a Coriolis tutorial, Part 1:

the Coriolis force, inertial and geostrophic motion

James F. Price
Woods Hole Oceanographic Institution
Woods Hole, Massachusetts, 02543
https://www2.whoi.edu/staff/jprice/ jprice@whoi.edu

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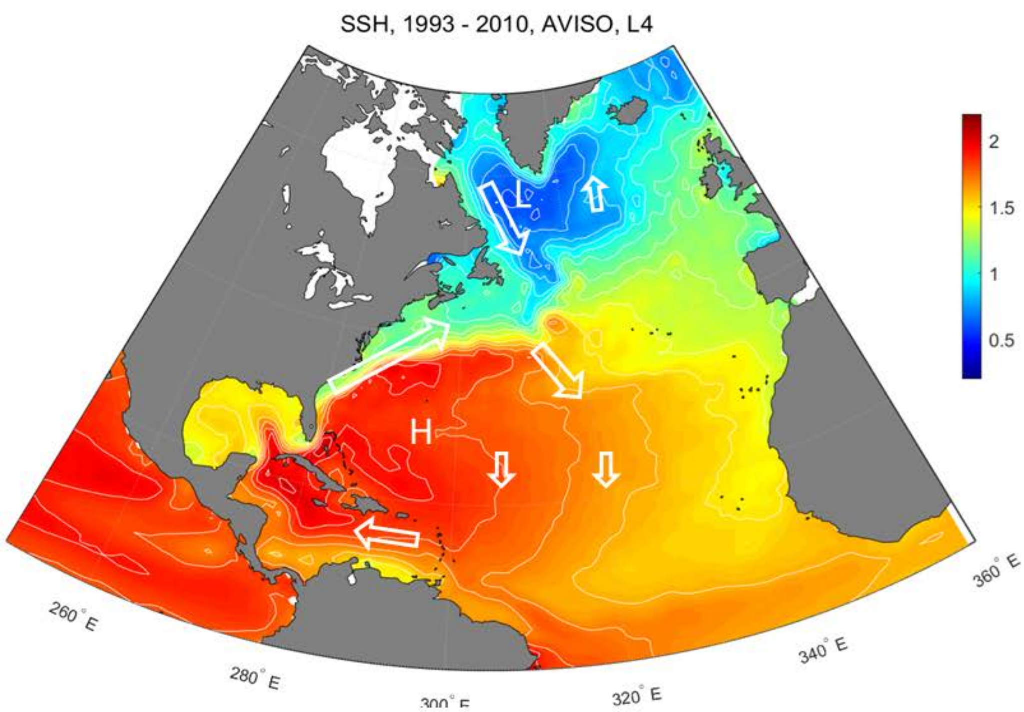

Figure 1: The decadal average sea surface height (SSH) of the North Atlantic. Colorbar at right is in meters. The principal features are a high over the subtropics and a low over the subpolar region. The inferred geostrophic current is sketched at a few locations. Geostrophic currents are almost parallel to lines of SSH, with higher SSH to the right of the current in the northern hemisphere. The upper-ocean circulation of the subtropical basin is thus a clockwise turning, horizontal gyre. The first goal of this essay is to understand how Earth's rotation leads to the Coriolis force, and the second goal is to begin to consider some of the consequences of rotation, including a very simple analog of the geostrophic relationship between SSH and upper-ocean currents.

Abstract: This essay is the first of a five part introduction to the Coriolis force and its consequences for the atmosphere and ocean. It is intended for students who are near the beginning of a quantitative study of geophysical fluid dynamics and who have some background in classical mechanics and applied mathematics.

The equation of motion appropriate to a steadily rotating reference frame includes two terms that account for accelerations that arise from the rotation of the reference frame, a centrifugal force and a Coriolis force. In the special case of an Earth-attached reference frame, the centrifugal force is subsumed into the gravity field. The Coriolis force has a very simple mathematical form, $-2\mathbf{\Omega}\times\mathbf{V}'M$, where $\mathbf{\Omega}$ is Earth's rotation vector, $\mathbf{V}'$ is the velocity observed from the rotating frame and M is the parcel mass. The Coriolis force is perpendicular to the velocity and so tends to change velocity direction, but not velocity amplitude. Hence the Coriolis force does no work. Nevertheless the Coriolis force has a profound importance for the circulation of the atmosphere and oceans.

Two direct consequences of the Coriolis force are considered here. If the Coriolis force is the only force acting on a moving parcel, then the velocity vector of the parcel will be turned anti-cyclonically (clockwise in the northern hemisphere) at the rate $-f$, where $f = 2\Omega \sin(\text{latitude})$ is the Coriolis parameter. These free motions, often termed inertial oscillations, are a first approximation of the upper ocean currents generated by a transient wind event. If the Coriolis force is balanced by a steady force, say a horizontal component of gravity as in Fig.1, then the associated geostrophic wind or current will be in a direction that is perpendicular to the gradient of the SSH and thus parallel to isolines of SSH. In the northern hemisphere, higher SSH is to the right of the current. This is said to be a geostrophic balance, and is the defining characteristic of the large scale, low frequency, extra-tropical circulation of the atmosphere and the oceans.

More on Figure 1: The mean of sea surface height (SSH) over almost two decades as observed by satellite altimetry and compiled by the Aviso project, http://www.aviso.oceanobs.com/duacs/ SSH is a constant pressure surface that is displaced slightly but significantly from level and hence there is a horizontal component of gravity along this surface that is proportional to the gradient of SSH. What keeps the SSH displaced away from level? We can be confident that the horizontal gravitational force associated with this tilted SSH is balanced locally (at a given point) by the Coriolis force acting upon currents that flow nearly parallel to isolines of SSH. This geostrophic relationship is a central topic of this essay. Notice that by far the largest gradients of SSH and so the largest geostrophic currents are found on the western boundary of the gyres. This east-west asymmetry is a consequence of Earth's rotation and nearly spherical shape, taken up in Part 4 of this series.

Contents

1 Large-scale, low frequency flows of the atmosphere and ocean

The large-scale, low frequency flows of Earth's atmosphere and ocean take the form of circulations around centers of high or low gravitational potential (the height of a constant pressure surface relative to a known level, the sea surface height, SSH, of Fig. (1), or the 500 hPa height of Fig. (2). Ocean circulation features of this sort include gyres that extend across ocean basins, and in the atmosphere, a broad belt of westerly wind that encircles the mid-latitudes in both hemispheres. Smaller scale circulations often dominate the weather. Hurricanes and mid-latitude storms have a more or less circular flow around a low, and many regions of the ocean are filled with slowly revolving eddies having a diameter of several hundred kilometers. The height anomaly that is associated with these circulation features is the direct result of a mass excess or deficit (high or low height anomaly).

What is at first surprising and deserving of an explanation is that large scale mass anomalies implicit in the SSH and height fields of Figs. (1) and (2) persist for many days or weeks, even in the absence of an external momentum or energy source. The winds and currents that would be expected to accelerate down the height gradient (in effect, downhill) and disperse the associated mass anomaly do not occur. Instead, large-scale, low frequency winds and currents outside of the near-equatorial region are observed to flow in a direction almost parallel to lines of constant height; the sense of the flow is clockwise around highs (northern hemisphere) and anti-clockwise around lows. The flow direction is reversed in the southern hemisphere, anti-clockwise around highs and clockwise around lows. From

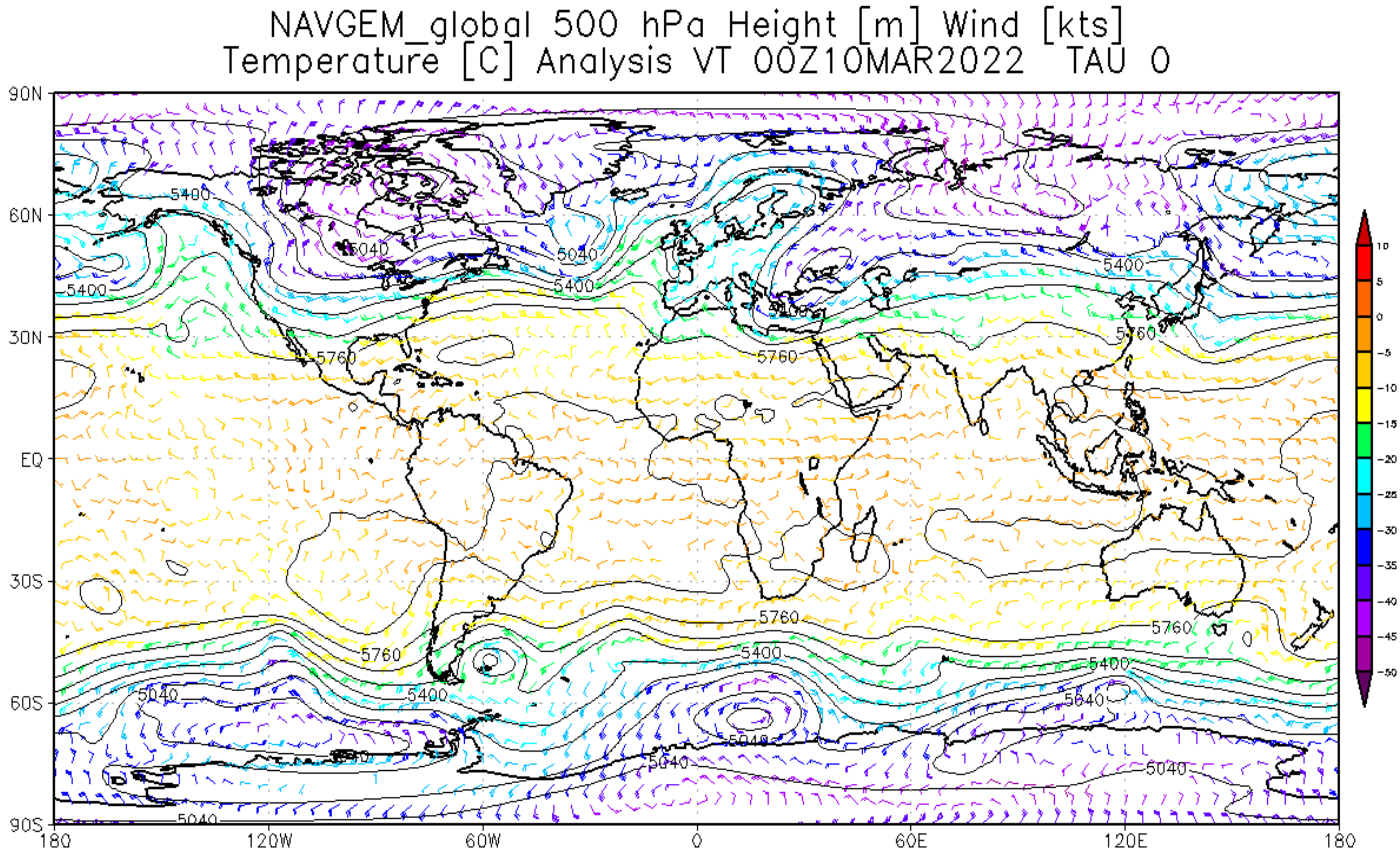

Figure 2: A weather chart at 500 hPa, a middle level of the atmosphere, on 10 March, 2022 produced by the NAVy Global Environmental Model, NAVGEM (thanks to the Naval Research Laboratory, Monterey CA, and Fleet Numerical Oceanography Center, https://www.nrlmry.navy.mil/metoc/nogaps/). The solid contours are the height (meters) of the 500 hPa pressure surface above sea level contoured at 120 m intervals. The horizontal wind is shown as barbs (one thin barb = 10 knots $\approx$ 5 m s^{-1}, one heavy barb = 50 knots). The color is temperature, deg C, with a scale at right. Several important phenomena are evident on this map: (1) The largest scale of variation is a high at low latitudes, roughly 5800 m near the equator, and a low at either pole, roughly 5050 m. (2) The zonal winds at mid-latitudes are mainly westerly in both hemispheres, i.e., from west to east, though with considerable variability in the north-south component, e.g., a prominent ridge over central Europe on this day. The broad band of westerly winds includes the jet stream(s), where wind speed is typically $\approx$ 30 m s^{-1}. (3) Within the westerly wind band, the 500 hPa surface generally slopes upward toward lower latitude, roughly 200 m per 1000 km. There is thus a small, but significant component of gravity along the 500 hPa surface directed from south to north in the northern hemisphere, and reversed in the southern hemisphere. (4) The wind and height fields outside of near-equatorial regions (roughly $\pm$ 10 degrees of latitude) thus exhibit a geostrophic relationship: wind vectors are nearly parallel to the contours of constant height, greater height is to the right of the wind vector in the northern hemisphere and to the left in the southern hemisphere, and faster winds are found in conjunction with larger height gradients.

this we can infer that the horizontal gravitational force along a pressure surface must be balanced approximately by a second force that acts to deflect horizontal winds and currents to the right of the velocity vector in the northern hemisphere and to the left of the velocity vector in the southern hemisphere (you should make a sketch of this[1]). This deflecting force is called the Coriolis force,[2] and is the theme of this essay. A quasi-steady balance between the horizontal gravitational force (or equivalently, the horizontal component of the pressure gradient) and the Coriolis force is called a geostrophic balance. An approximate or quasi- geostrophic balance is the defining characteristic of large scale, low frequency atmospheric and oceanic flows.[3]

We attribute profound physical consequences to the Coriolis force, and yet cannot point to a physical interaction as the cause of the Coriolis force in the straightforward way that height anomalies are related to the mass field. Rather, the Coriolis force arises from motion itself, combined with the necessity that we observe the atmosphere and ocean from an Earth-attached and thus rotating, noninertial reference frame. In this respect the Coriolis force is very different from other important forces acting on geophysical fluids, e.g., friction and gravity, that arise from an interaction of physical objects. The goals of this essay (elaborated below) are, first, to explain the origin and basic properties of the Coriolis force, and then to examine some of its direct consequences for the atmosphere and ocean, including a very simple analogue of geostrophic motion.

1.1 Models and reference frames

These essays proceed inductively, developing and adding new concepts one by one rather than deriving them from a comprehensive starting point. In that spirit, the first physical model considered here in Part 1 will be a single, isolated fluid particle, or 'parcel'. This is a very drastic and for most purposes untenable idealization of a fluid. Winds and currents, like all macroscopic fluid flows, are effectively a continuum of parcels that interact in three-dimensions; the motion of any one parcel is connected by pressure gradients and by friction to the motion of essentially all of the other parcels that make up the flow. This global dependence is at the very heart of fluid mechanics, but can be set aside here because the Coriolis force on a given parcel depends only upon the velocity of that parcel. What will have to go missing in this single parcel model is that the external forces on a parcel (the **F** below) must be

[1] And then you will surely be wondering, What's it do right on the equator?, taken up in Sec. 6, Homework Problem 12.

[2] After the French physicist and engineer, Gaspard-Gustave de Coriolis, 1792-1843, whose seminal contributions include the systematic derivation of the rotating frame equation of motion and the development of the gyroscope. An informative history of the Coriolis force is by A. Persson, 'How do we understand the Coriolis force?', *Bull. Am. Met. Soc.*, **79**(7), 1373-1385 (1998).

[3] To be sure, it's not quite this simple. This 'large scale, low frequency' is a shorthand for (1) large spatial scale, (2) low frequency, (3) extra-tropical, and (4) outside of frictional boundary layers. It is important to have a quantitative sense what is meant by each of these (which turn out to be linked in interesting ways) and we will come to this in Parts 2 and 3. For now, suffice it to say that this present use of 'large scale' encompasses everything that you can readily see in Figs. 1 and 2, except for the equatorial region, roughly ± 10 deg of latitude.

prescribed in a way that can take no account of, e.g., the mass and pressure fields that evolve as part of a real fluid flow. The phenomena that may be treated with a single parcel model are thus severely limited, but are nevertheless a recognizable subset of the phenomena that arise in more realistic fluid models (coming in Part 2) and in the real ocean and atmosphere (Figs. 1 and 2, and in Sec. 5).

1.1.1 Classical mechanics observed from an inertial reference frame

If a parcel is observed from an inertial reference frame[4] then the classical (Newtonian) equation of motion is just

$$\frac{d(M\mathbf{V})}{dt} = \mathbf{F} + \mathbf{g}_* M,$$

where d/dt is an ordinary time derivative, $\mathbf{V}$ is the velocity in a three-dimensional space, and M is the parcel's mass. The parcel mass will be presumed constant in all that follows, and the equation of motion rewritten as

$$\frac{d\mathbf{V}}{dt} = \mathbf{F}/M + \mathbf{g}_*, \tag{1}$$

where $\mathbf{F}$ is the sum of the forces that we can specify *a priori* given the complete knowledge of the environment, e.g., frictional drag with the sea floor, and $\mathbf{g}_*$ is gravitational mass attraction.[5] These are said to be central forces insofar as they act in a radial direction between parcels, or in the case of gravitational mass attraction, between parcels and the center of mass of the Earth.

In this essay we are especially concerned with identifying the acceleration that would be observed in a given reference frame. Unless it is noted otherwise, this acceleration will be written on the left-hand side of an equation of motion, as in Eqn. (1), even if the acceleration is considered to be the known quantity. The forces, i.e., everything else, will then be written be on the right-hand side of the equation as

$$observed\ acceleration \;\; = \;\; sum\ of\ the\ forces/M. \tag{2}$$

[4]'Inertia' has Latin roots *in+artis* meaning without art or skill and secondarily, resistant to change. Since Newton's *Principia*, physics usage has emphasized the latter: a parcel having inertia will remain at rest, or if in motion, continue without change unless subjected to an external force. A 'reference frame' is comprised of a coordinate system that serves to arithmetize the position of parcels, a clock to tell the time, and an observer who makes an objective record of positions and times as seen from that reference frame. A reference frame may or may not be attached to a physical object. In this essay we suppose purely classical physics so that measurements of length and of time are identical in all reference frames; measurements of position, velocity and acceleration *are* reference frame-dependent, as discussed in Section 2. This common sense view of space and time begins to fail when velocities approach the speed of light, which is not an issue here. An 'inertial reference frame' is one in which all parcels have the property of inertia and in which the total momentum is conserved, i.e., all forces occur as action-reaction force pairs. How this plays out in the presence of gravity is discussed briefly in Sec. 3.1.

[5]In all of this it is taken for granted that inertial mass (the mass property that is inversely proportional to acceleration for a given force) is exactly equal to gravitational mass (the mass property that is relevant in gravitational attraction). This is oftentimes referred to as the Equivalence Principal, and sometimes as the Universality of Free Fall. The simplest and most important rationalization for this is that inertial mass and gravitational mass are not merely equal numerically, but are one and the same thing.

The left and right-hand sides of an equation of motion will be called 'acceleration' and 'force(s)', even though all of the terms necessarily have the physical dimensions of acceleration, $[length\ time^{-2}]$. (This is not the usual convention in geophysical fluid dynamics, where the Coriolis force/M is most often written on the left side of the equation of motion.)

This inertial frame equation of motion (1) has two fundamental properties that are noted here because we are about to give them up:

Global conservation. For each of the central forces acting on the parcel there will be a corresponding reaction force acting on the environment that sets up the force. Thus the global time rate of change of momentum (global means the parcel plus the environment) due to the sum of all of the central forces $\mathbf{F}+\mathbf{g}_*M$ is zero, and so the global momentum is conserved. Usually our attention is focused on the local problem, i.e., the parcel only, with this global conservation taken for granted and not analyzed explicitly.

Invariance to Galilean transformation. Eqn. (1) should be invariant to a steady, linear translation of the reference frame, often called a Galilean transformation, because only relative motion has physical significance. Thus a constant velocity added to $\mathbf{V}$ will cause no change in the time derivative, and should as well cause no change in the forces $\mathbf{F}$ or $\mathbf{g}{*}M$. Like the global balance just noted, this fundamental property is not invoked frequently, but is a powerful guide to the form of the forces $\mathbf{F}$. For example, a frictional force that satisfies Galilean invariance should depend upon the difference of the parcel velocity with respect to a surface or adjacent parcels, and not the parcel velocity only.

1.1.2 Classical mechanics observed from a rotating, noninertial reference frame

When it comes to the analysis of the atmosphere or ocean, we always use a reference frame that is attached to the rotating Earth — true (literal) inertial reference frames are not accessible to most kinds of observation and wouldn't be desirable even if they were. Some of the reasons for this are discussed in a later section, 4.3, but for now we are concerned with the consequence that, because of the Earth's rotation, Fig. 3, an Earth-attached reference frame is significantly *noninertial* for the large-scale, low-frequency motions of the atmosphere and ocean: Eqn. (1) does not hold for large scale winds and ocean currents, even as a first approximation. The equation of motion appropriate to an Earth-attached, rotating reference frame (derived in detail in Sections 2 and 4.1) is instead

$$\frac{d\mathbf{V}'}{dt} = -\,2\boldsymbol{\Omega}\times\mathbf{V}' + \mathbf{F}'/M + \mathbf{g}, \tag{3}$$

where the prime on a vector indicates that it is observed from the rotating frame, $\boldsymbol{\Omega}$ is Earth's rotation vector (Fig. 3), which has the dimensions $[time^{-1}]$, and $\mathbf{g}$ is gravitational mass attraction plus the centrifugal force associated with Earth's rotation and called simply 'gravity' (discussed further in Section 4.1). Our obsession here will be the new term, $-2\boldsymbol{\Omega}\times\mathbf{V}'$, often called the Coriolis force in geophysics (though here it has the dimensions $[length\ time^{-2}]$ of an acceleration).

