.

c. What assumptions have you made about $y(t)$ and (more importantly) dy/dt in your answers to parts (a) and (b)? Are the assumptions reasonable?

THE CHAIN RULE ON A CALCULATOR

a. On your calculator, graph $Y_1 = \sin(X)$ and its slope $Y_2 = \text{nDeriv}(Y_1, X, X)$. Make sure you are in radian mode, and select the trigonometric viewing window.

1. What function does Y_2 appear to be?
2. Change Y_1 to $Y_1 = \sin(2X)$. Now what function does Y_2 appear to be? Check your guess by graphing the true derivative.
3. Finally, change Y_1 to $Y_1 = \sin(3X)$. What does Y_2 appear to be this time?
4. Conjecture: If k is some constant, then the derivative of $\sin(kx)$ is _____ .

b. Those functions are *chains* (also called *compositions*). They can be written in the form $Y = f(g(x))$. For $\sin(kx)$ the outer function is $f(x) =$ _____ and the inner function is $g(x) =$ _____ .

c. So far the inner function $g(x)$ has been linear, but it doesn't have to be. Let $Y = \sin(\sqrt{x})$.

Conjecture: $\dfrac{dY}{dx} =$ _____ when $g(x) = \sqrt{x}$.

Check your conjecture by graphing Y and comparing to the graph of the numerical derivative.

d. Now we generalize. Suppose $g(x)$ is any function. If $y = \sin(g(x))$, then $dy/dx =$ _____ .

e. There is nothing magical about the sine function. Whenever we have a composition of an outer and an inner function, the chain rule applies. Predict the following derivatives and check by graphing the numerical derivative on your calculator.

1. $y = (2x+4)^3$; $dy/dx =$
2. $y = \cos^2 x = (\cos x)^2$; $dy/dx =$
3. $y = \cos(x^2)$; $dy/dx =$
4. $y = [\sin(x^2+1)]^3$; $dy/dx =$

COMPUTING IN CALCULUS

Software is available for calculus courses—a lot of it. The packages keep getting better. Which program to use (if any) depends on cost and convenience and purpose. *How* to use it is a much harder question. These pages identify some of the goals. Our aim is to support, with examples, the effort to use computing to help learning.

For calculus, ***the greatest advantage of the computer is to offer graphics***. You see the function, not just the formula. As you watch, $f(x)$ reaches a maximum or a minimum or zero. A separate graph shows its derivative. Those statements are not 100% true, as everybody learns right away—as soon as a few functions are typed in. But the power to *see this subject* is enormous, because it is adjustable. If we don't like the picture we change to a new viewing window.

This is computer-based graphics. It combines ***numerical*** computation with ***graphical*** computation. You get pictures as well as numbers—a powerful combination. The computer offers the experience of actually working with a function. The domain and range are not just abstract ideas. *You choose them*. May I give a few examples.

EXAMPLE 1 Certainly x^3 equals 3^x when $x=3$. ***Do those graphs ever meet again*** ? At this point we don't know the full meaning of 3^x, except when x is a nice number. (Neither does the computer.) Checking at $x=2$ and 4, the function x^3 is smaller both times: 2^3 is below 3^2 and $4^3=64$ is below $3^4=81$. If x^3 is always less than 3^x we ought to know—these are among the basic functions of mathematics.

The computer will answer numerically or graphically. At our command, it solves $x^3=3^x$. At another command, it plots both functions—this shows more. The screen proves a point of logic (or mathematics) that escaped us. If the graphs cross once, they must cross again—because 3^x is higher at 2 and 4. A crossing point near 2.5 is seen by zooming in. I am less interested in the exact number than its position—it comes before $x=3$ rather than after.

A few conclusions from such a basic example:

1. A supercomputer is not necessary.
2. High-level programming is not necessary.
3. We can do mathematics without completely understanding it.

The third point doesn't sound so good.Write it differently: ***We can learn mathematics while doing it***. The hardest part of teaching calculus is to turn it from a spectator sport into a workout. The computer makes that possible.

EXAMPLE 2 (mental computer) Compare x^2 with 2^x. The functions meet at $x=2$. Where do they meet again ? Is it before or after 2 ?

That is mental computing because the answer happens to be a whole number (4). Now we are on a different track. Does an accident like $2^4=4^2$ ever happen again ? Can the machine tell us about integers ? Perhaps it can plot the solutions of $x^b=b^x$. I asked *Mathematica* for a formula, hoping to discover x as a function of b—but the program just gave back the equation. For once the machine typed HELP instead of the user.

Well, mathematics is not helpless. I am proud of calculus. There is a new exercise at the end of Section 6.4, to show that we never see whole numbers again.

EXAMPLE 3 Find the number b for which $x^b=b^x$ has only ***one*** solution (at $x=b$).

When b is 3, the second solution is below 3. When b is 2, the second solution (4) is above 2. If we move b from 2 to 3, there must be a special "double point"—where the graphs barely touch but don't cross. For that particular b—and only for that one value—the curve x^b never goes above b^x.