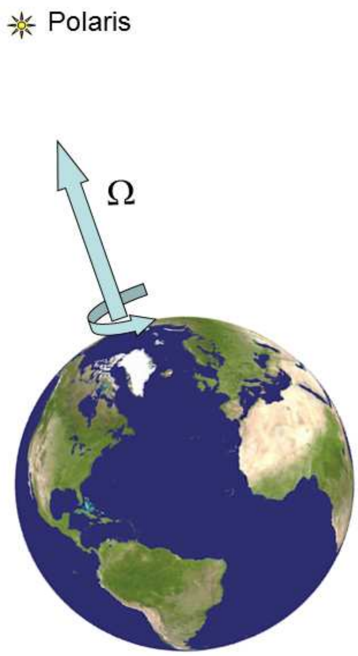

Figure 3: Earth's rotation vector, $\mathbf{\Omega}$, maintains a nearly steady bearing close to Polaris, commonly called the Pole Star or North Star. Earth thus has a specific orientation with respect to the universe at large, and, in consequence, all directions are not equal. This is manifest as a marked anisotropy of most large-scale circulation phenomena, e.g., the east-west asymmetry of ocean gyres noted in Fig. 1 and the westward propagation of low frequency waves and eddies studied in Part 3.

1.2 The goals and the plan of this essay

Eqn. (2) applied to geophysical flows is not the least bit controversial and so the practical thing to do is to accept the Coriolis force as given (as we do many other concepts) and get on with the applications. You can do that here by going directly to Sec. 5. However, that shortcut is very likely to leave you wondering ... **What is the Coriolis force?** ... in the conceptual and in the physical sense, and for example, is it appropriate to call $-2\mathbf{\Omega}\times\mathbf{V}'$ the Coriolis 'force'? The classical mechanics literature applies a bewildering array of names to this term, that it is the Coriolis effect, a pseudo force, a virtual force, an apparent force, an inertial force (we will use this as well), a noninertial force (which makes more literal sense), and most equivocal of all, a fictitious correction force.[6] A case can be made for each of these terms, but the choice here will be just plain Coriolis force, since we are going to be most concerned with what the Coriolis force does in the context of geophysical flows.

The plan for Secs. 1 - 4 is to take a slow and careful journey from Eqn. (1) to Eqn. (2) so that at the end we should be able to explain the origin and basic properties of the Coriolis force, the first goal of

[6] The latter is by by J. D. Marion, *Classical Mechanics of Particles and Systems* (Academic Press, NY, 1965), who describes the plight of a rotating observer as follows (the double quotes are his): '... the observer must postulate an additional force - the centrifugal force. But the "requirement" is an artificial one; it arises solely from an attempt to extend the form of Newton's equations to a non inertial system and this may be done only by introducing a fictitious "correction force". The same comments apply for the Coriolis force; this "force" arises when attempt is made to describe motion relative to the rotating body.' Rotating observers do indeed have to contend with inertial forces that are not found in otherwise comparable inertial frames, but these inertial forces are not *ad hoc* corrections as Marion's quote (taken out of context) might seem to imply.

this essay.[7] We have already noted that the Coriolis force arises from the rotation of an Earth-attached reference frame. The origin of the Coriolis force is thus a matter kinematics, i.e., mathematics, rather than physics, and taken up in Section 2. This is part of the reason why the Coriolis force can be hard to grasp, conceptually.[8] Several very simple applications of the rotating frame equation of motion are considered in Section 3. These illustrate the often marked difference between inertial and rotating frame descriptions of the same phenomenon, and they also show that the rotating frame equation of motion (2) does *not* retain the fundamental properties of the inertial frame Eqn. (1) noted above. Eqn. (2) applies on a rotating Earth or a planet, where the centrifugal force associated with planetary rotation is canceled (Section 4). The rotating frame equation of motion thus treats only the comparatively small relative velocity, i.e., winds and currents. This is a huge advantage compared with the inertial frame equation of motion, which has to treat all of the motion, including that due to Earth's rotation, and more than compensates for the admittedly peculiar properties of the Coriolis force which comes along with the rotating frame.

The second goal of this essay is to begin to address ... **What are the consequences of Earth's rotation and the Coriolis force for the circulation of the atmosphere and ocean?** This is an almost open-ended question that makes up much of the field of geophysical fluid dynamics. A first small step is taken in Section 5 by analyzing the motion of a parcel released onto a sloping surface, e.g., the sea surface or 500 hPa pressure surface (if they are considered to be fixed), and including a simplified form of friction. The resulting motion may include inertial oscillations, the free oscillation of the rotating frame momentum equations, and a steady motion, called geostrophic flow, that is a first approximation of the currents and winds of Figs. (1) and (2).

1.3 About these essays

This essay is the first in a five part series,

Part 1: the Coriolis force, inertial and geostrophic motion,

Part 2: a rotating shallow water model, potential vorticity conservation and geostrophic adjustment,

[7] 'Explanation is indeed a virtue; but still, less a virtue than an anthropocentric pleasure.' B. van Frassen, 'The pragmatics of explanation', in *The Philosophy of Science*, Ed. by R. Boyd, P. Gasper and J. D. Trout. (The MIT Press, Cambridge Ma, 1999). This pleasure of understanding is the deep motivation of this essay, but clearly the Coriolis force has great practical significance for the atmosphere and ocean, and for those of us who seek to understand and predict their motions.

[8] All this talk of 'forces, forces, forces' seems a little quaint and it is certainly becoming tedious. Modern dynamics is more likely to be developed around the concepts of energy, action and minimization principles, which are very useful in some special classes of fluid flow. However, it remains that the majority of fluid mechanics proceeds along the path of Eqn. (1) laid down by Newton. In part this is because mechanical energy is not conserved in most real fluid flows and in part because the interaction between a fluid parcel and its surroundings is often at issue, friction for example, and is usually best-described in terms of forces. Sometimes, just to avoid saying Coriolis force yet again, we will use instead 'rotation'.

Part 3: beta-effects, westward propagation,

Part 4: the Sverdrup relation and wind-driven ocean circulation, and,

Part 5: on the seasonally-varying Arabian Sea.

These have been written for students who are near the beginning of a quantitative study of geophysical fluid dynamics. Some background in classical mechanics and applied mathematics is assumed, roughly second year, undergraduate level physics and applied mathematics.

This Part 1 essay introduces the basics — the kinematics of a rotating reference frame and the properties of the resulting Coriolis force. Rotating reference frames are a topic included in many classical mechanics courses and textbooks and there is nothing fundamental and new regarding the Coriolis force added here.[9] The hope is that this essay will make a useful supplement to these sources by providing more of the relevant physics background and greater mathematical detail than is possible in a wide-ranging fluid dynamics text, and by emphasizing geophysical phenomena that are missed or outright misconstrued in many classical mechanics texts.[10,11] Ocean and atmospheric sciences are all about fluids in motion on a grand scale, and the electronic version of this essay includes links to animations and to source codes of numerical models that provide a more vivid depiction of these motions than is possible in a hardcopy.

The first version of this essay was released in 2003, and since then the text and models have been revised and expanded many times. The most up-to-date version of the essays and codes will be posted to the author's web site, https://www2.whoi.edu/staff/jprice and look for the page Education Projects. Comments and questions are encouraged and may be addressed to jprice@whoi.edu

[9] Classical mechanics texts in order of increasing level: A. P. French, *Newtonian Mechanics* (W. W. Norton Co., 1971); A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw-Hill, NY, 1990); C. Lanczos, *The Variational Principles of Mechanics* (Dover Pub., NY, 1949). Textbooks on geophysical fluid dynamics emphasize mainly the consequences of Earth's rotation; excellent introductions at about the level of this essay are by J. R. Holton, *An Introduction to Dynamic Meteorology, 3rd Ed.* (Academic Press, San Diego, 1992), and by B. Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics* (Prentice Hall, Engelwood Cliffs, New Jersey, 1994). Somewhat more advanced and highly recommended for the topic of geostrophic adjustment is A. E. Gill, *Atmosphere-Ocean Dynamics* (Academic Press, NY, 1982), for waves generally, J. Pedlosky, *Waves in the Ocean and Atmosphere*, (Springer, 2003) and also J. C. McWilliams, *Fundamentals of Geophysical Fluid Dynamics*, (Cambridge Univ. Press, 2006).

[10] There are several essays or articles that, like this one, aim to clarify the Coriolis force. A fine treatment in great depth is by H. M. Stommel and D. W. Moore, *An Introduction to the Coriolis Force* (Columbia Univ. Press, 1989); the present Section 4.1 owes a great deal to their work. A detailed analysis of dropped particle motion is by M. S. Tiersten and H. Soodak, 'Dropped objects and other motions relative to a noninertial earth', *Am. J. Phys.*, **68**(2), 129–142 (2000). A useful web page for general science students is http://www.ems.psu.edu/%7Efraser/Bad/BadFAQ/BadCoriolisFAQ.html

[11] The Coriolis force also has engineering applications; it is exploited to measure the angular velocity required for vehicle control systems, http://www.siliconsensing.com, and to measure mass transport in fluid flow, http://www.micromotion.com.

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2 Noninertial reference frames

The first step toward understanding the origin of the Coriolis force is to describe the origin of inertial forces in the simplest possible context, a pair of reference frames that are represented by displaced coordinate axes, Fig. (4). Frame one is labeled X and Z and frame two is labeled X' and Z'. It is helpful to assume that frame one is stationary and that frame two is displaced relative to frame one by a time-dependent vector, $\mathbf{X_o}(t)$. The measurements of position, velocity, etc. of a given parcel will thus be different in frame two vs. frame one. Just how the measurements differ is a matter of kinematics; there is no physics involved until we define the acceleration of frame two, and then use the accelerations to write an equation of motion appropriate to frame 2, e.g., Eqn. (2).

2.1 Kinematics of a linearly accelerating reference frame

If the position vector of a given parcel is $\mathbf{X}$ when observed from frame one, then from within frame two the same parcel will be observed at the position

$$\mathbf{X}' = \mathbf{X} - \mathbf{X_o}.$$

The position vector of a parcel thus depends upon the reference frame. Suppose that frame two is translated and possibly accelerated with respect to frame one, while maintaining a constant orientation (rotation will be considered shortly). If the velocity of a parcel observed in frame one is $d\mathbf{X}/dt$, then in

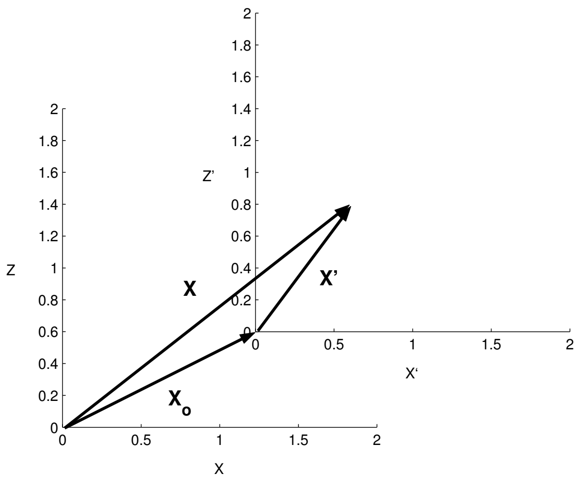


Figure 4: Two reference frames are represented by coordinate axes that are displaced by the vector $\mathbf{X_o}$ that is time-dependent. In this Section 2.1 we consider only a relative translation, so that frame two maintains a fixed orientation with respect to frame one. The rotation of frame two will be considered beginning in Section 2.2.

frame two the same parcel will be observed to have velocity

$$\frac{d\mathbf{X}'}{dt} = \frac{d\mathbf{X}}{dt} - \frac{d\mathbf{X_o}}{dt}.$$

The accelerations are similarly $d^2\mathbf{X}/dt^2$ and

$$\frac{d^2\mathbf{X}'}{dt^2} = \frac{d^2\mathbf{X}}{dt^2} - \frac{d^2\mathbf{X_o}}{dt^2}. \tag{4}$$

We are going to assume that frame one is an inertial reference frame, i.e., that parcels observed in frame one have the property of inertia so that their momentum changes only in response to a force, $\mathbf{F}$, i.e., Eqn. (1). From Eqn. (1) and from Eqn. (4) we can easily write down the equation of motion for the parcel as it would be observed from frame two:

$$\frac{d^2\mathbf{X}'}{dt^2} = -\frac{d^2\mathbf{X_o}}{dt^2} + \mathbf{F}/M + \mathbf{g}_*. \tag{5}$$

The term $-(d^2\mathbf{X_o}/dt^2)$ appearing in the frame two equation of motion (5) will be called an 'inertial force', and when this term is nonzero, frame two is said to be 'noninertial'. As an example, suppose that frame two is subject to a constant acceleration, $d^2\mathbf{X_o}/dt^2 = \mathbf{A}$ that is upward and to the right in Fig. (4). From Eqn. (4) it is evident that all parcels observed from within frame two would then appear to accelerate with a magnitude and direction $-\mathbf{A}$, downward and to the left, and which is, of course, exactly opposite the acceleration of frame two with respect to frame one. Clearly, the origin of the inertial force is the acceleration of the reference frame, $\mathbf{A}$, and not an interaction with other objects, e.g., by friction or collisions.

Inertial forces that arise in this way are exactly proportional to the mass of the object, and in this key respect are indistinguishable from gravitational mass attraction. If an inertial force is dependent only upon position, as is the centrifugal force due to Earth's rotation (Section 4.1), then it might as well be added with gravitational mass attraction $\mathbf{g}_*$ to give a single, time-independent acceleration field, usually termed 'gravity' and denoted by $\mathbf{g}$. Indeed, this gravity field — mass attraction plus centrifugal acceleration — is the acceleration field that may be observed directly, for example by observing a plumb line.[12]

Since there is no interaction between particles involved in an inertial force, there is also no action-reaction force pair associated with an inertial force. Global momentum conservation thus does not hold in the presence of inertial forces. There is something highly equivocal about the phenomenon we are calling an 'inertial force', and so you can appreciate why some authors have deemed them to be 'virtual' or 'fictitious correction' forces.[6]

Whether an inertial force is problematic or not depends entirely upon whether the acceleration $d^2\mathbf{X_o}/dt^2$ is known or not. If it should happen that the acceleration of frame two is not known, then all bets are off. For example, imagine observing the motion of a plumb bob within an enclosed trailer that was moving along in irregular, stop-and-go traffic. The bob would be observed to lurch forward and backward unexpectedly, and we would soon conclude that studying dynamics in such an uncontrolled, noninertial reference frame was going to be a very difficult endeavor. Inertial forces could be blamed if it was observed that all of the physical objects in the trailer, observers included, experienced exactly the same unaccounted acceleration. In many cases the relevant inertial forces are known well enough to use noninertial reference frames with great precision, e.g., the topography of Earth's gravity field must be known to within a few centimeters to interpret sea surface altimetry data of the kind seen in Fig. (1)[13] and the Coriolis force can be readily calculated as in Eqn. (2) knowing only Earth's rotation vector and the parcel velocity.

In the specific example of a translating reference frame sketched in Fig. (4), one could just as well transform the observations made from frame two back into the inertial frame one, use the inertial frame equation of motion to make a calculation, and then transform back to frame two if required. By that tactic we could avoid altogether the seeming delusion of an inertial force. However, when it comes to the observation and analysis of Earth's atmosphere and ocean, there is really no choice but to use an Earth-attached and thus rotating and noninertial reference (discussed in Section 4.3). That being so, we have to contend with the Coriolis force, an inertial force that arises from the rotation of an Earth-attached frame. The kinematics of rotation add a small complication that is taken up in the next Section 2.2. But the key point is this — if you followed the development of Eqn. (5), then you already

[12] A plumb bob is nothing more than a weight, the bob, that hangs from a string, the plumb line (and *plumbum* is the Latin for lead, Pb). When a plumb bob is at rest in a given reference frame, the plumb line is parallel to the local acceleration field of that reference frame. If the bob is displaced and released, it will oscillate as a simple pendulum. The observed period of small amplitude oscillations, P, can be used to infer the magnitude of the acceleration, $g = L/(P/2\pi)^2$, where L is the length of the plumb line. If the reference frame is attached to the rotating Earth (and it probably will be!), then the motion of the bob will be effected also by the Coriolis force, in which case the device is often termed a Foucault pendulum discussed in a later section, 4.5.

[13] Earth's gravity field is the object of extensive and ongoing survey by some truly remarkable technology and methods, see http://www.csr.utexas.edu/grace/

understand the origin of inertial forces, including the Coriolis force.

2.2 Kinematics of a rotating reference frame

The Coriolis force will arise in the equivalent of Eqn. (5) for the case of a steadily rotating reference frame. Reference frame one will again be assumed to be stationary and defined by a triad of orthogonal unit vectors, $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ (Fig. 5). A parcel P can then be located by a position vector $\mathbf{X}$

$$\mathbf{X} = \mathbf{e_1}x_1 + \mathbf{e_2}x_2 + \mathbf{e_3}x_3, \tag{6}$$

where the Cartesian (rectangular) components, x_i, are the projection of $\mathbf{X}$ onto each of the unit vectors in turn. It is useful to rewrite Eqn. (6) using matrix notation. The unit vectors are made the elements of a row matrix,

$$\mathbb{E} = [\mathbf{e_1}\ \mathbf{e_2}\ \mathbf{e_3}], \tag{7}$$

and the components x_i are taken to be the elements of a column matrix,

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{8}$$

Eqn. (6) may then be written in a way that conforms with the usual matrix multiplication rules as

$$\mathbf{X} = \mathbb{E}\mathbb{X}. \tag{9}$$

The vector $\mathbf{X}$ and its time derivatives are presumed to have an objective existence, i.e., they represent something physical that is unaffected by our arbitrary choice of a reference frame. Nevertheless, the way these vectors appear clearly does depend upon the reference frame (Fig. 5) and for our purpose it is essential to know how the position, velocity and acceleration vectors will appear when they are observed from a steadily rotating reference frame. In a later part of this section we will identify the rotating reference frame as an Earth-attached reference frame and the stationary frame as one aligned on the distant fixed stars. It is assumed that the motion of the rotating frame can be represented by a time-independent rotation vector, $\mathbf{\Omega}$. The $\mathbf{e_3}$ unit vector can be aligned with $\mathbf{\Omega}$ with no loss of generality, Fig. (5a). We can go a step further and align the origins of the stationary and rotating reference frames because the Coriolis force is independent of position (Section 2.2).

2.2.1 Transforming the position, velocity and acceleration vectors

Position: Back to the question at hand: how does this position vector look when viewed from a second reference frame that is rotated through an angle θ with respect to the first frame? The answer is

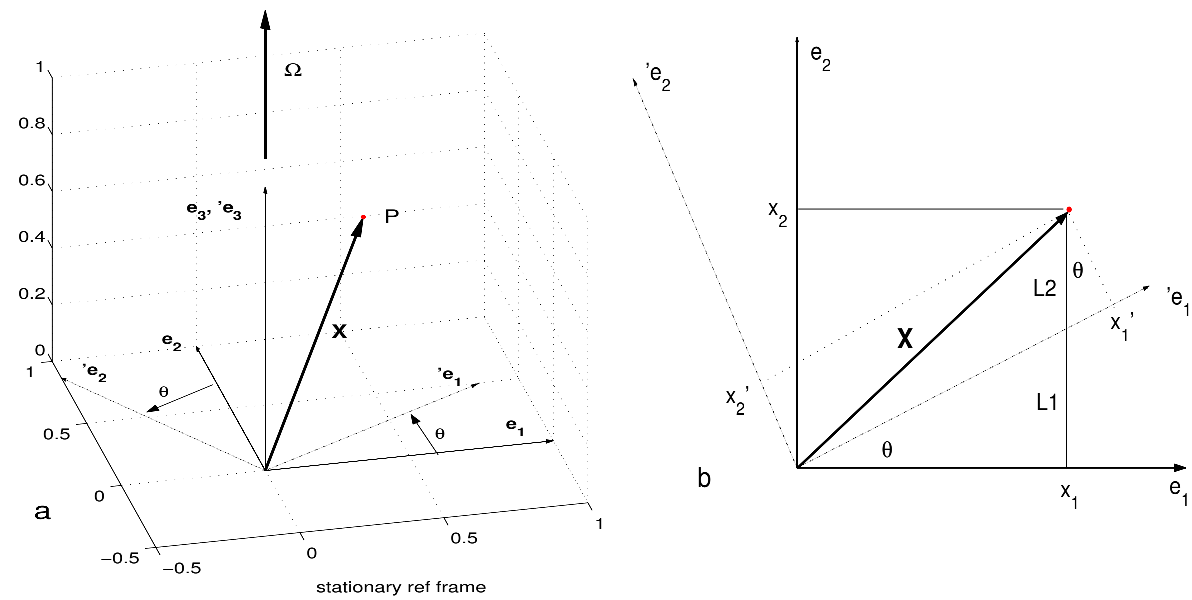


Figure 5: **(a)** A parcel P is located by the tip of a position vector, **X**. The stationary reference frame has solid unit vectors that are presumed to be time-independent, and a second, rotated reference frame has dashed unit vectors that are labeled $\grave{}\mathbf{e_i}$. The reference frames have a common origin, and rotation is about the $\mathbf{e_3}$ axis. The unit vector $\mathbf{e_3}$ is thus unchanged by this rotation and so $\grave{}\mathbf{e_3} = \mathbf{e_3}$. This holds also for $\mathbf{\Omega}' = \mathbf{\Omega}$, and so we will use $\mathbf{\Omega}$ exclusively. The angle θ is counted positive when the rotation is counterclockwise. **(b)** The components of **X** in the stationary reference frame are x_1, x_2, x_3, and in the rotated reference frame they are x'_1, x'_2, x'_3.

that the vector 'looks' like the components appropriate to the rotated reference frame, and so we need to find the projection of **X** onto the unit vectors that define the rotated frame. The details are shown in Fig. (5b); notice that $x_2 = L1 + L2$, $L1 = x_1 \tan\theta$, and $x'_2 = L2\cos\theta$. From this it follows that $x'_2 = (x_2 - x_1\tan\theta)\cos\theta = -x_1\sin\theta + x_2\cos\theta$. By a similar calculation we can find that $x'_1 = x_1\cos\theta + x_2\sin\theta$. The component x'_3 that is aligned with the axis of the rotation vector is unchanged, $x'_3 = x_3$, and so the set of equations for the primed components may be written as a column vector

$$\mathbb{X}' = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta \\ x_3 \end{bmatrix}. \tag{10}$$

By inspection this can be factored into the product

$$\mathbb{X}' = \mathbb{R}\mathbb{X}, \tag{11}$$

where $\mathbb{X}$ is the matrix of stationary frame components and $\mathbb{R}$ is the rotation matrix,[14]

$$\mathbb{R}(\theta) \;=\; \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

This θ is the angle displaced by the rotated reference frame and is positive counterclockwise. The position vector observed from the rotated frame will be denoted by $\mathbf{X}'$; to construct $\mathbf{X}'$ we sum the rotated components, $\mathbb{X}'$, times a set of unit vectors that are fixed and thus

$$\mathbf{X}' = \mathbf{e_1}x_1' + \mathbf{e_2}x_2' + \mathbf{e_3}x_3' = \mathbb{E}\mathbb{X}' \tag{13}$$

For example, the position vector $\mathbf{X}$ of Fig. (5) is at an angle of about 45° counterclockwise from the $\mathbf{e_1}$ unit vector and the rotated frame is at $\theta = 30°$ counterclockwise from the stationary frame one. That being so, the position vector viewed from the rotated reference frame, $\mathbf{X}'$, makes an angle of 45° - 30° = 15° with respect to the $\mathbf{e_1}$ (fixed) unit vector seen within the rotated frame, Fig. (6). As a kind of verbal shorthand we might say that the position vector has been 'transformed' into the rotated frame by Eqs. (10) and (13). But since the vector has an objective existence, what we really mean is that the components of the position vector are transformed by Eqn. (10) and then summed with fixed unit vectors to yield what should be regarded as a new vector, $\mathbf{X}'$, the position vector that we observe from the rotated (or rotating) reference frame.

Velocity: The velocity of parcel P seen in the stationary frame is just the time rate of change of the position vector seen in that frame,

$$\frac{d\mathbf{X}}{dt} = \frac{d}{dt}\mathbb{E}\mathbb{X} = \mathbb{E}\frac{d\mathbb{X}}{dt},$$

since $\mathbb{E}$ is time-independent. The velocity of parcel P as seen from the rotating reference frame is similarly

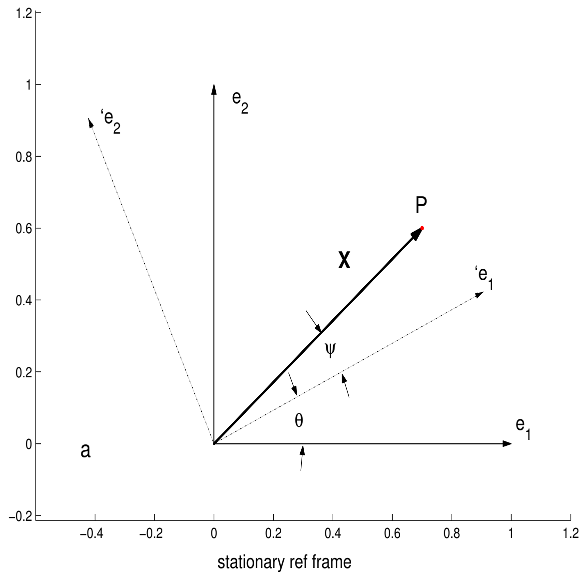
$$\frac{d\mathbf{X}'}{dt} = \frac{d}{dt}\mathbb{E}\mathbb{X}' = \mathbb{E}\frac{d\mathbb{X}'}{dt},$$

which indicates that the time derivatives of the rotated components are going to be very important in what follows. For the first derivative we find

$$\frac{d\mathbb{X}'}{dt} = \frac{d(\mathbb{R}\mathbb{X})}{dt} = \frac{d\mathbb{R}}{dt}\mathbb{X} + \mathbb{R}\frac{d\mathbb{X}}{dt}. \tag{14}$$

The second term on the right side of Eqn. (14) represents velocity components from the stationary frame that have been transformed into the rotating frame, as in Eqn. (11). If the rotation angle θ was

[14] A concise and clear reference on matrix representations of coordinate transformations is by J. Pettofrezzo *Matrices and Transformations* (Dover Pub., New York, 1966). An excellent all-around reference for undergraduate-level applied mathematics including coordinate transformations is by M. L. Boas, *Mathematical Methods in the Physical Sciences, 2nd edition* (John Wiley and Sons, 1983).



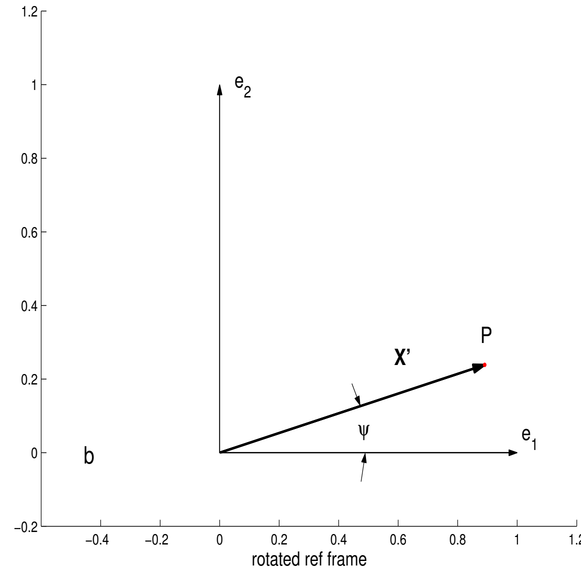


Figure 6: **(a)** The position vector $\mathbf{X}$ seen from the stationary reference frame. **(b)** The position vector as seen from the rotated frame, denoted by $\mathbf{X}'$. Note that in the rotated reference frame the unit vectors are labeled $\mathbf{e_i}$ since they are fixed; when these unit vectors are seen from the stationary frame, as on the left, they are labeled $\grave{\mathbf{e}}_\mathbf{i}$. If the position vector is stationary in the stationary frame, then $\theta + \psi = constant$. The angle ψ then changes as $d\psi/dt = -d\theta/dt = -\Omega$, and thus the vector $\mathbf{X}'$ appears to rotate at the same rate but in the opposite sense as does the rotating reference frame.

constant so that $\mathbb{R}$ was independent of time, then the first term on the right side would vanish and the velocity components would transform exactly as do the components of the position vector. In that case there would be no Coriolis force.

When the rotation angle is time-varying, as it will be here, the first term on the right side of Eqn. (14) is non-zero and represents a velocity component that is induced solely by the rotation of the reference frame. For an Earth-attached reference frame

$$\theta = \theta_0 + \Omega t,$$

where Ω is Earth's rotation rate measured with respect to the distant stars, effectively a constant defined below (and θ_0 is unimportant). Though Ω may be presumed constant, the associated reference frame is nevertheless accelerating and is noninertial in the same way that circular motion at a steady speed is accelerating because the direction of the velocity vector is continually changing (cf. Fig. 10). Given this $\theta(t)$, the time-derivative of the rotation matrix is

$$\frac{d\mathbb{R}}{dt} = \Omega \begin{bmatrix} -\sin\theta(t) & \cos\theta(t) & 0 \\ -\cos\theta(t) & -\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

which has the elements of $\mathbb{R}$, but shuffled around. By inspection, this matrix can be factored into the product of a matrix $\mathbb{C}$ and $\mathbb{R}$ as

$$\frac{d\mathbb{R}}{dt} = \Omega\, \mathbb{C}\mathbb{R}(\theta(t)), \tag{16}$$

where the matrix $\mathbb{C}$ is

$$\mathbb{C} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2). \tag{17}$$

Multiplication by $\mathbb{C}$ acts to knock out the component $(\ )_3$ that is parallel to $\boldsymbol{\Omega}$ and causes a rotation of $\pi/2$ in the plane perpendicular to $\boldsymbol{\Omega}$. Substitution into Eqn. (14) gives the velocity components appropriate to the rotating frame

$$\frac{d(\mathbb{R}\mathbb{X})}{dt} \;=\; \Omega\mathbb{C}\mathbb{R}\mathbb{X} + \mathbb{R}\frac{d\mathbb{X}}{dt}, \tag{18}$$

or using the prime notation $(\ )'$ to indicate multiplication by $\mathbb{R}$, then

$$\frac{d\mathbb{X}'}{dt} = \Omega\mathbb{C}\mathbb{X}' + \left(\frac{d\mathbb{X}}{dt}\right)' \tag{19}$$

The second term on the right side of Eqn. (19) is just the rotated velocity components and is present even if Ω vanished (a rotated but not a rotating reference frame). The first term on the right side represents a velocity that is induced by the rotation rate of the rotating frame; this induced velocity is proportional to Ω and makes an angle of $\pi/2$ radians to the right of the position vector in the rotating frame (assuming that $\Omega > 0$).

To calculate the vector form of this term we can assume that the parcel P is stationary in the stationary reference frame so that $d\mathbf{X}/dt = 0$. Viewed from the rotating frame, the parcel will appear to move clockwise at a rate that can be calculated from the geometry (Fig. 7). Let the rotation in a time interval δt be given by $\delta\psi = -\Omega\delta t$; in that time interval the tip of the vector will move a distance $|\delta\mathbf{X}'| = |\mathbf{X}'|\sin(\delta\psi) \approx |\ \mathbf{X}'|\delta\psi$, assuming the small angle approximation for $\sin(\delta\psi)$. The parcel will move in a direction that is perpendicular (instantaneously) to $\mathbf{X}'$. The velocity of parcel P as seen from the rotating frame and due solely to the coordinate system rotation is thus $\lim_{\delta t\to 0}\frac{\delta\mathbf{X}'}{\delta t} = -\boldsymbol{\Omega}\times\boldsymbol{X'}$, the vector cross-product equivalent of $\Omega\mathbb{C}\mathbb{X}'$ (Fig. 8). The vector equivalent of Eqn. (19) is then,

$$\boxed{\frac{d\mathbf{X}'}{dt} = -\boldsymbol{\Omega}\times\boldsymbol{X'} + \left(\frac{d\mathbf{X}}{dt}\right)'} \tag{20}$$

This and a handful of other equations upcoming are boxed to indicate their special importance.

The relation between time derivatives given by Eqn. (20) applies to velocity vectors, acceleration vectors, etc., and may be written as an operator equation,

$$\frac{d(\ )'}{dt} = -\boldsymbol{\Omega}\times(\ \ )' + \left(\frac{d(\ )}{dt}\right)' \tag{21}$$

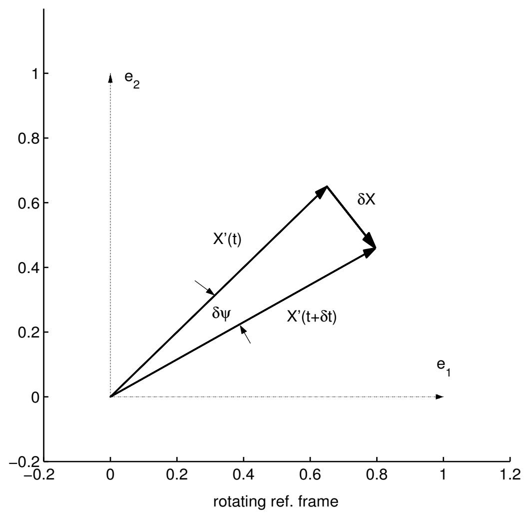


Figure 7: The position vector $\mathbf{X}'$ seen from the rotating reference frame. The unit vectors that define this frame, $\grave{\mathbf{e}}_\mathbf{i}$, appear to be stationary when viewed from within this frame, and hence we label them with $\mathbf{e_i}$ (not primed). Assume that $\Omega > 0$ so that the rotating frame is turning counterclockwise with respect to the stationary frame, and assume that the parcel P is stationary in the stationary reference frame so that $d\mathbf{X}/dt = 0$. Parcel P as viewed from the rotating frame will then appear to move clockwise on a circular path.

that is valid for all vectors regardless of their position with repsect to the axis of rotation.[15] From left to right the terms are: 1) the time rate of change of a vector as seen in the rotating reference frame, 2) the cross-product of the rotation vector with the vector and 3) the time rate change of the vector as seen in the stationary frame and then rotated into the rotating frame. Notice that the time rate of change and prime operators of (21) do not commute, the difference being the cross-product term which represents a time rate change in the *direction* of the vector, but not its magnitude. The left hand side, term 1), is the time rate of change that we observe directly or seek to solve when working from the rotating frame.

Acceleration: Our goal here is to relate the accelerations seen in the two reference frames and so differentiating Eqn. (19) once more and after rearrangement of the kind used above

$$\frac{d^2\mathbb{X}'}{dt^2} = 2\Omega\mathbb{C}\frac{d\mathbb{X}'}{dt} + \Omega^2\mathbb{C}^2\mathbb{X}' + \left(\frac{d^2\mathbb{X}}{dt^2}\right)' \tag{22}$$

The middle term on the right includes multiplication by the matrix $\mathbb{C}^2 = \mathbb{C}\mathbb{C}$,

$$\mathbb{C}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi/2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbb{R}(\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that knocks out the component parallel to the rotation vector $\mathbf{\Omega}$ and reverses the other two components; the vector equivalent of $\Omega^2\mathbb{C}^2\mathbb{X}'$ is thus $-\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}'$ (Fig. 8). The vector equivalent of Eqn. (22) is

[15] Imagine arrows taped to a turntable with random orientations. Once the turntable is set into (solid body) rotation, all of the arrows will necessarily rotate at the same rotation rate regardless of their position or orientation. The rotation will, of course, cause a translation of the arrows that depends upon their location, but the rotation rate is necessarily uniform, and this holds regardless of the physical quantity that the vector represents. This is of some importance for our application to a rotating Earth, since Earth's motion includes a rotation about the polar axis, as well as an orbital motion around the Sun and yet we represent Earth's rotation by a single vector.

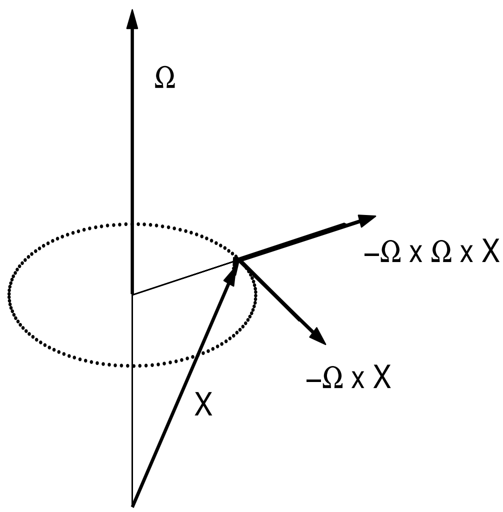


Figure 8: A schematic showing the relationship of a vector **X**, and various cross-products with a second vector **Ω** (note the signs). The vector **X** is shown with its tail perched on the axis of the vector **Ω** as if it were a position vector. This helps to visualize the direction of the cross-products, but it is important to note that the relationship among the vectors and vector products shown here holds for all vectors, regardless of where they are defined in space or the physical quantity, e.g., position or velocity, that they represent.

then[16]

$$\boxed{\frac{d^2\mathbf{X}'}{dt^2} = -2\mathbf{\Omega}\times\frac{d\mathbf{X}'}{dt} - \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}' + \left(\frac{d^2\mathbf{X}}{dt^2}\right)'} \tag{23}$$

Note the similarity with Eqn. (3). From left to right the terms are 1) the acceleration as seen in the rotating frame, 2) the Coriolis term, 3) the centrifugal[17] term, and 4) the acceleration as seen in the stationary frame and then rotated into the rotating frame. As before, term 1) is the acceleration that we directly observe or seek to solve for when working from the rotating reference frame.

2.2.2 Stationary ⇒ Inertial; Rotating ⇒ Earth-Attached

The third and final step in this derivation of the Coriolis force is to define the inertial reference frame one, and then the rotation rate of frame two. To make frame one inertial it is presumed that the unit

[16]The relationship between the stationary and rotating frame velocity vectors given by Eqs. (19) and (20) is clear visually and becomes intuitive given just a little experience. It is not so easy to intuit the corresponding relationship between the accelerations given by Eqs. (22) and (23). To understand the transformation of acceleration there is really no choice but to understand (be able to reproduce and then explain) the mathematical steps going from Eqn. (19) to Eqn. (22) and/or from Eqn. (20) to Eqn. (23).

[17]'Centrifugal' and 'centripetal' have Latin roots, *centri+fugere* and *centri+peter*, meaning center-fleeing and center-seeking, respectively. Taken literally these would indicate merely the sign of a radial force, for example. However, they are very often used to mean specifically a term of the sort $\Omega^2 r$, seen on the right side of Eq. (23), i.e., the centrifugal force in an equation of motion written for a rotating, non-inertial reference frame. The same kind of term, though with the rotation rate written as ω and referring to the angular coordinate of the parcel rather than the reference frame, will also arise as the acceleration observed from an inertial reference frame. In that case $\omega^2 r$ is the centripetal acceleration that must accompany every curving trajectory. This seeming change of identity from an inertial force to an acceleration will be discussed further in Sec. 3.2.

vectors $\mathbf{e_i}$ could in principle be aligned on the distant, 'fixed stars'.[18] The rotating frame two is presumed to be attached to Earth, and the rotation rate Ω is then given by the rate at which the same fixed stars are observed to rotate overhead, one revolution per *sidereal* day (Latin for from the stars),

$$\boxed{\Omega \;=\; 2\pi/(23 \text{ hrs}, 56 \text{ min and } 4.09 \text{ sec}) \;=\; 7.2921\times 10^{-5} \text{ rad s}^{-1}} \tag{24}$$

A sidereal day is only about four minutes less than a solar day, and so in a purely numerical sense, $\Omega \approx \Omega_{solar} = 2\pi/24$ hours = 7.2722×10^{-5} rad s^{-1} which is certainly easier to remember than is Eqn. (24). For the purpose of a rough estimate, the small numerical difference between Ω and Ω_{solar} is not significant. However, the difference between Ω and Ω_{solar} can be told in numerical simulations and in well-resolved field observations. And too, on Mach's Principle,[18] the difference between Ω and Ω_{solar} is highly significant.

Earth's rotation rate is very nearly constant, and the axis of rotation maintains a nearly steady bearing on a point on the celestial sphere that is close to the North Star, Polaris (Fig. 3). The Earth's rotation vector thus provides a definite orientation of Earth with respect to the universe, and Earth's rotation rate has an absolute magnitude. The practical evidence of this comes from rotation rate sensors[11] that read out Earth's rotation rate with respect to the fixed stars as a kind of gage pressure, called 'Earth rate'.[19]

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[18]Fixed is a matter of degree. The Sun and the planets certainly do not qualify as fixed, but even some nearby stars move detectably over the course of a year. The intent is that the most distant stars should serve as sign posts for the spatially-averaged mass of the universe as a whole on the hypothesis that inertia arises whenever there is an acceleration (linear or rotational) with respect to the mass of the universe. This grand idea was expressed most forcefully by the Austrian philosopher and physicist Ernst Mach, and is often termed Mach's Principle (see, e.g., J. Schwinger, *Einstein's Legacy* Dover Publications, 1986; M. Born, *Einstein's Theory of Relativity*, Dover Publications, 1962). Mach's Principle seems to be in accord with all empirical data, including the magnitude of the Coriolis force. Mach's principle is probably best thought of as a (profound) relationship, rather than the fundamental mechanism of inertia. A new hypothesis takes the form of so-called vacuum stuff (or Higgs field) that is presumed to pervade all of space and so provide a local mechanism for resistance to accelerated motion (see P. Davies, 'On the meaning of Mach's principle', http://www.padrak.com/ine/INERTIA.html). The debate between Newton and Leibniz over the reality of absolute space — which had seemed to go in favor of relative space, Leibniz and Mach's Principle — has been renewed in the search for a physical origin of inertia. When this is achieved, we can then point to a physical origin of the Coriolis force.

Observations on the fixed stars are a very precise means to define rotation rate, but can not, in general, be used to define the linear translation or acceleration of a reference frame. The only way to know if a reference frame that is aligned on the fixed stars is inertial is to carry out mechanics experiments and test whether Eqn.(1) holds and global momentum is conserved. If yes, the frame is inertial.