This special point b can be found with computer-based graphics. In many ways it is the "***center point of calculus***." Since the curves touch but don't cross, they are tangent. They have the same slope at the double point. Calculus was created to work with slopes, and we already know the slope of x^2. Soon comes x^b. Eventually we discover the slope of b^x, and identify the most important number in calculus.

The point is that this number can be discovered first by experiment.

EXAMPLE 4 Graph $y(x)=e^x-x^e$. Locate its minimum. Zoom in near $x=e$.

From the derivatives of e^x and x^e, show that $dy/dx=0$ at $x=e$.

If you try, you can also find the next derivative d^2y/dx^2. Can you see why $d^2y/dx^2>0$ at $x=e$?

The next example was proposed by Don Small. Solve $x^4 - 11x^3 + 5x - 2 = 0$. The first tool is algebra—try to factor the polynomial. That succeeds for quadratics, and then gets extremely hard. Even if the computer can do algebra better than we can, factoring is seldom the way to go. In reality we have two good choices:

1. (***Mathematics***) Use the derivative. Solve by Newton's method.
2. (***Graphics***) *Plot the function and zoom in*.

Both will be done by the computer. Both have potential problems! Newton's method is fast, but that means it can fail fast. (It is usually terrific.) Plotting the graph is also fast—but solutions can be outside the viewing window. This particular function is zero only once, in the standard window from -10 to 10. The graph seems to be leaving zero, but mathematics again predicts a second crossing point. So we zoom out before we zoom in.

The use of the zoom is the best part of graphing. Not only do we *choose* the domain and range, we *change* them. The viewing window is controlled by four numbers. They can be the limits $A \leqslant x \leqslant B$ and $C \leqslant y \leqslant D$. They can be the coordinates of two opposite corners: (A,C) and (B,D). They can be the center position (a,b) and the scale factors c and d. Clicking on opposite corners of the zoom box is the fastest way, unless the center is unchanged and we only need to give scale factors. (Even faster: Use the default factors.) Section 3.4 discusses the ***centering transform*** and ***zoom transform***—a change of picture on the screen and a change of variable within the function.

EXAMPLE 5 Find all real solutions to $x^4 - 11x^3 + 5x - 2 = 0$.

EXAMPLE 6 Zoom out and in on the graphs of $y = \cos 40x$ and $y = x \sin(1/x)$. Describe what you see.

EXAMPLE 7 What does $y = (\tan x - \sin x)/x^3$ approach at $x = 0$? For small x the machine eventually can't separate $\tan x$ from $\sin x$. It may give $y = 0$. Can you get close enough to see the limit of y as $x \to 0$?

SYMBOLIC COMPUTATION

In symbolic computation, answers can be *formulas* as well as numbers and graphs. The derivative of $y = x^2$ is seen as "$2x$." The derivative of $\sin t$ is "$\cos t$." The slope of b^x is known to the program. The computer does more than substitute numbers into formulas—it operates directly on the formulas. We need to think where this fits with learning calculus.

In a way, symbolic computing is close to what we ourselves do. Maybe too close—there is some danger that symbolic manipulation is *all* we do. With a higher-level language and enough power, a computer can print the derivative of $\sin(x^2)$. So why learn the chain rule? Because mathematics goes deeper than "algebra with formulas." We deal with *ideas*.

I want to say clearly: Mathematics is not formulas or computations or even proofs, but ideas. The symbols and pictures are the language. The book and the professor and the computer can join in teaching it. The computer should be non-threatening (like this book and your professor)—you can work at your own pace. Your part is to learn by doing.

EXAMPLE 8 A computer algebra system quickly finds 100 factorial. This is $100! = (100)(99)(98)\ldots(1)$. The number has 158 digits (not written out here). The last 24 digits are zeros. For $10! = 3628800$ there are seven digits and two zeros. Between 10 and 100, and beyond, are simple questions that need ideas:

1. How many digits (approximately) are in the number $N!$?
2. How many zeros (exactly) are at the end of $N!$?

For Question **1**, the computer shows more than N digits when $N = 100$. It will never show more than N^2 digits, because none of the N terms can have more than N digits. A much tighter bound would be $2N$, but is it true? *Does* $N!$ *always have fewer than 2N digits?*

For Question **2**, the zeros in 10! can be explained. One comes from 10, the other from 5 times 2. (10 is also 5 times 2.) Can you explain the 24 zeros in 100!? An idea from the card game blackjack applies here too: *Count the fives*.

Hard question: How many zeros at the end of 200!?

Writing in Calculus May I emphasize the importance of writing? We totally miss it, when the answer is just a number. A one-page report is harder on instructors as well as students—but much more valuable. You can't write sentences without being forced to organize ideas—and part of yourself goes into it.

I will propose a writing exercise with options. If you have computer-based graphing, follow through on Examples $1-4$ above and report. Without a computer, pick a paragraph from this book that should be clearer and *make it clearer*. Rewrite it with examples. Identify the key idea at the start, explain it, and come back to express it differently at the end. Ideas are like surfaces—they can be seen many ways.

Mathematics can be learned by *talking* and *writing*—it is a human activity. Our goal is not to test but to teach and learn.

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Resource: Calculus
Gilbert Strang

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Resource: Calculus
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