[19]For our present purpose $\mathbf{\Omega}$ may be presumed constant. In fact, there are small but observable variations of Earth's rotation rate due mainly to changes in the atmospheric and oceanic circulation and due to mass distribution within the cryosphere, see B. F. Chao and C. M. Cox, 'Detection of a large-scale mass redistribution in the terrestrial system since 1998,' Science, **297**, 831–833 (2002), and R. M. Ponte and D. Stammer, 'Role of ocean currents and bottom pressure variability on seasonal polar motion,' *J. Geophys. Res.*, **104**, 23393–23409 (1999). The direction of $\mathbf{\Omega}$ with respect to the celestial sphere also varies detectably on time scales of tens of centuries on account of precession, so that Polaris has not always been the pole star (Fig. 3), even during historical times. The slow variation of Earth's orbital parameters (slow enough to be assumed to vanish for our purpose) are an important element of climate, see e.g., J. A. Rial, 'Pacemaking the ice ages by frequency modulation of

Assume that the inertial frame equation of motion is

$$\frac{d^2\mathbb{X}}{dt^2} = \mathbb{F}/M + \mathbb{G}_* \;\text{ and }\; \frac{d^2\mathbf{X}}{dt^2} = \mathbf{F}/M + \mathbf{g}_* \tag{25}$$

($\mathbb{G}_*$ is the component matrix of $\mathbf{g}*$). The acceleration and force can always be viewed from another reference frame that is rotated (but not rotating) with respect to the first frame,

$$\left(\frac{d^2\mathbb{X}}{dt^2}\right)' = \mathbb{F}' + \mathbb{G}'_* \quad\text{and}\quad \left(\frac{d^2\mathbf{X}}{dt^2}\right)' = \mathbf{F}' + \mathbf{g}'_*, \tag{26}$$

as if we had chosen a different set of fixed stars or multiplied both sides of Eqn. (22) by the same rotation matrix. This equation of motion preserves the global conservation and Galilean transformation properties of Eqn. (25). To find the rotating frame equation of motion, eliminate the rotated acceleration from Eqn. (26) using Eqs. (22) and (23) and then solve for the acceleration seen in the rotating frame: the components are

$$\frac{d^2\mathbb{X}'}{dt^2} = 2\Omega\mathbb{C}\frac{d\mathbb{X}'}{dt} - \Omega^2\mathbb{C}^2\mathbb{X}' + \mathbb{F}' + \mathbb{G}'_* \tag{27}$$

and the vector equivalent is

$$\boxed{\frac{d^2\mathbf{X}'}{dt^2} = -\,2\mathbf{\Omega}\times\frac{d\mathbf{X}'}{dt} \;-\; \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}' + \mathbf{F}' + \mathbf{g}_*{}'.} \tag{28}$$

Eqn. (28) has the form of Eqn. (4), the difference being that the noninertial reference frame is rotating rather than translating. If the origin of this noninertial reference frame was also accelerating, then there would be a third inertial force term, $-(d^2\mathbf{X_o}/dt^2)$. Notice that we are not yet at Eqn. (2); in Section 4.1 the centrifugal force and gravitational mass attraction terms will be combined into the time-independent inertial force **g**.

Earth's orbital eccentricity,' *Science*, **285**, 564–568 (1999).

As well, Earth's motion within the solar system and galaxy is much more complex than a simple spin around a perfectly stable polar axis. Among other things, the Earth orbits the Sun in a counterclockwise direction with a rotation rate of 1.9910 $\times 10^{-7}$ s^{-1}, which is about 0.3% of the rotation rate Ω. Does this orbital motion enter into the Coriolis force, or otherwise affect the dynamics of the atmosphere and oceans? The short answer is no and yes. We have already accounted for the rotation of the Earth with respect to the fixed stars. Whether this rotation is due to a spin about an axis centered on the Earth or due to a solid body rotation about a displaced center is not relevant for the Coriolis force *per se*, as noted in the discussion of Eqn. (21). However, since Earth's polar axis is tilted significantly from normal to the plane of the Earth's orbit around the Sun (the tilt implied by Fig. 3), we can ascribe Earth's rotation Ω to spin alone. The orbital motion about the Sun combined with Earth's finite size gives rise to tidal forces, which are small but important spatial variations of the centrifugal/gravitational balance that holds for the Earth-Sun and for the Earth-Moon as a whole (described particularly well by French[9], and see also Tiersten, M. S. and H Soodak, 'Dropped objects and other motions relative to the noninertial earth', *Am. J. Phys.*, **68** (2), Feb. 2000, 129-142).

2.2.3 Remarks on the transformed equation of motion

Once the transformation rule for accelerations, Eqn. (23), is in hand, the path to the rotating frame equation of motion is short and direct — if Eqn. (26) holds in a given reference frame (say an inertial frame, but that is not essential) then Eqs. (27) and (28) hold exactly in a frame that rotates at the constant rate and direction given by $\mathbf{\Omega}$ with respect to the first frame. The rotating frame equation of motion includes two terms that are dependent upon the rotation vector, the Coriolis term, $-2\mathbf{\Omega}\times(d\mathbf{X}'/dt)$, and the centrifugal term, $-\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}'$. These terms are sometimes written on the left side of an equation of motion as if they were going to be regarded as part of the acceleration, i.e.,

$$\frac{d^2\mathbf{X}'}{dt^2} + 2\mathbf{\Omega}\times\frac{d\mathbf{X}'}{dt} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}' = \mathbf{F}'/M + \mathbf{g}*'. \tag{29}$$

Comparing the left side of Eqn. (29) with Eqn. (23), it is evident that the rotated acceleration is equal to the rotated force,

$$\left(\frac{d^2\mathbf{X}}{dt^2}\right)' = \mathbf{F}'/M + \mathbf{g}*',$$

which is well and true and the same as Eqn. (26).[20] However, it is crucial to understand that the left side of Eqn. (29), $(d^2\mathbf{X}/dt^2)'$ is *not* the acceleration that is observed from the rotating reference frame, $d^2\mathbf{X}'/dt^2$. When Eqn. (29) is solved for $d^2\mathbf{X}'/dt^2$, it follows that the Coriolis and centrifugal terms are, figuratively or literally, sent to the right side of the equation of motion where they are interpreted as if they were forces.

When the Coriolis and centrifugal terms are regarded as forces — and it is argued here that they should be when observing from a rotating reference frame — they have all of the peculiar properties of inertial forces noted in Section 2.1. From Eqn. (29) (and Eqn. 4) it is evident that the centrifugal and Coriolis terms are exactly proportional to the mass of the parcel observed, whatever that mass may be. The acceleration associated with these inertial forces arises from the rotational acceleration of the reference frame, combined with relative velocity for the Coriolis force. They differ from central forces $\mathbf{F}$ and $\mathbf{g}*M$ in the respect that there is no physical interaction that causes the Coriolis or centrifugal force and hence there is no action-reaction force pair. As a consequence the rotating frame equation of motion does not retain the global conservation of momentum that is a fundamental property of the inertial frame equation of motion and central forces (an example of this nonconservation is described in Section 3.4). Similarly, we note here only that invariance to Galilean transformation is lost since the Coriolis force involves the velocity rather than velocity derivatives. Thus $\mathbf{V}'$ is an absolute velocity in the rotating reference frame of the Earth. If we need to call attention to these special properties of the Coriolis force, then the usage Coriolis *inertial* force seems appropriate because it is free from the taint

[20] Recall that $\mathbf{\Omega} = \mathbf{\Omega}'$ and so we could put a prime on every vector in this equation. That being so, it would be better to remove the visually distracting primes and then make note that the resulting equation holds in a steadily rotating reference frame. We will keep the primes for now, since we will be considering both inertial and rotating reference frames until Section 5.

of unreality that goes with 'virtual force', 'fictitious correction force', etc., and because it gives at least a hint at the origin of the Coriolis force. It is important to be aware of these properties of the rotating frame equation of motion, and also to be assured that in most analysis of geophysical flows they are of no great practical consequence. What is most important is that the rotating frame equation of motion offers a very significant gain in simplicity compared to the inertial frame equation of motion, discussed further in Section 4.

The Coriolis and centrifugal forces taken individually have simple interpretations. From Eqn. (28) it is evident that the Coriolis force is normal to the velocity, $d\mathbf{X}'/dt$, and to the rotation vector, $\mathbf{\Omega}$. The Coriolis force will thus tend to cause the velocity to change direction but not magnitude, and is appropriately termed a deflecting force as noted in Section 1 (the purest example of this deflection occurs in an important phenomenon called inertial motion, described in Section 5.2.) The centrifugal force is in a direction perpendicular to and directed away from the axis of rotation. It is independent of time and is dependent upon position. How these forces effect dynamics in simplified conditions will be considered in Sections 3, 4.3 and 5.

2.3 Problems

(1) It is important that Eqs. (10) through (13) have an immediate and concrete meaning for you. Some questions/assignments to help you along: Verify Eqs. (10) and (13) by some direct experimentation, i.e., try them and see. Show that the transformation of the vector components given by Eqs. (11) and (12) leaves the magnitude of the vector unchanged, i.e., $|\mathbf{X}'| = |\mathbf{X}|$. Verify that $\mathbb{R}(\theta_1)\mathbb{R}(\theta_2) = \mathbb{R}(\theta_1 + \theta_2)$ and that $\mathbb{R}\theta^{-1} = \mathbb{R}(-\theta)$, where $\mathbb{R}^{-1}$ is the inverse (and also the transpose) of the rotation matrix.

(2) Show that the unit vectors that define the rotated frame can be related to the unit vectors of the stationary frame by $\grave{}\mathbb{E} = \mathbb{E}\mathbb{R}^{-1}$ and hence the unit vectors observed from the stationary frame turn the opposite direction of the position vector observed from the rotating frame (and thus the reversed prime). The components of an ordinary vector (a position vector or velocity vector) are thus said to be *contravariant*, meaning that they rotate in a sense that is opposite the rotation of the coordinate system. What, then, can you make of $\grave{}\mathbb{E}\mathbb{X}' = \mathbb{E}\mathbb{R}^{-1}\mathbb{R}\mathbb{X}$?

(3) What is the rotation rate of the Moon? Hint, make a sketch of the Earth-Moon orbital system and consider what we observe of the Moon from Earth. What would the Coriolis and centrifugal forces be on the Moon?

3 Inertial and noninertial descriptions of elementary motions

The object of this section is to evaluate the equations of motion (25) and (28) for several examples of elementary motions. The goal will be to understand how the accelerations and the inertial forces — gravity, centrifugal and Coriolis — depend upon the reference frame. Though the motions considered

A characterization of the forces on geophysical flows.

	central?	inertial?	Galilean invariant?	position only?
contact forces	yes	no	yes	no
grav. mass attraction	yes	yes	yes	yes
centrifugal	no	yes	yes	yes
Coriolis	no	yes	no	no

Table 1: Contact forces on fluid parcels include pressure gradients (normal to a surface) and frictional forces (mainly tangential to a surface). The centrifugal force noted here is that associated with Earth's rotation. 'position only' means dependent upon the parcel position but not the parcel velocity, for example. This table ignores electromagnetic forces since they are usually small.

here are truly elementary, nevertheless the analysis is slightly subtle in that the acceleration and inertial force terms may change identity, as if be fiat, when observing from one reference frame or another. To appreciate that there is more to this analysis than an arbitrary relabeling of terms, it will be very helpful for you to make a sketch of each case, starting with the observed acceleration.

3.1 Switching sides

3.1.1 One-dimensional, vertical motion with gravity

Consider a parcel of fixed mass M that is in contact with the ground and at rest. For the purpose here, a reference frame that is attached to the ground may be considered to be inertial. The vertical component of the equation of motion is then, in general,

$$\frac{d^2z}{dt^2} = F_z/M - g,$$

where the observed acceleration is written on the left hand side and the forces are listed on the right side. The forces acting on this parcel include the vertical component of a contact force, F_z, that acts over the surface of the parcel. To measure the contact force, the parcel could (in principal) be enclosed in a wrap-around strain gage that reads out the tangential and normal stresses acting on the surface of the parcel. In this case, the strain gauge will read a contact force that is upwards, $F_z > 0$. The other force acting on this parcel is due to gravity, $\mathbf{g}M$, an inertial force that acts throughout the body of the parcel (in this section there is no need to make a distinction between g and $g*$) (Table 1). To make an independent measure of $\mathbf{g}$, the direction may be observed as the direction of a stationary plumb line, and the magnitude of $\mathbf{g}$ may be inferred from the period of small oscillations.[12] For the conditions prescribed, a parcel at rest, the equation of motion for a ground-attached reference frame is just

$$\text{ground} - \text{attached frame}: \quad 0 = F_z - gM, \tag{30}$$

a static force balance between the upward contact force, F_z, and the downward force due to gravity, i.e., the parcel's weight (you were warned that this would be elementary).

Now suppose that the same parcel is observed from a reference frame that is in free-fall and accelerating downwards at the rate $-g$ with respect to the ground-attached frame.[21] When viewed from this free-falling reference frame, the parcel is observed to be accelerating upward at the rate

$$\frac{d^2z'}{dt^2} = g > 0.$$

In this free-falling frame, there is no gravitational force (imagine astronauts free-falling (floating) in space and attempting pendulum experiments ..... 'Houston, we have a pendulum problem') and so the only force recognized as acting on the parcel is the upward contact force, F_z, which is unchanged from the case before, i.e., the contact force is invariant. The equation of motion for the parcel observed from this free-falling reference frame is then, with the observed acceleration on the left,

$$\text{free} - \text{falling frame}: \quad \frac{d^2z'}{dt^2} = g = F_z/M. \tag{31}$$

Notice that in going from the inertial frame and Eqn. (30) to the free-falling frame and Eqn. (31), the term involving g has seemed to switch sides; gM is the force of gravity (an inertial force) in the ground-attached reference frame, Eqn. (30), and appears as an acceleration in the free-falling reference frame, Eqn. (31). Exactly this kind of switching sides will obtain when we consider rotating reference frames and the centrifugal and Coriolis forces.

3.1.2 Two-dimensional, circular motion; polar coordinates

Now consider the purely horizontal motion of a parcel so that gravity and the vertical component of the motion are ignored. It will be advantageous to utilize polar coordinates, which are reviewed here briefly. If you are familiar with polar coordinates, then jump ahead to Eqns. (36) and (37).

Presume that the motion is confined to a plane defined by the usual cartesian coordinates x_1 and x_2 and unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$. Thus the position of any point in the plane may be specified by (x_1, x_2) and vectors by their projection onto $\mathbf{e_1}$ and $\mathbf{e_2}$. Alternatively, a position may also be defined by polar coordinates, the distance from the origin, r, and an angle, λ between the radius vector and (arbitrarily)

[21] Gravitational mass attraction is an inertial force and a central force that has a very long range. Consider two gravitating bodies and a reference frame attached to one of them, say parcel one, which will then be observed to be at rest. If parcel two is then found to accelerate towards parcel one, the total momentum of the system (parcel one plus parcel two) will not be conserved, i.e., in effect, gravity would not be recognized as a central force. A reference frame attached to one of the parcels is thus noninertial. To define an inertial reference frame in the presence of mutually gravitating bodies we can use the center of mass of the system, and then align on the fixed stars. This amounts to putting the entire system into free-fall with respect to any larger scale (external to this system) gravitational mass attraction. For more on gravity and inertial reference frames see http://plato.stanford.edu/entries/spacetime-iframes/.

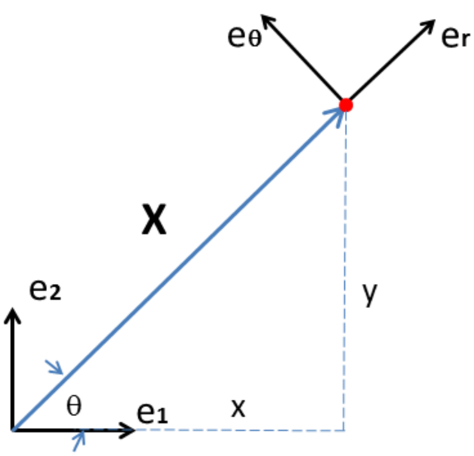

Figure 9: The unit vectors $\mathbf{e_1}, \mathbf{e_2}$ define a cartesian reference frame. The unit vectors for a polar coordinate system, $\mathbf{e_r}$ and $\mathbf{e}_\lambda$, are defined at the position of a given parcel (red dot) with $\mathbf{e_r}$ in the direction of the line segment from the origin to the parcel position. These polar unit vectors are in general time-dependent because the angle λ is time-dependent.

$\mathbf{e_1}$. The angle λ increases anti-clockwise (Fig. 9). To insure that the polar coordinates are unique we will require that

$$r \geq 0 \quad \text{and} \quad 0 \leq \lambda < 2\pi.$$

The position vector is then

$$\mathbf{X} = r\mathbf{e_r},$$

where the unit vector $\mathbf{e_r}$ has an origin at the parcel position and is in the direction of a line segment from the origin to the parcel position. The direction of $\mathbf{e_r}$ is thus λ. The unit vector $\mathbf{e}_\lambda$ is orthogonal and to the left of $\mathbf{e_r}$. The conversion from cartesian to polar coordinates is

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \lambda = \tan^{-1}(y/x),$$

and back,

$$x = r\cos\lambda \quad \text{and} \quad y = r\sin\lambda.$$

The polar system unit vectors are time-dependent because λ is in general time-dependent. To find out how they vary with $\lambda(t)$ we start by writing their expression in terms of the time-independent cartesian unit vectors as

$$\mathbf{e_r} = \cos\lambda\,\mathbf{e_1} + \sin\lambda\,\mathbf{e_2}, \quad \text{and,} \quad \mathbf{e}_\lambda = -\sin\lambda\,\mathbf{e_1} + \cos\lambda\,\mathbf{e_2}. \tag{32}$$

From Eqn (32) the time rate changes are

$$\frac{d\mathbf{e_r}}{dt} = \omega \mathbf{e}_\lambda \quad \text{and} \quad \frac{d\mathbf{e}_\lambda}{dt} = -\omega \mathbf{e_r}, \tag{33}$$

where $\omega = d\lambda/dt$. The d/dt operating on a polar unit vector induces a rotation of 90 degrees in the direction of ω, and stretching by the factor ω. With these results in hand the parcel velocity is readily computed as

$$\frac{d\mathbf{X}}{dt} = \frac{dr}{dt}\mathbf{e_r} + r\frac{d\mathbf{e_r}}{dt} = \frac{dr}{dt}\mathbf{e_r} + r\omega\mathbf{e}_\lambda \tag{34}$$

which shows the polar velocity components

$$U_r = \frac{dr}{dt} \quad \text{and} \quad U_\lambda = r\omega.$$

A second, similar differentiation yields the the acceleration,

$$\frac{d^2\mathbf{X}}{dt^2} \; = \; \left(\frac{d^2r}{dt^2} - r\omega^2\right)\mathbf{e_r} + \left(2\omega\frac{dr}{dt} + r\frac{d\omega}{dt}\right)\mathbf{e}_\lambda, \tag{35}$$

and the equation of motion sorted into radial and tangential components,

$$\frac{d^2r}{dt^2} - r\omega^2 \; = \; F_r/M, \tag{36}$$

$$2\omega\frac{dr}{dt} + r\frac{d\omega}{dt} \; = \; F_\lambda/M. \tag{37}$$

Notice that there are terms $-r\omega^2$ and $2\omega\frac{dr}{dt}$ on the left-hand side of (36) and (37) that have the form of centrifugal and Coriolis terms and are sometimes said to be such, e.g., the excellent applied mathematics text by Boas.[14] However, the angular velocity ω in these equations is that of the parcel position, not the rotation rate of the reference frame, and these terms are an essential part of the acceleration seen in the inertial reference frame. Casual labeling may be harmless in most contexts, but for the goals here it is an error: these equations have been written for an inertial reference frame where centrifugal and Coriolis forces ***do not arise***.

To see this last important point, consider uniform circular motion, $r = const$ and $\omega = d\lambda/dt = const$. The radial acceleration is then from Eqn (36), $-r\omega^2 < 0$, which is the centripetal (center-seeking) acceleration of uniform circular motion (d/dt operating twice on $\mathbf{e_r}$ times a constant r, or, Fig. 10). To say it a little more emphatically, $-r\omega^2$ is the entire acceleration observed in the case of uniform circular motion. This radial acceleration implies a centripetal radial force, $F_r \; = \; -r\omega^2 M < 0$, and the radial component balance Eqn (36) reduces to

$$\text{uniform circular motion, inertial frame :} \quad -r\omega^2 \; = \; F_r/M. \tag{38}$$

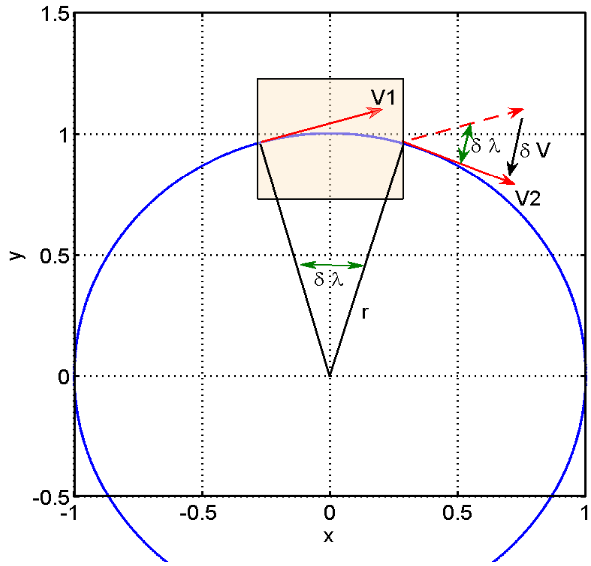

Figure 10: The velocity at two times along a circular trajectory (thin blue line) having radius r and frequency ω. The angular distance between the two times is $\delta\lambda = \delta t\omega$ and the velocity change is $\delta\mathbf{V} = \mathbf{V_2} - \mathbf{V_1}$. In the limit $\delta t \to 0$, the time rate change of velocity $\delta\mathbf{V}/\delta t$ is toward the center of curvature, i.e., a *centripetal acceleration*. If the motion is steady and circular, then $d\mathbf{V}/dt = -|\mathbf{V}|\omega\mathbf{e_r} = -r\omega^2\mathbf{e_r}$, where $\mathbf{e_r}$ is the radial unit vector. The centripetal acceleration may also be written $-(U_\lambda^2/r)\mathbf{e_r}$, where $U_\lambda = \omega r$ is the azimuthal speed. The shaded rectangle is a control volume used in a later problem to find the equivalent of centripetal acceleration in cartesian coordinates, $u\partial v/\partial x$, for the particular position shown here.

The azimuthal component Eqn. (37) vanishes term by term.

We can rewrite Eqns. (36) and (37) in a way that will help develop a physical interpretation by noting that $r\omega^2 = U_\lambda^2/r$ and that the angular momentum is $L = rU_\lambda M$ and thus

$$\boxed{\frac{d^2r}{dt^2} - \frac{U_\lambda^2}{r} = F_r/M} \tag{39}$$

and

$$\boxed{\frac{1}{M}\frac{dL}{dt} = rF_\lambda/M} \tag{40}$$

Two points: 1) The centripetal acceleration of Eqn.(39) depends quadratically upon the tangential velocity, U_λ, times the radius of curvature, $1/r$, and 2) The angular momentum of Eqn.(40) can change only if there is a torque, rF_λ, exerted upon the parcel, with the moment arm being the distance to the origin, r.

It is straightforward to find the corresponding rotating reference frame equation of motion. The origin of the rotating frame may be set at the origin of the fixed frame, and hence the radius is the same, $r' = r$. The unit vectors are identical since they are defined at the location of the parcel, $\mathbf{e_r'} = \mathbf{e_r}$ and $\mathbf{e_\lambda'} = \mathbf{e_\lambda}$. The components of the force F are also identical in the two frames, $F_r' = F_r$ and $F_\lambda' = F_\lambda$. Differences arise when the angular velocity ω of the parcel is decomposed into the presumed constant angular velocity of the rotating frame, Ω, and a relative angular velocity of the parcel when viewed from the rotating frame, i.e., ω', i.e.,

$$\omega = \Omega + \omega'.$$

An observer in the rotating reference frame will see the parcel motion associated with the relative angular velocity, but not the angular velocity associated with rotation of the reference frame, Ω. Substituting this into the inertial frame equations of motion above, and rearrangement to keep the observed acceleration on the left hand side while moving terms containing Ω to the right hand side yields the rather formidable-looking rotating frame equations of motion:

$$\frac{d^2r'}{dt^2} - r'\omega'^2 = r'\Omega^2 + 2\Omega\omega' r' + F'_r/M, \tag{41}$$

$$2\omega'\frac{dr'}{dt} + r'\frac{d\omega'}{dt} = -2\Omega\frac{dr'}{dt} + F'_\lambda/M. \tag{42}$$

We can write these using the rotating frame velocity components, $U'_r = dr'/dt$ and $U'_\lambda = \omega' r'$ and angular momentum, $L' = r'U'_\lambda M$, as

$$\boxed{\frac{d^2r'}{dt^2} - \frac{U'^2_\lambda}{r} = r'\Omega^2 + 2\Omega U'_\lambda + F'_r/M} \tag{43}$$

$$\boxed{\frac{1}{rM}\frac{dL'}{dt} = -2\Omega U'_r + F'_\lambda/M} \tag{44}$$

There is a genuine centrifugal force term $\propto \Omega^2 > 0$ in the radial component (43), and there are Coriolis force terms, $\propto 2\Omega$, on the right hand sides of both (43) and (44). This makes the third time that we have derived the centrifugal and Coriolis terms — in Cartesian coordinates, Eqn. (27), in vector form, Eqn. (28), and here in polar coordinates. It is worthwhile for you to verify the steps leading to these equations, as they are perhaps the most direct derivation of the Coriolis force and most easily show how the factor of 2 arises in the Coriolis term.

Now let's use these rotating polar coordinates to analyze the simple but important example of uniform circular motion whose inertial frame description was Eqn (38). Assume that the reference frame rotation rate is ω, the angular velocity of the parcel seen in the inertial frame. Thus $d\omega'/dt = 0$, and the parcel is stationary in the rotating frame; we might call this a co-rotating frame. It follows that $d(\ )/dt = U_\lambda = U_r = 0$ and so the azimuthal component Eqn. (42) vanishes term by term. All that is left of the radial component Eqn. (43) is

$$\mathrm{co-rotating,\ non-inertial\ frame:}\quad 0 = r'\omega^2 + F'_r/M \tag{45}$$

and recall that $r' = r$. The term $r'\omega^2 > 0$ is a centrifugal (center fleeing) force that is balanced by a centripetal contact force, F'_r, which is the same contact force observed in the inertial frame, $F'_r = F_r = -r'\omega^2 M$, consistent with Eqn. (45). Thus Eqns. (38) and (45) comprise another example of switching sides: an acceleration seen in an inertial frame — in this case a centripetal acceleration on the left side of Eqn. (38) — is transformed into an inertial force — a centrifugal force on the right side of (45) — when the same parcel is observed from a non-inertial, co-rotating reference frame.

Before moving on to an application below, it may be prudent to note that a rotating frame description is not always so adept as it may appear above. For example, assume that the parcel is at rest in the inertial frame, and that the horizontal component of the contact force vanishes. The inertial frame equation of motion in polar coordinates Eqns. (36) and (37) vanishes term by term; clearly, nothing is happening in an inertial frame. Now suppose that the same parcel is viewed from a steadily rotating reference frame, say rotating at a rate Ω, and at a distance r' from the origin. Viewed from this frame, the parcel will appear to be moving in a circle of radius $r' = constant$ and in a direction opposite the rotation of the reference frame. The parcel's rotation rate is $\omega' = -\Omega$, just as in Figure (7). With these conditions the tangential component equation of motion vanishes term by term ($\mathbf{F} = 0$), but three of the radial component terms are nonzero,

$$-r'\omega'^2 \;=\; r'\Omega^2 + 2\Omega\omega' r', \tag{46}$$

and indicate an interesting balance between the centripetal acceleration, $-r'\omega'^2$ (the observed acceleration is listed on the left hand side), and the sum of the centrifugal and Coriolis inertial forces (the right hand side, divided by M, and note that $\omega' = -\Omega$). Interesting perhaps, but also a bit alarming; a parcel that is at rest and doing nothing in an inertial frame has acquired a rather complex momentum balance when observed from a rotating reference frame. It is tempting to deem the Coriolis and centrifugal terms that arise in this example to be 'virtual', or 'fictitious, correction' forces to acknowledge this discomfort.[6] But labeling terms this way wouldn't add anything useful, and it might obscure the fundamental issue — accelerations and inertial forces are relative to a reference frame. This applies just as well to centrifugal and Coriolis forces as it does to gravitational mass attraction.

3.2 To get a feel for the Coriolis force

A centrifugal force is often encountered in daily life. For example, a runner having a speed $V = 5$ m s^{-1} and making a moderately sharp turn, radius $R = 15$ m, will easily feel the centrifugal force, $(V^2/R)M \approx 0.15gM$ or 15% of her weight. It is unlikely that a runner would think of this centrifugal force as virtual or fictitious.

The Coriolis force associated with Earth's rotation is by comparison very, very small, only about $2\Omega VM \approx 10^{-4}gM$ for the same runner, and not noticeable. To experience the Coriolis force in the same direct way that we can experience (feel) the centrifugal force, requires a platform having a rotation rate that exceeds Earth's rotation rate by a factor of about 10^4. A typical merry-go-round has a rotation rate $\Omega = 2\pi/12$ rad s^{-1} = 0.5 rad s^{-1} that is ideal.

In what follows, the heretofore nondescript 'parcel' is presumed to be a sentient being i.e., a rider (is that you?). Motion is observed by a separate observer who can be either external to the merry-go-round, and thus is an inertial reference frame, or riding on the merry-go-round, and thus in a non-inertial reference frame. To calculate the forces on this rider, assume a mass, $M = 75$ kg, the standard airline passenger before the era of super-sized meals and passengers.

3.2.1 At rest in the rotating frame

To start, let's presume that we are standing quietly at the radius $r = 6$ m of a merry-go-round that it is rotating at a steady rate, $\Omega = 0.5$ rad s^{-1}. How does the description of our motion depend upon the reference frame, and what are the forces involved?

Viewed from an inertial frame outside of the merry-go-round, the radial component balance Eqn. (37) is, with $\omega = \Omega$ and $dr/dt = d\omega/dt = F_\theta = 0$

$$-r\Omega^2 M = F_r, \tag{47}$$

in which a centripetal acceleration ($\times M$) is balanced by an inward-directed contact force, $F_r = -r\Omega^2 M = -112$ N, equivalent to the weight of a mass $F_r/g = 11.5$ kg (equivalent to about 28 lbs) and which is quite noticeable. This contact force is exerted by the merry-go-round on us, say through a hand rail.

Viewed from a rotating, non-inertial reference frame, i.e., from the merry-go-round, there is no acceleration, and the radial force balance is Eqn. (45) with $r' = r$,

$$0 = r'\Omega^2 M + F_r'. \tag{48}$$

The physical conditions are unchanged and thus the contact force exerted by the merry-go-round is exactly as before, $F_r' = F_r = -112$ N. As noted by examples in Sec. 3.1, the acceleration seen in the inertial frame Eqn. (47) has become an inertial force, $r'\Omega^2 M$, a centrifugal force, in the rotating frame, Eqn. (48). Within the rotating frame, the centrifugal force is quite vivid; it appears that we are being pushed outwards, or centrifugally, by a force that is distributed throughout our body. To maintain a fixed position, this centrifugal force must be opposed by a centripetal contact force, F_r', exerted by the hand rail.

3.2.2 A cautious walk in the rotating frame

Most merry-go-rounds have signs posted that caution riders to remain in their seats after the ride begins. This is a good and prudent rule, of course. But if the goal is to get a feel for the Coriolis force, then we may decide to go for a very cautious walk on the merry-go-round.

Azimuthal relative velocity. Let's assume that we begin to walk azimuthally, so that $r = 6$ m and constant. A reasonable walking pace under the circumstance is about $U_w = 1.5$ m s^{-1}, which corresponds to a relative rotation rate $\omega' = 0.25$ rad s^{-1}, and recall that $\Omega = 0.5$ rad s^{-1}. If the direction is in the direction of the merry-go-round rotation, then $\omega = \Omega + \omega' = 0.75$ rad s^{-1}. From the inertial frame, Eqn. (37), the centripetal force required to maintain $r = constant$ when moving at this greater angular velocity is

$$F_r \;=\; -r\omega^2 M \;=\; -r(\Omega + \omega')^2 M \;=\; -253 \text{ N},$$

which is roughly twice the centripetal force experienced when stationary. If we then reverse direction and walk at the same speed against the direction of rotation of the merry-go-round, $\omega' = -0.25$ rad s^{-1}, then F_r is reduced to about -28 N. This pronounced variation of F_r with ω_w is a straightforward consequence of the quadratic dependence of centripetal acceleration upon azimuthal velocity.

When our motion is viewed and analyzed from within the rotating frame of the merry-go-round, we distinguish between the rotation rate of the merry-go-round, Ω, and the relative rotation rate, ω', due to the presumed relative motion. The radial component of the rotating frame equation of motion (41) reduces to

$$-r'\omega' M \;=\; r'\Omega^2 M + 2\Omega\omega' r' M + F'_r. \tag{49}$$

The term on the left is a centripetal force, the first term on the right is the centrifugal force due to the merry-go-round's rotation, and the second term on the right, $2\Omega\omega' M$, is a Coriolis force. For these conditions, the Coriolis force is substantial, $2r'\Omega\omega' M \pm 112$ N, with the sign determined by the direction of motion relative to Ω. If $\Omega > 0$ and $\omega' > 0$, i.e., walking in the anti-clockwise direction of the merry-go-round rotation, then the radial Coriolis force is positive and to the right of the relative velocity.

Radial relative velocity. Now consider going for a very cautious walk in the radial direction so that $\omega' = 0$ and the rotation rate remains constant at $\Omega = 0.5$ rad sec^{-1}. Presume a modest radial speed $dr'/dt = 1$ m s^{-1} and positive. In practice, even this slow pace is difficult to maintain for more than a few steps, but that will suffice.

Viewed from an inertial frame, the azimuthal component of the equation of motion, Eqn. (37), reduces to

$$2\Omega \frac{dr}{dt} M \;=\; F_\lambda, \tag{50}$$

where $F_\lambda \approx 75$ N (or about 16 pounds) for the given data. The sense of F_λ is positive, or anti-clockwise. The left hand side of (50) has the form of a Coriolis force, but don't jump to that conclusion; this is an inertial frame description, so there is no Coriolis force. The inertial frame description may be cast as a balance of angular momentum, $L = r^2\Omega M$ and hence $L \propto r^2$ since Ω and M are constant in this case. When $dr/dt > 0$ the angular momentum is necessarily increasing and must be provided by a positive torque, i.e., $rF_\lambda > 0$. If instead the radial motion was inward, so that $dr/dt < 0$, then the angular momentum would be decreasing, and F_λ would be negative. Be sure that the sense (the direction) of F_λ is clear before going on to consider this motion from the rotating frame.

From within the rotating frame, and given that the motion is constrained to be radial only, the azimuthal component of the equation of motion reduces to a force balance,

$$0 = -2\Omega\frac{dr'}{dt}M + F'_\lambda, \tag{51}$$

where $-2\Omega\frac{dr'}{dt}M$ is now a Coriolis force and $F'_\lambda = F_\lambda$ is the same contact force as before. For example, if the radial motion is outward, $\frac{dr'}{dt} \geq 0$, then the azimuthal Coriolis force is clockwise, $-2\Omega\frac{dr'}{dt}M \leq 0$, which is to the right of and normal to the radial velocity. If the radial motion is inward, then the Coriolis force is reversed, and again is to the right of the radial motion.

Be careful! If you have the opportunity to do this experiment you will learn with the first few steps whether the Coriolis force is better described as a real force, or as a fictitious correction force. Be sure to ask permission of the merry-go-round operator before you start walking around, and exercise genuine caution. The Coriolis force is an inertial force and so is distributed throughout your body, unlike the contact force which acts only where you are in contact with the merry-go-round, i.e., through a secure grip on a hand rail. The radial Coriolis force associated with azimuthal motion feels much like an increase or slackening of the centrifugal force, and is not especially difficult to compensate. Be warned, though, that the azimuthal Coriolis force associated with radial motion is startling, even presuming that you are the complete master of this analysis. (If you do not have access to a merry-go-round or if you judge that this experiment is unwise, then see Stommel and Moore[10] for alternate ways to accomplish some of the same things.)

3.3 An elementary projectile problem

A very simple projectile problem analyzed from inertial and rotating reference frames can reveal some other aspects of rotating frame dynamics. Assume that a projectile is launched with velocity $(U_0, V_0, W_0) = (0, 1, 1)$ and from the origin $(x, y) = (0, 0)$. The only force presumed to act on the projectile after launch is the downward force of gravity, $-gM\mathbf{e_3}$, which is the same in either reference frame.

From the inertial frame. The equations of motion and initial conditions in Cartesian components are linear and uncoupled;

$$\frac{d^2x}{dt^2} = 0; \quad x(0) = 0, \quad \frac{dx}{dt} = 0, \tag{52}$$

$$\frac{d^2y}{dt^2} = 0; \quad y(0) = 0, \quad \frac{dy}{dt} = V_0,$$

$$\frac{d^2z}{dt^2} = -g; \quad z(0) = 0, \quad \frac{dz}{dt} = W_0,$$

where M has been divided out. These are readily integrated to yield the solution for the time interval when the parcel is in flight;

$$\begin{aligned} 0 \;<\; &t \;<\; \frac{2W_0}{g}, \\ x(t) &= 0, \\ y(t) &= y_0 + tV_0, \\ z(t) &= t(W_0 - \frac{1}{2}gt). \end{aligned} \tag{53}$$

The horizontal displacement (x, y) is sketched as the blue curve of Fig. (11), a linear displacement toward positive y until to $t = 2\pi$ when the parcel returns to the ground. The vertical displacement (not shown) is a simple up and down, with constant downward acceleration.

From the rotating frame. How would this same motion look when viewed from a rotating reference frame? Set the origin of the rotating frame coincident with the origin of the inertial frame and assume that the rotation is about the $\mathbf{e_3}$ (vertical, or z) axis at a constant Ω. The equations of motion, with $\mathbf{F} = 0$, are (Eqn. (28),

$$\begin{aligned} \frac{d^2x'}{dt^2} &= -2\Omega v' + x'\Omega^2; \quad x'(0) = 0, \quad \frac{dx'}{dt} = 0, \\ \frac{d^2y'}{dt^2} &= 2\Omega u' + y'\Omega^2; \quad y'(0) = 0, \quad \frac{dy'}{dt} = V_0, \\ \frac{d^2z'}{dt^2} &= -g; \quad z'(0) = 0, \quad \frac{dz'}{dt} = W_0. \end{aligned} \tag{54}$$

The z component equation is unchanged since the rotation axis was aligned with z. This is quite general; motion that is parallel to the rotation vector $\mathbf{\Omega}$ is unchanged by rotation.

The horizontal components of the rotating frame equations (54) include Coriolis and centrifugal force terms that are coupled but linear, and so we can integrate this system almost as easily as the inertial frame counterpart,

$$\begin{aligned} 0 \;<\; &t \;<\; \frac{2W_0}{g}, \\ x'(t) &= -tV_0 \sin(-\Omega t), \\ y'(t) &= tV_0 \cos(-\Omega t), \\ z'(t) &= t(W_0 - \frac{1}{2}gt) \end{aligned} \tag{55}$$

and find the black trajectory of Fig. (11). The rotating frame trajectory rotates clockwise, or opposite the reference frame rotation, and makes a complete rotation in time = $2\pi/\Omega$. When it intersects the inertial frame trajectory we are reminded that the distance from the origin (radius) is not changed by

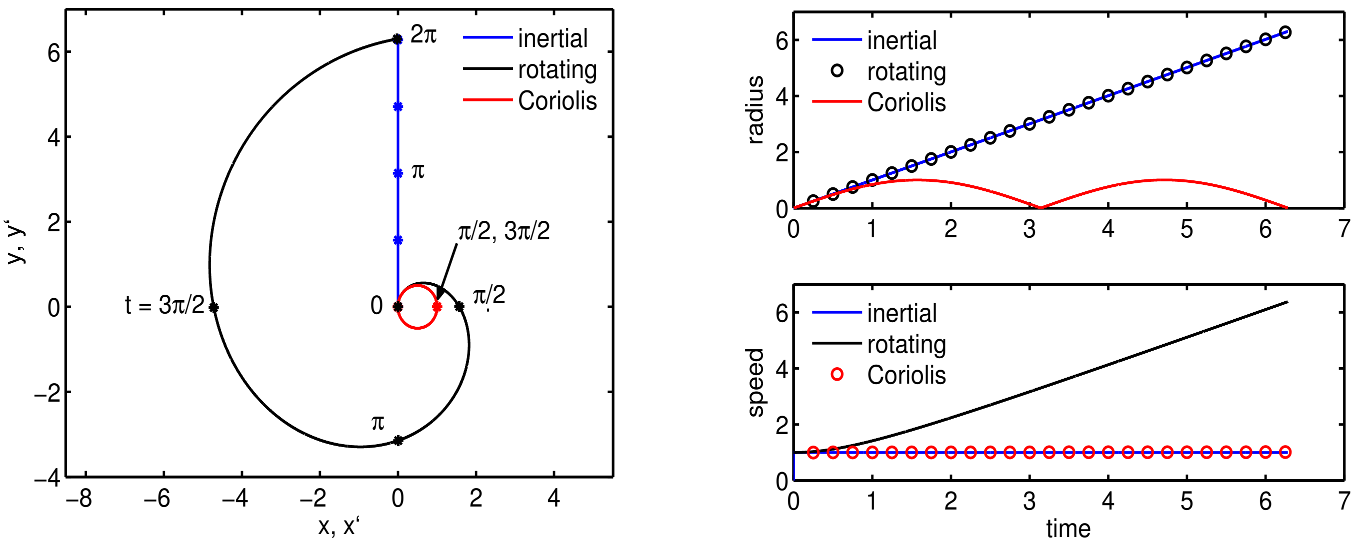

Figure 11: **(left)** The horizontal trace of a parcel launched from $(0,0)$ in the positive y-direction as seen from an inertial reference frame (blue line) and as seen from a rotating frame (black line). The elapsed time is marked at intervals of $\pi/2$. The rotating frame was turning anti-clockwise with respect to the inertial frame, and hence the black trajectory turns clockwise with time at the same rate, though in the opposite direction. For comparison, the red trajectory was computed with the Coriolis force only (no centrifugal force; the motion for this will come in Sec. 4). This an inertial motion that makes two complete clockwise orbits in time = 2π, twice the rate of the reference frame rotation. Videos from comparable laboratory experiments may be viewed at https://spinlab.ess.ucla.edu/ **(right) (upper)** The radius (distance from origin) and **(lower)** speed for the three trajectories. Notice that 1) the inertial and rotating trajectories have equal radius, while the radius of the Coriolis trajectory is much less, and 2) the inertial and Coriolis trajectories show the same, constant speed, while the rotating trajectory has a greater and increasing speed on account of the centrifugal force.

rotation, $r' = r$, since the coordinate systems have coincident origins. We know the inertial frame radius, $r = tV_0$, and hence we also know

$$r' = tV_0. \tag{56}$$

The angular position of the parcel in the inertial frame is $\lambda = \pi/2$ and constant, since the motion is purely radial. The relative rotation rate of the parcel seen from the rotating frame is $\omega' = -\Omega$, and thus

$$\lambda' = \pi/2 - \Omega t, \tag{57}$$

which, together with Eqn. (56), gives the polar coordinates of the parcel position. Both the radius and the angle increase linearly in time, and the rotating frame trajectory is Archimedes spiral.

When viewed from the rotating frame, the projectile is observed to be deflected to the right which we can attribute to the Coriolis force. Notice that the horizontal speed and thus the kinetic energy increase with time (Fig. 11, right). This cannot be attributed to the Coriolis force, which is always perpendicular to the velocity and so can do no work. The rate of increase of rotating frame kinetic energy (per unit mass) is

$$\begin{aligned} \frac{d\mathbf{V'^2}/2}{dt} &= \frac{d(V_0^2 + r'^2\Omega^2)/2}{dt} \\ &= \frac{dr'}{dt} r'\Omega^2 \end{aligned} \tag{58}$$

which may be interpreted as the work done by the centrifugal force, $r'\Omega^2$, on the radial velocity, dr'/dt. In fact, if the projectile had not returned to the ground, its speed (observed from the rotating reference frame) would have increased without limit so long as the radius increased. It was noted earlier that a rotating, non-inertial reference frame does not, in general, conserve global momentum, and now it is apparent that energy is also not conserved. Nevertheless, we can provide a complete and internally consistent accounting of the energy changes seen in a rotating frame, as in Eqn. (58).

3.4 Appendix to Section 3; Spherical Coordinates

Spherical coordinates can be very useful when motion is more or less confined to the surface of a sphere, e.g., the Earth, approximately. We will have occasion to use spherical coordinates later on, and so will go ahead and write them down here while polar coordinates are still fresh and pleasing(?). The method for finding the equation of motion in spherical coordinates is exactly as above, though with the need for an additional angle. There are many varieties of spherical coordinates; we will use 'geographic' spherical coordinates in which the longitude (also called azimuth) is measured by λ, where $0 \le \lambda \le 2\pi$, increasing anti-clockwise (Figure 12), the latitude (also called elevation) is measured by ϕ, where $-\pi/2 \le \phi \le \pi/2$, increasing anti-clockwise and with a zero at the equator and distance from the origin by r. The conversion from spherical to cartesian coordinates is:

$$x = r\cos^2\phi, \quad y = r\cos\phi\sin\lambda, \quad z = r\sin\phi,$$

and the reverse,

$$\lambda = \tan^{-1}(y/x), \quad \phi = \sin^{-1}(z/\sqrt{x^2+y^2+z^2}), \quad r = \sqrt{x^2+y^2+z^2}.$$

The spherical system unit vectors (Fig. 13) written in Cartesian coordinates are:

$$\mathbf{e}_\lambda = -\sin\lambda\,\mathbf{e_1} + \cos\lambda\,\mathbf{e_2}, \tag{59}$$

$$\mathbf{e}_\phi = -\cos\lambda\sin\phi\,\mathbf{e_1} - \sin\lambda\sin\phi\,\mathbf{e_2} + \cos\phi\,\mathbf{e_3}, \tag{60}$$

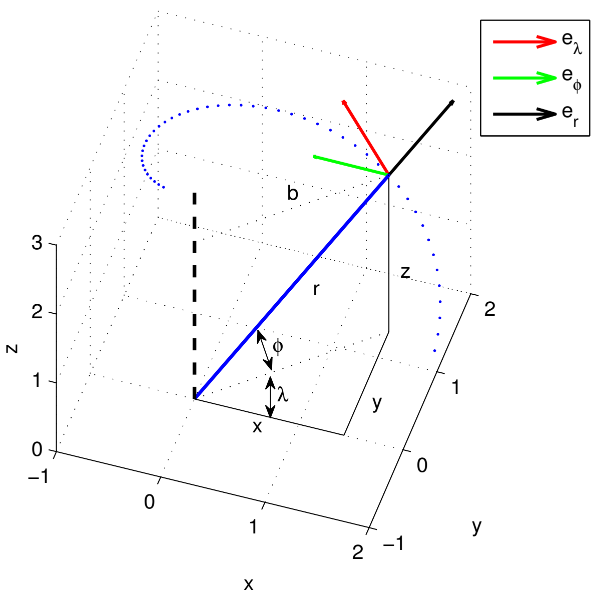

Figure 12: A three-dimensional trajectory (blue dots) with, for one point only, the radius (blue line) and the spherical unit vectors (red, green and black). The spherical system coordinates are: (1) the longitude, λ, the angle between the projection of the radius onto the (x,y) plane and the x axis; (2) the latitude, ϕ, the angle between the radius and the (x,y) plane, and (3) the radius magnitude, r. The black dashed center line will be the axis of rotation (pole) when reference frame rotation is considered. The perpendicular distance from the pole to a given point, labeled b, is then very important. The (x,y,z) components of this point are also shown.

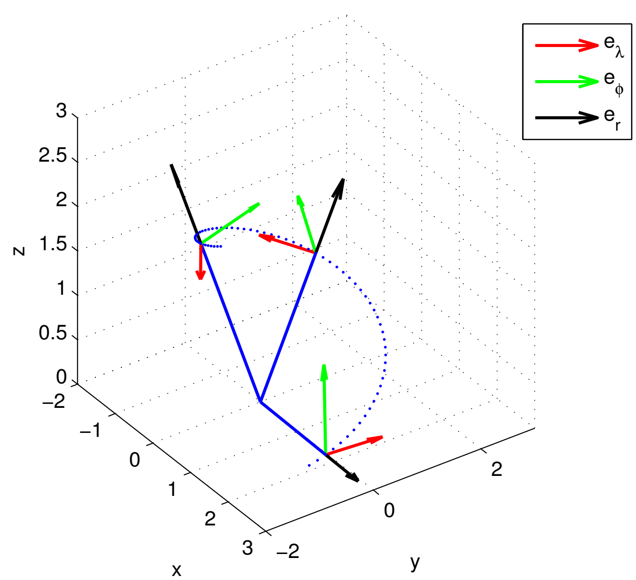

Figure 13: A three-dimensional trajectory (blue dots) that begins at lower center and then turns counterclockwise as it moves toward positive z. Radials from the origin $(0,0,0)$ are the blue lines shown at three points along the trajectory. The spherical system unit vectors are in red, green and black at the same points. Notice that these change direction along the trajectory and that the black vector, $\mathbf{e_r}$, remains aligned with the radial.

$$\mathbf{e_r} = \cos\lambda\cos\phi\mathbf{e_1} + \sin\lambda\cos\phi\mathbf{e_2} + \sin\phi\mathbf{e_3}. \tag{61}$$

Notice that when $\phi = 0$ these reduce to the polar coordinate system.

The position and velocity vectors are

$$\mathbf{X} = r\mathbf{e_r}, \tag{62}$$

and

$$\frac{d\mathbf{X}}{dt} = \frac{dr}{dt}\mathbf{e_r} + r\frac{d\phi}{dt}\mathbf{e}_\phi + r\cos\phi\frac{d\lambda}{dt}\mathbf{e}_\lambda, \tag{63}$$

where the velocity components are

$$U_\lambda = r\cos\phi\frac{d\lambda}{dt}, \quad U_\phi = r\frac{d\phi}{dt}, \quad \text{and} \quad U_r = \frac{dr}{dt}.$$

These bear obvious similarity to the now familiar polar velocity, though with the moment arm $r\cos\phi = b$ in the longitudinal component in place of r only. Continuing on to find the acceleration and then the equation of motion in λ, ϕ and r components:

$$(2\frac{dr}{dt}\frac{d\lambda}{dt}\cos\phi \;-\; 2r\frac{d\phi}{dt}\frac{d\lambda}{dt}\sin\phi \;+\; r\cos\phi\frac{d^2\lambda}{dt^2})M \;=\; F_\lambda, \tag{64}$$

$$(2\frac{dr}{dt}\frac{d\phi}{dt} + r\frac{d^2\phi}{dt^2} \;+\; r\cos\phi(\frac{d\lambda}{dt})^2\sin\phi)M \;=\; F_\phi, \tag{65}$$

$$(\frac{d^2r}{dt^2} \;-\; r\cos\phi(\frac{d\lambda}{dt})^2\cos\phi \;-\; r(\frac{d\phi}{dt})^2)M \;=\; F_r. \tag{66}$$

These may be rewritten in a more compact and revealing form be defining angular momentum for the λ and ϕ coordinates:

$$L_\lambda = (r\cos\phi)^2\frac{d\lambda}{dt}M, \;\text{and}\; L_\phi = r^2\frac{d\phi}{dt}M,$$

and centripetal accelerations ($\times M$) for the λ and ϕ components:

$$C_\lambda \;=\; -r\cos\phi(\frac{d\lambda}{dt})^2 M \;\text{and}\; C_\phi \;=\; -r(\frac{d\phi}{dt})^2 M.$$

In these variables the equations of motion are:

$$\frac{1}{r\cos\phi}\frac{dL_\lambda}{dt} \;=\; F_\lambda, \tag{67}$$

$$\frac{1}{r}\frac{dL_\phi}{dt} \;-\; C_\lambda\sin\phi \;=\; F_\phi, \tag{68}$$

$$\frac{d^2r}{dt^2}M \; + \; C_\lambda \cos\phi + C_\phi \; = \; F_r. \tag{69}$$

The rotating frame equations follow from the substitution

$$\frac{d\lambda}{dt} = \Omega + \frac{d\lambda'}{dt},$$

and rearranging the way we did for the polar coordinates:

$$(2\frac{dr'}{dt}\frac{d\lambda'}{dt}\cos\phi' \; + \; r\cos\phi' \; \frac{d^2\lambda'}{dt^2} \; - \; 2r'\frac{d\phi'}{dt}\frac{d\lambda'}{dt}\sin\phi')M \; = \; -2\Omega\frac{dr'}{dt}\cos\phi' \; + \; 2\Omega r'\frac{d\phi'}{dt}\sin\phi' \; + \; F'_\lambda, \tag{70}$$

$$(2\frac{dr'}{dt}\frac{d\phi'}{dt} \; + \; r'\cos\phi' \; (\frac{d\lambda'}{dt})^2\sin\phi' \; + \; r'\frac{d^2\phi'}{dt^2})M \; = \; -\, r'\cos\phi'\Omega^2\sin\phi' \; - \; 2\Omega \; r\cos\phi' \; \frac{d\lambda'}{dt}\sin\phi' \; + \; F'_\phi, \tag{71}$$

$$(\frac{d^2r'}{dt^2} \; - \; r'\cos\phi' \; (\frac{d\lambda'}{dt})^2\cos\phi' \; - \; r'(\frac{d\phi'}{dt})^2)M \; = \; r'\cos\phi' \; \Omega^2\cos\phi' \; + \; 2\Omega r'\cos\phi'\frac{d\lambda'}{dt}\cos\phi' \; + \; F'_r. \tag{72}$$

We can tidy these up a little by rewriting in terms of $L'_\lambda = (r'\cos\phi')^2\frac{d\lambda'}{dt}M$, etc.,

$$\frac{1}{r'\cos\phi'}\frac{dL'_\lambda}{dt} \; = \; -2\Omega U'_r\cos\phi M \; + \; 2\Omega\sin\phi U'_\phi M + F'_\lambda, \tag{73}$$

$$\frac{1}{r'}\frac{dL'_\phi}{dt} \; - \; C'_\lambda\sin\phi \; = \; -r'\cos\phi'\sin\phi'\Omega^2 M - 2\Omega\sin\phi' U_\lambda M \; + \; F'_\phi, \tag{74}$$

$$\frac{d^2r'}{dt^2}M \; + \; C_\lambda\cos\phi \; + \; C_\phi \; = \; r'\cos^2\phi\Omega^2 M \; + \; 2\Omega\cos\phi U_\lambda M \; + \; F'_r. \tag{75}$$

3.5 Problems

(1) Given that we know the inertial frame trajectory, Eqns. (53), show that the rotating frame trajectory may be computed by applying a time-dependent rotation operation via Eqn. (12), $\mathbb{X}' = \mathbb{R}\mathbb{X}$ and with $\theta = \Omega t$, with the result Eqns. (55). So for this case — a two-dimensional planar domain and rotation vector normal to the plane, we can either integrate the rotating frame equations of motion, or, rotate the inertial frame solution. This will not be the case when we finally get to an Earth-attached, rotating frame.

(2) In the example of Sec. 3.2, walking on a merry-go-round, it was suggested that you would be able to feel the Coriolis force directly. Imagine that you are riding along on the projectile of Sec 3.3 (don't try this one at home) — would you be able to feel the Coriolis force?

(3) The centrifugal force produces a radial acceleration on every object on the merry-go-round and thus contributes to the direction and magnitude of the time-independent acceleration field observed in a rotating frame, an important point returned to in Section 4.1. For example, show that a plumb line makes an angle to the vertical of $\arctan(r'\Omega^2/g)$, where the vertical direction and g are in the absence of rotation.

(4) Your human pinball experiments on the merry-go-round (Sec. 3.2) were illuminating, and something you were eager to share with your father, Gustav-Gaspard, and younger brother, Gustav-Gaspard Jr. Your father is old school — he doesn't believe in ghosts or magic or virtual forces — and engaged in a heated debate with GG Jr. regards just what happened on the merry-go-round: was it a Coriolis force that pushed every moving object sideways — this is GG Jr.'s assertion — or was it simply a torque required to change angular momentum, as your father insists?

(5) The spherical system equations (64) - (66) are fairly forbidding upon a first or second encounter and you certainly can not expect to spot errors without considerable experience (and in fact, errors (probably typographical) are common in the literature). How can we check that the equations listed here are correct? One straightforward if slightly tedious way to check the equations is to define a 3-dimensional trajectory in the spherical system, $\mathbf{X}(\lambda, \phi, r)$, convert to the familiar $\mathbf{X}(x, y, z)$ coordinates, and compute the velocity, acceleration, Coriolis force, etc. in the cartesian coordinates. Then compute the same quantities using the spherical system, and compare the results directly. The script sphere_check.m (Sec. 6.3) does just this. You can use that script to define a new trajectory (your choice), and check the results for yourself.

4 A reference frame attached to the rotating Earth

4.1 Cancellation of the centrifugal force by Earth's (slightly chubby) figure

If Earth was a perfect, homogeneous sphere (it is not), the gravitational mass attraction at the surface, $\mathbf{g}*$, would be directed towards the center (Fig. 14). Because the Earth is rotating, every parcel on the surface is also subject to a centrifugal force

$$\boldsymbol{C} = -\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{X} \tag{76}$$

of magnitude $\Omega^2 R_E \cos\phi$, where R_E is Earth's nominal radius, and ϕ is the latitude. The vector $\boldsymbol{C}$ is perpendicular to the Earth's rotation axis, and is directed away from the axis. This centrifugal force has a component parallel to the surface, a shear force, Eqn. (71),

$$C_\phi = \Omega^2 R_E \cos\phi \sin\phi, \tag{77}$$

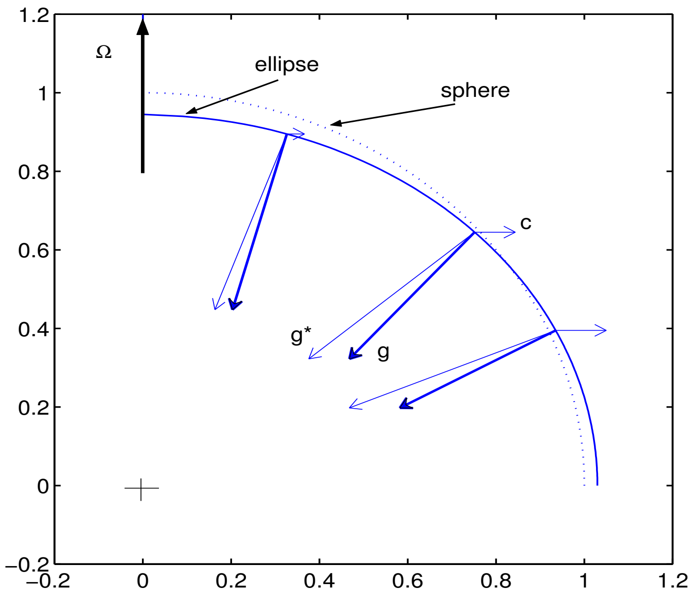

Figure 14: Cross-section through a hemisphere of a gravitating and rotating planet. The gravitational acceleration due to mass attraction is shown as the vector $\mathbf{g}*$ that points to the center of a spherical, homogeneous planet. The centrifugal acceleration, $\mathbf{C}$, associated with the planet's rotation is directed normal to and away from the rotation axis, and is to scale for the planet Saturn. The combined gravitational and centrifugal acceleration is shown as the heavier vector, $\mathbf{g}$. This vector is in the direction of a plumb line, and defines vertical. A surface that is normal to $\mathbf{g}$ similarly defines a level surface, and has the approximate shape of an oblate spheroid (the solid curve). The ellipse of this diagram has a flatness $F = 0.1$ that approximates Saturn; for Earth, $F = 0.0033$.

that is directed towards the equator (except at the equator where the 3-d vector centrifugal force is vertical).[22] C_ϕ is very small compared to $g*$, $C_\phi/g* \approx 0.002$ at most, but it has been present since the Earth's formation. A fluid can not sustain a shear without deforming, and over geological time this holds as well for the Earth's interior and crust. Thus it is highly plausible that the Earth long ago settled into a rotational-gravitational equilibrium configuration in which this C_ϕ is exactly balanced by a component of the gravitational (mass) attraction that is parallel to the displaced surface and poleward, i.e., centripetal.

To make what turns out to be a pretty rough estimate of the displaced surface, η_Ω, assume that the gravitational mass attraction remains that of a sphere and that the meridional slope $(1/R_E)\partial\eta_\Omega/\partial\phi$ times the gravitational mass attraction is in balance with the tangential component of the centrifugal force (Eqn. 71),

$$\frac{g*}{R_E}\frac{\partial\eta}{\partial\phi} = \Omega^2 R_E \cos\phi \sin\phi. \qquad (78)$$

[22]Ancient critics of the rotating Earth hypothesis argued that loose objects on a spinning sphere should fly off into space, which clearly does not happen. Even so, given the persistent centrifugal force due to Earth's rotation it is plausible that we might drift towards the equator. Alfred Wegner proposed just this as the engine of Earth's moving continents, which may have helped delay the acceptance of his otherwise remarkable inference that continents move (see D. McKenzie, 'Seafloor magnetism and drifting continents', in *A Century of Nature*, 131-137. Ed. by L. Garwin and T. Lincoln, The Univ. of Chicago Press, Chicago, Il, 2003.).

This may then be integrated with latitude to yield the equilibrium displacement,

$$\begin{aligned}\eta_\Omega(\phi) &= \int_0^\phi \frac{\Omega^2 R_E^2}{g*}\cos\phi\sin\phi d\phi \\ &= \frac{\Omega^2 R_E^2}{2g*}\sin\phi^2 + constant.\end{aligned} \tag{79}$$

When this displacement is added onto a sphere the result is an oblate (flattened) spheroid, Fig. (14), which is consistent qualitatively (but not quantitatively) with the observed shape of the Earth.[23] A convenient measure of flattening is $J = (R_{eqt} - R_{pol})/R_{eqt}$, where the subscripts refer to the equatorial and polar radius. Earth's flatness is $J = 0.0033$, which seems quite small, but is nevertheless highly significant in ways beyond that considered here.[24,25]

Closely related is the notion of 'vertical'. A direct measurement of vertical can be made by means of a plumb line; the plumb line of a plumb bob that is at rest is parallel to the local gravity and defines the direction vertical. Following the discussion above we know that the time-independent, acceleration field of the Earth is made up of two contributions, the first and by far the largest being mass attraction, $\mathbf{g}*$, and the second being the centrifugal acceleration, $\mathbf{C}$, associated with the Earth's rotation, Fig. (14). Just as on the merry-go-round, this centrifugal acceleration adds with the gravitational mass attraction to give the net acceleration, called 'gravity', $\mathbf{g} = \mathbf{g}* + \mathbf{C}$, a time-independent vector (field) whose direction is observable with a stationary plumb line and whose magnitude may be inferred by observing the period of small amplitude oscillations when the plumb bob is displaced and released, i.e., a pendulum. A surface that is normal to the gravitational acceleration vector is said to be a level surface in as much as the acceleration component parallel to that surface is zero. A resting fluid can sustain a

[23] The idea behind Eqn. (79) is a good start, but doesn't go far enough. The pole-to-equator rise given by Eqn. (79) is about 11 km whereas precise observations show that Earth's equatorial radius, $R_{eqt} = 6378.2$, is greater than the polar radius, $R_{pol} = 6356.7$ km, by about 21.5 km. The calculation (79) is a first approximation insofar as it ignores the gravitational mass attraction of the equatorial bulge, which is toward the equator and thus also has a centrifugal component. Thus still more mass must be displaced equatorward in order to increase η_Ω enough to reach a rotational-gravitational equilibrium. The net result is roughly a factor of two greater amplitude than Eqn. (79) indicates.

A comprehensive source for physical data on the planets is C. F. Yoder, 'Astrometric and geodetic data on Earth and the solar system,' Ch. 1, pp 1–32, of *A Handbook of Physical Constants: Global Earth Physics (Vol. 1).* American Geophysical Union (1995).

[24] To note just two: 1) Earth's ellipsoidal shape must be accounted for in highly precise, long range navigation systems (GPS), while shorter range or less precise systems can approximate the Earth as spherical. 2) Because the Earth is not perfectly spherical, the gravitational tug of the Sun, Moon and planets can exert a torque on the Earth and thereby perturb Earth's rotation vector. The resulting, slow variations lead to dramatic changes in Earth's climate.[19]

[25] The flatness of a rotating planet is given roughly by $J \approx \Omega^2 R/g$. If the gravitational acceleration at the surface, g, is written in terms of the planet's mean radius, R, and density, ρ, then $J = \Omega^2/(\frac{4}{3}\pi G\rho)$, where $G = 6.67\times10^{-11}$ m^3 kg^{-1} s^{-2} is the universal gravitational constant. The rotation rate and the density vary a good deal among the planets, and consequently so does J. The gas giant, Saturn, has a rotation rate a little more than twice that of Earth and a very low mean density, about one eighth of Earth's. The result is that Saturn's flatness is large enough, $J \approx 0.10$, that it can be discerned through a good backyard telescope or in a figure drawn to scale, Fig. (14).

normal stress, i.e., pressure, but not a shear stress. Thus a level surface can also be defined by observing the free surface of a water body that is at rest in the rotating frame.[26] In sum, the measurements of vertical and level that we can readily make necessarily lump together gravitational mass attraction with the centrifugal force due to Earth's rotation.

4.2 The equation of motion for an Earth-attached reference frame

Now we are going to apply the inference made above, that there exists a tangential component of gravitational mass attraction that exactly balances the centrifugal force due to Earth's rotation and that we define vertical in terms of the measurements that we can readily make; thus

$$\mathbf{g} = \mathbf{g}* + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{X}. \tag{80}$$

The equations of motion for a rotating/gravitating planet are then,

$$\boxed{\frac{d\mathbf{V}'}{dt} = -2\mathbf{\Omega}\times\mathbf{V}' + \mathbf{F}'/M + \mathbf{g}} \tag{81}$$

which is Eqn. (2), at last! The happy result is that the rotating frame equation of motion applied in an Earth-attached reference frame does not include the centrifugal force associated with Earth's rotation (and neither do we tend to roll towards the equator).

Vector notation is handy for many derivations and for visualization, but when it comes time to do a calculation we will need the component-wise equations, usually Earth-attached, rectangular coordinates. The east unit vector is $\mathbf{e_x}$, north is $\mathbf{e_y}$, and the horizontal is defined by a tangent plane to Earth's surface. The vertical direction, $\mathbf{e_z}$, is thus radial with respect to the (approximately) spherical Earth. The rotation vector $\mathbf{\Omega}$ makes an angle ϕ with respect to the local horizontal x', y' plane, where ϕ is the latitude of the coordinate system and thus

$$\mathbf{\Omega} = \Omega\cos\phi\,\mathbf{e_y} + \Omega\sin\phi\,\mathbf{e_z}.$$

If $\mathbf{V}' = u'\mathbf{e_x} + v'\mathbf{e_y} + w'\mathbf{e_z}$, then the full, three-dimensional Coriolis force is

$$-2\mathbf{\Omega}\times\mathbf{V}' = -(2\Omega\cos\phi\, w' - 2\Omega\sin\phi\, v')\mathbf{e_x} - 2\Omega\sin\phi\, u'\mathbf{e_y} + 2\Omega\cos\phi\, u'\mathbf{e_z}. \tag{82}$$

4.3 Coriolis force on motions in a thin, spherical shell

Application to geophysical flows is made somewhat simpler by noting that large scale geophysical flows are very flat in the sense that the horizontal component of wind and current are very much larger

[26]The ocean and atmosphere are *not* at rest, and the observed displacements of constant pressure surfaces, e.g., the sea surface and 500 hPa surface, are invaluable, indirect measures of that motion that may be inferred via geostrophy, Sec 5.

than the vertical component, $u' \propto v' \gg w'$, in part because the oceans and the atmosphere are quite thin, having a depth to width ratio of about 0.001. As well, the ocean and atmosphere are stably stratified in the vertical, which greatly inhibits the vertical component of motion. For large scale (in the horizontal) flows, the Coriolis term multiplying w' in the x component of Eqn. (82) is thus very much smaller than the terms multiplied by u' or v' and as an excellent approximation the w' terms may be ignored; very often they are ignored with no mention made. The Coriolis term that appears in the vertical component is usually much, much smaller than the gravitational acceleration, and it too is often dropped without mention. The result is the thin fluid approximation of the Coriolis force in which only the horizontal Coriolis force acting on horizontal motions is retained,

$$\boxed{-2\mathbf{\Omega}\times\mathbf{V}' \approx -\mathbf{f}\times\mathbf{V}' = fv'\mathbf{e_x} - fu'\mathbf{e_y}} \tag{83}$$

where $\mathbf{f} = f\mathbf{e_z}$, and f is the very important Coriolis parameter,

$$\boxed{f = 2\Omega \sin\phi} \tag{84}$$

and ϕ is the latitude. Notice that f varies with the sine of the latitude, having a zero at the equator and maxima at the poles; $f < 0$ in the southern hemisphere. The horizontal, component-wise momentum equations written for the thin fluid form of the Coriolis force are:

$$\boxed{\begin{aligned} \frac{du}{dt} &= fv - g\frac{\partial\eta}{\partial x} \\ \frac{dv}{dt} &= -fu - g\frac{\partial\eta}{\partial y} \end{aligned}} \tag{85}$$

where the force associated with a tilted constant pressure surface is included on the right.[27]

For problems that involve parcel displacements, L, that are very small compared to the radius of the Earth, R_E, a simplification of f itself is often appropriate. The Coriolis parameter may be expanded in a Taylor series about a central latitude ϕ_0 where the north coordinate $y = y_0$,

$$f(y) = f(y_0) + (y - y_0)\frac{df}{dy}\,|_{y_0} + HOT. \tag{86}$$

If the second term involving the first derivative $df/dy = 2\Omega\cos\phi/R_E$, often written as $df/dy = \beta$, is demonstrably much smaller than the first term, which follows if $L \ll R_E$, then the second and higher terms may be dropped to leave

$$f = f(y_0), \tag{87}$$

and thus f is taken as constant. This is called the f-plane approximation. While the f-plane approximation is very useful in a number of contexts, there is an entire class of low frequency motions

[27]This system has what will in general be three unknowns: u, v and η. For now we will take η as known, i.e., the height of the sea floor in Sec. 5. In a genuine fluid model, which we will come to in Part 2, η will be connected to the flow by the conservation of mass.

known as Rossby waves that go missing and which are of great importance for the real atmosphere and ocean. We will come to this phenomena in Part 3 by keeping the second order term of (86), and thus represent $f(y)$ by

$$f(y) = f(y_0) + \beta(y - y_0), \tag{88}$$

often called a β-plane approximation.

4.4 One last look at the inertial frame equations

We have noted that the rotating frame equation of motion has some inherent awkwardness, viz., the loss of Galilean invariance and global momentum conservation that accompany the Coriolis force. Why, then, do we insist upon using the rotating frame equations for nearly all of our analyses of geophysical flow?

The reasons are several, any one of which would be compelling, but beginning with the fact that the definition and implementation of an inertial frame (outside of the Earth) is simply not a viable option; whatever conceptual clarity might be gained by avoiding the Coriolis force would be more than offset by difficulty with observation. Consider just one aspect of this: the inertial frame velocity,

$$\mathbf{V} = \mathbf{V}_{\Omega} + \mathbf{V}', \tag{89}$$

is dominated by the planetary velocity due to the solid-body rotation $V_{\Omega} = \Omega R_E \cos\phi$, where R_E is earth's nominal radius, 6365 km, and thus $V_{\Omega} \approx 450$ m s^{-1} near the equator. A significant wind speed at mid-level of the atmosphere is $V' \approx 30$ m s^{-1} (the westerlies of Fig. 2) and a fast ocean current is $V' \approx 1$ m s^{-1} (the western boundary current of Fig. 1). An inertial frame description must account for $\mathbf{V}_{\Omega}$ and the associated, very large centripetal force, and yet our interest is almost always the comparatively small relative motion of the atmosphere and ocean, $\mathbf{V}'$, since it is the relative motion that transports heat and mass over the Earth. In that important regard, the planetary velocity $\mathbf{V}_{\Omega}$ is invisible to us Earth-bound observers, no matter how large it is. To say it a little differently — it is the relative velocity that we measure when observe from Earth's surface, and it is the relative velocity that we seek to know for almost every practical purpose. The Coriolis force follows.

The reservations regards practical use of the inertial frame equations apply mainly to observations. Given that we presume to know exactly the centripetal force required to balance the planetary velocity, then in principle a calculation based upon the inertial frame equations should be quite doable. To illustrate this, and before we turn away completely and finally from the inertial frame equations, it is instructive to analyze some very simple motions using the inertial frame, spherical equations of motion (Sec. 3.4). This is partly repetitious with the discussions of Secs. 3.2 and 3.3. It will differ importantly insofar as the setting will be a rotating planet, Fig. (15). As before we will analyze the motion of a single parcel, but just for the sake of visualization it is helpful to imagine that this parcel is part of a torus of fluid, Fig. (15), that encircles a rotating planet. It is presumed that the torus will move in a completely coherent way, so that the motion of any one parcel will be the same as all other parcels.

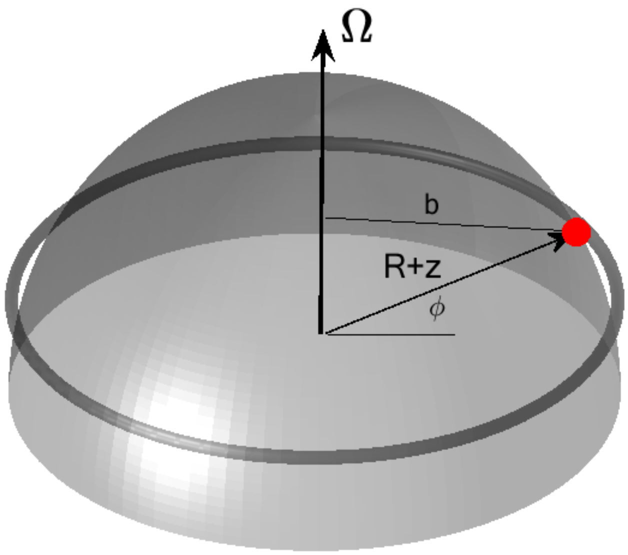

Figure 15: A schematic showing a rotating planet and an encircling tube of fluid whose motion includes a rotation at the same rate as the underlying planet, i.e., a planetary velocity. A single parcel whose motion is identical with the tube at large is denoted by the red dot. This analysis will use spherical coordinates, Sec. 3.4. Here the radial distance from the center will be written $r = R + z$, where $z \ll R$. Not shown here is the longitude (or azimuth) coordinate, λ, which is the same as in the spherical system.

The only two forces acknowledged here will be gravity, certainly in the vertical component, and also the horizontal gravitational acceleration associated with Earth's oblate figure (equatorial bulge). The basic state velocity is that due to planetary rotation, $U_\lambda = (R+z)\cos\phi\,\Omega$ and which is azimuthal, or eastward. With these in mind, the inertial frame, spherical system equations of motion are:

$$\frac{1}{(R+z)\cos\phi}\frac{dL_\lambda}{dt} = 0, \tag{90}$$

$$\frac{1}{(R+z)}\frac{dL_\phi}{dt} - C_\lambda \sin\phi = -(R+z)\cos\phi\,\Omega^2 \sin\phi, \tag{91}$$

$$\frac{d^2 z}{dt^2} + C_\lambda \cos\phi + C_\phi = -g. \tag{92}$$

Northward motion: For the first example, presume that the parcel stays in contact with a frictionless planet so that $r = R$ and constant. The longitudinal angular velocity may be written

$$\frac{d\lambda}{dt} = \Omega + \frac{d\lambda'}{dt}$$

and the tangential or λ-component angular momentum is

$$L_\lambda = (R\cos\phi)^2(\Omega + \frac{d\lambda'}{dt}).$$

The λ component equation of motion (Eqn. 67) is just conservation of this angular momentum,

$$\frac{dL_\lambda}{dt} \;=\; 0,$$

and hence

$$-2R\sin\phi\frac{d\phi}{dt}(\Omega+\frac{d\lambda'}{dt}) \;+\; R\cos\phi\frac{d^2\lambda'}{dt^2}=0.$$

Factoring out the Ω term and moving it to the right gives,

$$\begin{aligned}\frac{1}{R\cos\phi}\frac{dL'_\lambda}{dt} \;&=\; 2\Omega\sin\phi R\cos\phi\frac{d\phi}{dt}\\ &= fU_\phi,\end{aligned} \tag{93}$$

which is the corresponding rotating frame equation of motion. But the inertial frame interpretation is via angular momentum conservation: as the parcel (or torus) moves northward, $d\phi/dt \geq 0$ say, it acquires some positive or eastward L'_λ specifically because the perpendicular to the rotation axis, b, shrinks northward. The initial angular momentum includes a very large (dominant) contribution from the Earth's rotation, i.e., $\Omega \gg d\lambda'/dt$. You may very well feel that the inertial frame derivation is based upon much more familiar, 'physical' principles than is the rotating frame version. However, the inference of an eastward relative acceleration associated with northward motion is exactly the same from both perspectives, as it should be.

Eastward motion: The inertial frame ϕ component equation of motion includes a significant contribution from the planetary velocity and centripetal force; if in steady state, assuming that $U'_\phi = 0$ for the moment, then Eqn. (68) is just,

$$\begin{aligned}-C_\lambda\sin\phi \;&=\; F_\phi\\ &=\; -R\cos\phi\Omega^2\sin\phi,\end{aligned} \tag{94}$$

a steady balance between the ϕ component of the centripetal acceleration and the centripetal force associated with the equatorial bulging noted in Sec. 4.1. Now suppose that there is comparatively small relative λ component velocity so that

$$\frac{d\lambda}{dt} = \Omega \;+\; \frac{d\lambda'}{dt}$$

and substitute into the ϕ component equation of motion, Eqn. (68),

$$\frac{1}{r}\frac{dL_\lambda}{dt} \;+\; R\cos\phi(\Omega^2+2\Omega\frac{d\lambda'}{dt} \;+\; (\frac{d\lambda'}{dt})^2)\sin\phi \;=\; -R\cos\phi\Omega^2\sin\phi.$$

Rearranging and moving the 2Ω term to the right side yields

$$\begin{aligned}\frac{1}{R}\frac{dL'_\lambda}{dt} \;-\; C'_\lambda\sin\phi \;&=\; 2\Omega\sin\phi R\cos\phi\frac{d\phi'}{dt}\\ &=\; fU'_\phi.\end{aligned} \tag{95}$$

Again, this is the rotating frame equivalent. A significant difference with the example of northward motion noted above is that the induced acceleration comes from an out-of-balance centripetal force and acceleration. As in the previous case, the basic state is that due to Earth's rotation and resulting gravitational-rotational equilibrium.

Vertical motion: Imagine a parcel that is released from (relative) rest at a height h and allowed to free fall. The initially purely vertical motion has no appreciable consequences for either the ϕ or r component equations of motion, but it does appear in the λ component equation multiplied by Ω (Eqn. 67). The vertical acceleration, ignoring air resistance is just

$$\frac{d^2z}{dt^2} = -g, \tag{96}$$

with g the presumed constant acceleration of gravity, 9.8 m sec^{-2}. Integrating once to find the vertical velocity, $w = -gt$, and once more for the displacement, $z = h - 1/2gt^2$. The time of flight is just $T = \sqrt{2h/g}$.

The only force acting on the parcel is the radial force of gravity, and hence the parcel will conserve angular momentum. The λ-component angular momentum conservation, Eqn. (67), is then just

$$\frac{d}{dt}\left((R + z)^2 \cos^2\phi(\Omega + \frac{d\lambda'}{dt})\right) = 0. \tag{97}$$

Expanding the derivative and cancelling terms gives

$$2\frac{dz}{dt}\cos\phi(\Omega + \frac{d\lambda'}{dt}) + (R+z)\cos\phi\frac{d^2\lambda'}{dt^2} = 0$$

Rewriting in terms of $u' = R\cos\phi\frac{d\lambda'}{dt}$ and $w' = \frac{dz}{dt}$ and assuming that z is O(100), then $z \ll R$, and the relative speed u' is very, very small compared to the planetary rotation speed, $u' \ll \Omega R$. To an excellent approximation Eqn. (97) is

$$\frac{du'}{dt} \approx -2\Omega\cos\phi w'. \tag{98}$$

Thus, as the parcel falls, $w' \leq 0$, and moves into orbit closer to the rotation axis, it is accelerated to the east at a rate that is proportional to twice the rotation rate Ω and the cosine of the latitude. Viewed from an inertial reference frame, this eastward acceleration is the expected consequence of angular momentum conservation, where the angular momentum is that due to planetary rotation. The complementary rotating frame description of this motion is that eastward acceleration is due to the Coriolis force acting upon the relative vertical velocity.

4.5 Problems

(1) The rather formal notions of vertical and level raised in Sec. 4.2 turned out to have considerable practical importance beginning on a sweltering August afternoon when the University Housing

Office notified your dear younger brother, GG Jr., that because of an unexpectedly heavy influx of freshmen, his old and comfortable dorm room was not going to be available. As a consolation, they offered him the use of the merry-go-round (the one in Section 3.3, and still running) at the local, failed amusement park just gobbled up by the University. He shares your enthusiasm for rotation and accepts, eagerly. The centrifugal force, amusing at first, was soon a huge annoyance. GG suffered from recurring nightmares of sliding out of bed and over a cliff. Something had to be done, so you decide to build up the floor so that the tilt of the floor, combined with gravitational acceleration, would be just sufficient to balance the centrifugal force, as in Eqn. (78). What shape $\eta(r)$ is required, and how much does the outside edge ($r = 6$ m, $\Omega = 0.5$ rad s^{-1}) have to be built up? How could you verify success? Given that GG's bed is 2 m long and flat, what is the axial traction, or tidal force? Is the calibration of a bathroom scale effected? Guests are always impressed with GG's rotating dorm room, and to make sure they have the full experience, he sends them to the refrigerator for another cold drink. Describe what happens next using Eqn. (81). Is their experience route-dependent?

(2) In most of what follows the Coriolis force will be represented by the thin fluid approximation Eqn. (83) that accounts only for the horizontal Coriolis force due to horizontal velocity. This horizontal component of the Coriolis force is proportional to the Coriolis parameter, f, and thus vanishes along the equator. This is such an important and striking result that it can be easy to forget the three-dimensional Coriolis force. Given an eastward and then a northward relative velocity, make a sketch that shows the 3-d Coriolis force at several latitudes including the pole and the equator (and recall Fig. 8), and resolve into (local) horizontal and vertical components. The vertical component of the Coriolis force is negligible for most geophysical flow phenomena, but it is of considerable importance for gravity mapping, where it is called the Eotvos effect (see https://en.wikipedia.org/wiki/Eotvos_effect).

(3) The vertical component of the Coriolis force has a small but observable effect on the motion of free projectiles. Consider the Coriolis deflection of a long-range rifle shot, say range is $L = 1$ km and with a trajectory that is nearly flat. Assuming mid-latitude; estimate the horizontal deflection and show that it is given by $\delta y \approx \delta t\, f\, L/2$, where δt is the time of flight, 2 sec. Show that the vertical deflection is similar and given approximately by $\delta z \approx \delta t\, f_{vert}\, L\, cos(\psi)/2$, where $f_{vert} = 2\Omega cos\phi$ and ψ is the direction of the projectile motion with respect to east (north is $\pi/2$). How do these deflections vary with latitude, ϕ, and with the direction, ψ?

(4) The effect of Earth's rotation on the motion of a simple (one bob) pendulum, called a Foucault pendulum in this context, is treated in detail in many physics texts, e.g. Marion[6], and need not be repeated here. Foucault pendulums are commonly displayed in science museums, though seldom to large crowds (see *The Prism and the Pendulum* by R. P. Crease for a more enthusiastic appraisal). It is, however, easy and fun (!) to make and observe your own Foucault pendulum, nothing more than a simple pendulum having two readily engineered properties. First, the e-folding time of the motion due to frictional dissipation must be long enough that the precession will become apparent before the motion dies away, 20 min will suffice at mid-latitudes. This can be achieved using a dense, smooth and symmetric bob having a weight of about half a kilogram or more, and suspended on a fine, smooth monofilament line. It is helpful if line is several meters or more in length. Second, the pendulum should not interact appreciably with its mounting. This is harder to evaluate, but generally requires a very rigid support, and a bearing that can not exert appreciable torque, for example a fish hook bearing on a very hard steel surface. The precession is easily masked by any initial motion you might inadvertently impose, but after several careful trials

you will very likely begin to see the Earth rotate under your pendulum. Can you infer your latitude from the observations? The rotation effect is proportional to the rotation rate, and so you should plan to bring a simple and rugged pocket pendulum (a rock on a string will do) on your merry-go-round ride (Section 3.2). How do your observations (even if qualitative) compare with your solution for a Foucault pendulum? (Hint - consider the initial condition.)

(5) In Sec. 4.4 we used the spherical system equations of motion as the starting point for an analysis of some simple motions. The spherical system is an acquired taste, which I am betting you have not acquired. There is a simpler way to come to several of the results of that section that you may find more appealing. When observed from an inertial reference frame, the eastward velocity of the parcel is $U = \Omega b + u'$ where $b = (R+z)\cos\phi$ is the perpendicular distance to the rotation axis. The parcel has angular momentum associated with this eastward velocity, $L = Ub$. For what follows here we can think of the angular momentum as a scalar. Presume that the parcel motion is unforced, aside from gravity. Show that conservation of this angular momentum under changing ϕ and z leads immediately to the inference of a Coriolis force. In fact, you can think of this as your (partial) derivation of the Coriolis force (partial since it does not include the planetary centripetal acceleration, the second case considered in Sec. 4.4).

(6) The Coriolis force on most everyday (every weekend) motions is not noticeable. For example, golfers worry about about all kinds of environmental and fluid dynamical issues, but never the Coriolis force. A good golf drive may go 250 m, and stay in the air for about 5 sec. Can you explain quantitatively why this motion is not much affected by the Coriolis force? Contrast this with the very large effects upon winds and ocean currents.

5 A dense parcel released onto a rotating slope with friction

The second goal of this essay is to begin to understand the consequences of rotation for the atmosphere and ocean. As already noted in Sec. 1, the consequences of rotation are profound and wide ranging and will likely be an enduring topic of your study of the atmosphere and ocean. In this section we can take a rewarding and nearly painless first step toward understanding the consequences of rotation by analyzing the motion of a dense parcel that is released onto a rotating, sloping sea floor. This simple problem serves to illustrates two fundamental modes of the rotating momentum equations — inertial motion and geostrophic motion — that will recur in much more comprehensive models and in the real atmosphere and ocean.

The sea floor is presumed to be at a depth $z = -b(y)$ that increases uniformly in the y direction as $db/dy = \alpha$, a small positive constant, O(10^{-2}). The fixed buoyancy of the parcel is $g' = -g\frac{\delta\rho}{\rho_o}$, where $\delta\rho$ is the density anomaly of the parcel with respect to its surroundings, say 0.5 kg m^{-3}, and ρ_o is a nominal sea water density, 1030 kg m^{-3}. (Notice that a prime superscript is used here to denote buoyancy, or reduced gravity. The prime previously used to indicate rotating frame velocity will be omitted, with rotating frame understood.) The component of the buoyancy parallel to the sea floor, $g'\alpha$, thus provides a constant force (per unit mass, understood from here on) in the y direction. Absent rotation, the parcel would accelerate down hill in the positive y direction. With rotation, the parcel velocity V will be significantly altered in a time T_r in the scale analysis sense (rough magnitude only) that

$$f\,V\,T_r \approx V$$

and hence

$$\boxed{T_r = \frac{1}{f}} \tag{99}$$

The important time scale $1/f$ is dubbed the rotation time. For a mid-latitude, $1/f \approx 4$ hours. In other words, for rotation to be of first order importance, the motion has to persist for several hours or more. Thus the flight path of a golf ball (requiring about 3 seconds) is very little affected by Earth's rotation when compared to other curves and swerves, and as we knew from a more detailed calculation in Sec. 3. Given that the motion will be nearly horizontal and that we seek the simplest model, rotation will be modeled by the thin fluid form of the Coriolis force, and the Coriolis parameter f will be taken as constant (the f-plane approximation).

Since the parcel is imagined to be in contact with the bottom, it is plausible that the momentum balance should include bottom friction. Here the bottom friction will be represented by the simplest linear (or Rayleigh) law in which the friction is presumed to be proportional to and antiparallel to the velocity difference between the parcel velocity and the bottom, i.e., bottom friction $= -r(\mathbf{V} - \mathbf{V_{bot}})$. The ocean bottom is at rest in the rotating frame and hence $\mathbf{V_{bot}} = 0$ and omitted from here on. From observations of ocean density currents (looking ahead to Fig. 16), a reasonable order of magnitude of the friction coefficient is $r = \mathrm{O}(10^{-5})\ \mathrm{s}^{-1}$.[28]

The equations of motion for the parcel including rotation and this simplified bottom friction are

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{du}{dt} &= \quad f v - r u, \\
\frac{d^2y}{dt^2} &= \frac{dv}{dt} &= -f u - r v + g'\alpha,
\end{aligned} \tag{100}$$

with vector equivalent,

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k}\times\mathbf{V} - r\mathbf{V} + g'\nabla b. \tag{101}$$

Initial conditions on the position and the velocity components are

$$x(0) = X_0, \quad y(0) = Y_o \quad \text{and} \quad u(0) = U_o, \quad v(0) = 0. \tag{102}$$

[28] This use of a linear friction law is purely expedient. A linear friction law is most appropriate in a viscous, laminar boundary layer that is in contact with a no-slip boundary. In that case $\tau = \mu \frac{\partial U}{\partial z}$ within the laminar boundary layer, where μ is the viscosity of the fluid. However, the laminar boundary layer above a rough ocean bottom is very thin, $\mathrm{O}(10^{-3})$ m, and above this the flow will in general be turbulent. If the velocity that is used to estimate or compute friction is measured or computed within the much thicker turbulent boundary layer, as it almost always has to be, then the friction law is likely better approximated as independent of the viscosity and quadratic in the velocity, i.e., $\tau = \rho C_d U^2$, where C_d is the drag coefficient. Typically, $C_d = 1 - 3 \times 10^{-3}$, but depending upon bottom roughness, mean speed, and more.

In most of what follows we will presume $U_o = 0$. Integrating once gives the solution for the velocity components,

$$u(t) = \frac{g'\alpha}{r^2+f^2}\left[f - \exp\left(-rt\right)\left(f\cos\left(-ft\right) - r\sin\left(-ft\right)\right)\right], \tag{103}$$
$$v(t) = \frac{g'\alpha}{r^2+f^2}\left[r - \exp\left(-rt\right)\left(f\sin\left(-ft\right) + r\cos\left(-ft\right)\right)\right].$$

If the position (trajectory) is required, it may be computed by integrating the velocity

$$x(t) = X_o + \int_0^t u dt \quad \text{and} \quad y(t) = Y_o + \int_0^t v dt,$$

and if the depth is required,

$$z(t) = Z_o - \alpha y(t).$$

5.1 The nondimensional equations; Ekman number

The solution above is simple by the standards of fluid dynamics, but it does contain three parameters along with the time, and so has a fairly large parameter space. We will consider a couple of specific cases motivated by observations, but our primary intent is to develop some understanding of the effects of rotation and friction over the entire family of solutions. How can the solution be displayed to this end?

A very widely applicable approach is to rewrite the governing equations and (or) the solution using nondimensional variables. This will serve to reduce the number of parameters to the fewest possible while retaining everything that was present in the dimensional equations. Lets start with the x-component momentum equation, and hence u will be the single dependent variable and it has units length and time, l and t. Time is the sole independent variabil, and obviousl its units are em t. There are three independent parameters in the problem; 1) the buoyancy and bottom slope, $g'\alpha$, which always occur in this combination and so count as one parameter, an acceleration with units l and t and dimensions $l\ \ t^{-2}$. 2) the Coriolis parameter, f, an inverse time, dimensions t^{-1}, and 3) the bottom friction coefficient, r, also an inverse time scale, t^{-1}. Thus there are five variables or parameters having two fundamental units. Because we anticipate that rotation will be of great importance in the parameter space of most interest, the inverse Coriolis parameter or rotation time, will be used to scale time, i.e., $t_* = tf$. You can think of this as measuring the time in units of the rotation time. A velocity (speed) scale is then estimated as the product of this time scale and the acceleration $g'\alpha$,

$$\boxed{U_{geo} = \frac{g'\alpha}{f}} \tag{104}$$

the very important geostrophic speed. Measuring the velocity in these units then gives the nondimensional velocity, $u_* = u/U_{geo}$ and similarly for the v component. Rewriting the governing equations in terms of these nondimensional variables

$$\frac{du_*}{dt_*} = v_* - Eu_*, \tag{105}$$

$$\frac{dv_*}{dt_*} = -u_* - Ev_* + 1, \tag{106}$$

where E is the Ekman number,

$$\boxed{E = \frac{r}{f}} \tag{107}$$

the nondimensional ratio of the friction parameter to the Coriolis parameter. There are other forms of the Ekman number that follow from different forms of friction parameterization. They all have in common that small E indicates small friction compared to rotation. The initial condition is presumed to be a state of rest, $u_*(0) = 0,\ v_*(0) = 0$ and the solution of these equations is

$$\begin{aligned} u_*(t_*) &= \frac{1}{1+E^2}\left[1 - \exp(-Et_*)(\cos(-t_*) - E\sin(-t_*))\right], \\ v_*(t_*) &= \frac{1}{1+E^2}\left[E - \exp(-Et_*)(\sin(-t_*) + E\cos(-t_*))\right], \end{aligned} \tag{108}$$

and for completeness,

$$t_* = tf, \quad U_{geo} = \frac{g'\alpha}{f}, \quad u_* = \frac{u}{U_{geo}} \quad \text{and} \quad v_* = \frac{v}{U_{geo}}.$$

The geostrophic scale U_{geo} serves only to scale the velocity amplitude, and thus the parameter space of this problem has been reduced to a single independent, nondimensional variable, t_*, and one nondimensional parameter E.[29]

The solution Eqn. (108) can be written as the sum of a time-dependent part, termed an *i*nertial motion (or just as often, inertial 'oscillation') that is here damped by friction,

$$\begin{bmatrix} u_* \\ v_* \end{bmatrix}_i = -\frac{\exp(-Et_*)}{1+E^2}\begin{bmatrix} \cos(-t_*) - E\sin(-t_*) \\ \sin(-t_*) + E\cos(-t_*) \end{bmatrix}, \tag{109}$$

and a time-independent motion that is the single parcel equivalent of *g*eostrophic motion

$$\begin{bmatrix} u_* \\ v_* \end{bmatrix}_g = \frac{1}{1+E^2}\begin{bmatrix} 1 \\ E \end{bmatrix}, \tag{110}$$

[29] On first encounter, this kind of dimensional analysis is likely to seem abstract, arbitrary and abstruse, i.e., far more harmful than helpful. The method and the benefits of dimensional analysis will become clearer with experience, mainly, and an attempt to help that along is 'Dimensional analysis of models and data sets', by J. Price, *Am. J. Phys.*, **71**(5), 437–447 (2003) and available online in an expanded version from the author's web page (Sec. 1.3).

also damped by friction. Since the IC was taken to be a state of rest, $U_o = 0$, the dimensional amplitude is directly proportional to the geostrophic velocity scale, U_{geo}. Since the model and solution are linear, the form of the solution does not change with U_{geo}.

Our discussion of the solution will generally refer to the velocity, Eqns. (109) and (110), which are simple algebraically. However, the solution is considerably easier to visualize in the form of the parcel trajectory, computed by integrating the velocity in time (Fig. 16, left, and see the embedded animation or better, run the script partslope.m to make your own).

Immediately after the parcel is released from rest it accelerates down the slope. The Coriolis force acts to deflect the moving parcel to the right, and by about $t = 1/f$, or $t_* = 1$, the parcel has been turned by 1 radian, or about 50°, with respect to the buoyancy force. The time required for the Coriolis force to have an appreciable effect on a moving object is thus $1/f$, the very important rotation time scale noted previously. The Coriolis force continues to turn the parcel to the right, and by about $t_* = \pi$ the parcel velocity is directed up the slope. If $E = 0$ and there is no friction, the parcel will climb back to its starting depth at $t_* = 2\pi$ (or $t = 2\pi/f$) where it will stop momentarily, before repeating the cycle. In the meantime it will have moved a significant distance along the slope. When friction is present, $0 < E < 1$, the parcel still makes at least a few oscillations up and down slope, but with decreasing amplitude with time, and will gradually slide down the slope. The clockwise-turning looping motion is associated with near-inertial motion Eqn. (109) and the steadily growing displacement along the slope, in the positive x direction mainly, is associated with quasi-geostrophic motion, Eqn. (110). In fact, these specific trajectories may be viewed as nothing but the superposition of inertial and geostrophic motion, damped by friction when $E > 0$.

5.2 (Near-) Inertial motion

In Eqn. (109) we already have a solution for inertial motion, but it is helpful to take a step back to the dimensional form of the momentum equations, (4.3) and point out the subset that supports pure inertial motion:

$$\boxed{\begin{aligned} \frac{du}{dt} &= fv \\ \frac{dv}{dt} &= -fu \end{aligned}} \tag{111}$$

The Coriolis force can not generate a velocity, and so to get things started we have to posit an initial velocity, $u(t = 0) = U_o$ and $v(t = 0) = 0$. The solution is pure inertial motion,

$$u = U_o \cos(-ft), \text{ and } v = U_o \sin(-ft), \tag{112}$$

which is the free mode of the f-plane momentum equations, i.e., when the Coriolis force is left on it its own. The speed of a pure inertial motion is constant in time, and the velocity vector rotates at a steady

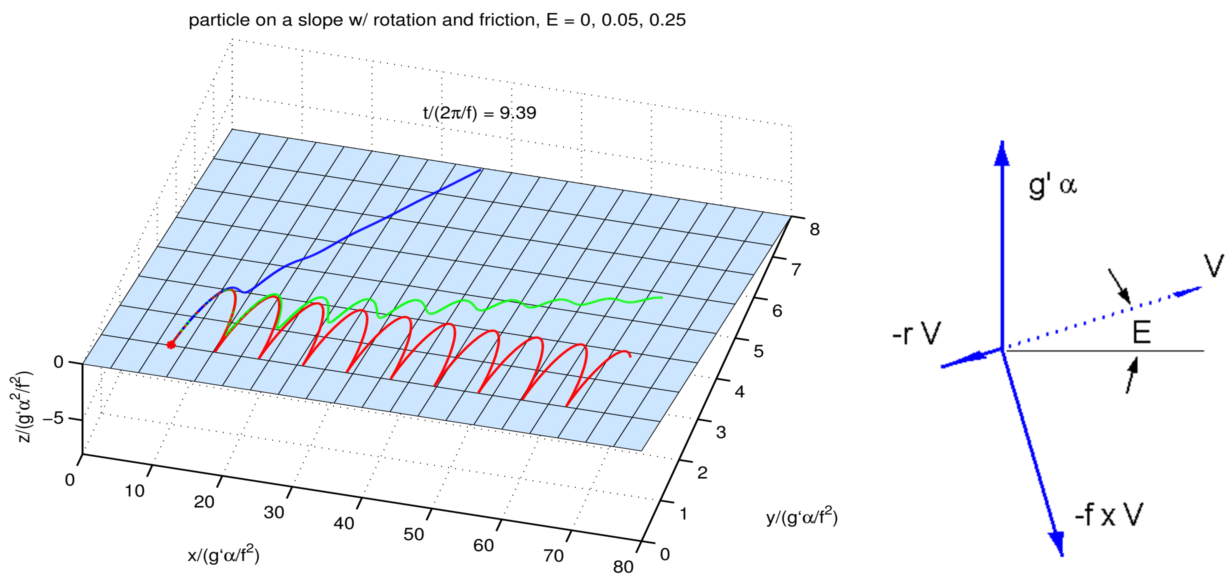

Figure 16: **(left)** Trajectories of three dense parcels released from rest onto a rotating slope. The buoyancy force is toward positive y (up in this figure). These parcels differ by having rather large friction (blue trajectory, $E = r/f = 0.25$), moderate, more or less realistic friction (green trajectory, $E = 0.05$) and no friction at all (red trajectory, $E = 0$). The elapsed time in units of inertial periods, $2\pi/f$, is at upper left. At mid-latitude, an inertial period is approximately one day, and hence these trajectories span a little more than one week. The along- and across-slope distance scales are distorted by a factor of almost 10 in this plot, so that the blue trajectory having $E = 0.25$ makes a much shallower descent of the slope than first appears here. Notice that for values of $E \ll 1$ (red and green trajectories), the motion includes a looping inertial motion, and a long-term displacement that is more or less along the slope, the analog of geostrophic motion. This is presumed to be a northern hemisphere problem, $f > 0$, so that shallower bottom depth is to the right when looking in the direction of the long-term motion. Experiments that test different r or different initial conditions may be carried out via the Matlab script partslope.m (linked in Sec. 6.3). **(right)** The time-mean horizontal velocity (the dotted vector) and the time-mean force balance (solid arrows) for the case $E = 0.25$ (the blue trajectory). The Coriolis force (/M) is labeled $-\mathbf{f}\times\mathbf{V}$. The angle of the velocity with respect to the isobaths is $E = r/f$, the Ekman number.

rate $f = 2\Omega \sin\phi$ in a direction opposite the rotation of the reference frame, Ω; inertial rotation is clockwise in the northern hemisphere and anti-clockwise in the southern hemisphere.

Inertial motion is a striking example of the non-conservation property inherent to the rotating frame equations: the velocity of the parcel is continually accelerated (deflected) with nothing else showing a reaction force; i.e., there is no evident physical cause for this acceleration, and global momentum is not conserved.[30, 31]

The trajectory of a pure inertial motion is circular (Fig. 11),

$$x(t) = \int u(t)dt = \frac{U_o}{f}\sin(-ft), \tag{113}$$

$$y(t) = \int v(t)dt = -\frac{U_o}{f}\cos(-ft), \tag{114}$$

up to a constant. The radius of the circle is $r = \sqrt{x^2+y^2} = \mid U_o \mid /f$. A complete orbit takes time $2\pi/f$, a so-called inertial period: just a few minutes less than 12 hrs at the poles, a little less than 24 hrs at 30 N or S, and infinite at the equator. (Infinite is, of course, unlikely physically, and suggests that something more will arise on the equator; more on this below). Though inertial motion rotates in the sense opposite the reference frame, it is clearly not just a simple rotation of the inertial frame solution (cf., Fig. 11). In most cases (equator aside) the displacement associated with an inertial motion is not large, typically a few kilometers in the mid-latitude ocean. Inertial motion thus does not, in general, contribute directly to what we usually mean by 'circulation', *viz.*, significant transport by fluid flow.

The centripetal acceleration associated with circular, inertial motion is $-U_o^2/r$ (Fig. 10). This centripetal acceleration is provided by the Coriolis force, and hence the radial momentum balance of this pure inertial motion is just

$$\frac{-U_o^2}{r} = fU_o. \tag{115}$$

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[30] To discern a physical cause of inertial motion we could analyze the inertial frame equivalent motion as in Sec. (3.4), a combination of angular momentum conservation (northward relative motion) and the slightly out of balance centripetal acceleration (eastward relative motion). See also D. R. Durran, 'Is the Coriolis force really responsible for the inertial oscillation?' *Bull. Am. Met. Soc.*, **74**(11), 2179–2184 (1993).

[31] The Coriolis force is isomorphic to the Lorentz force, $q\mathbf{V}\times\mathbf{B}$, on a moving charged particle having charge q and mass M in a magnetic field $\mathbf{B}$. The charged particle will be deflected into a circular orbit with the cyclotron frequency, qB/M, analogous to an inertial oscillation at the frequency f. A difference in detail is that geophysical flows are generally constrained to occur in the local horizontal plane, while a charged particle may have an arbitrary three dimensional velocity with respect to $\mathbf{B}$. What happens when $\mathbf{V}$ is parallel to $\mathbf{B}$? Where on Earth does it happen that $\mathbf{V}$ (a horizontal current) may be parallel to Ω? Still another example of such a force law comes from General Relativity which predicts that a rotating object will be accompanied by a gravitomagnetic field that gives rise to a Coriolis-like gravitational force on moving objects. The Gravity Probe B mission, one of the most challenging physics experiments ever conducted, has apparently confirmed the presence of a gravitomagnetic field around Earth, see http://einstein.stanford.edu/

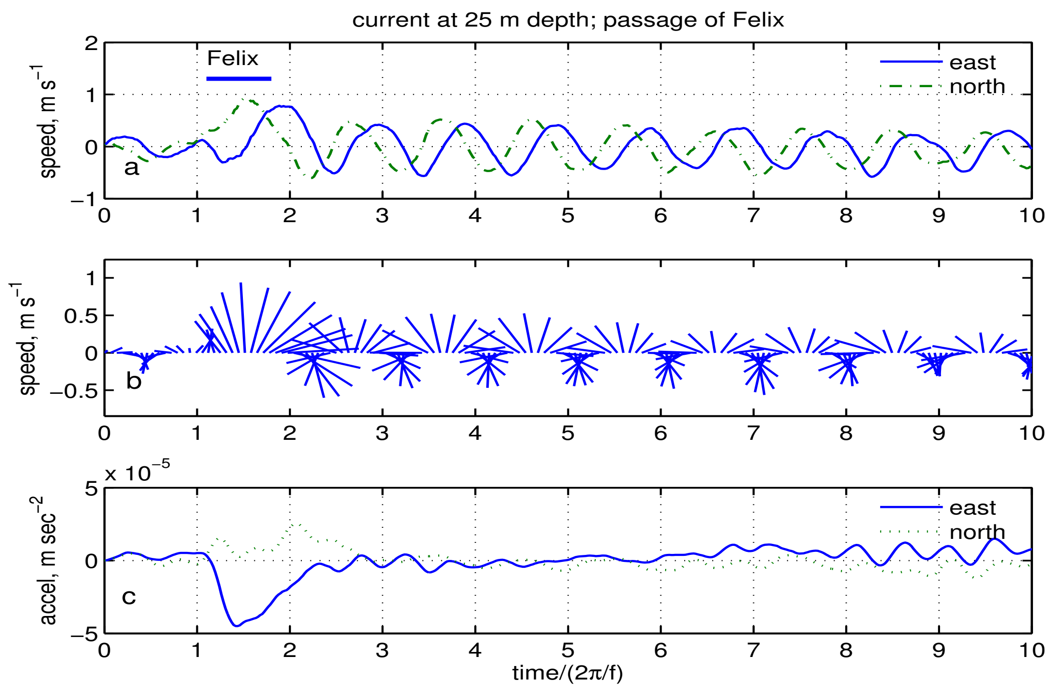

Figure 17: Ocean currents measured at a depth of 25 m by a current meter deployed southwest of Bermuda. The time scale is inertial periods, $2\pi/f$, which are nearly equal to days at this latitude. Hurricane Felix passed over the current meter mooring between $1 < t/(2\pi/f) < 2$ and the strong and rapidly changing wind stress produced energetic, clockwise rotating currents within the upper ocean. **(a)** East and north current components. Notice that the maximum north leads maximum east by about a quarter inertial period, and hence the velocity vector is rotating clockwise. **(b)** Current vectors, with north 'up'. To a first, the fluctuating current seen here is an inertial inertial oscillation. A more refined description is to note that it is a near-inertial oscillation; the frequency is roughly 5% percent higher than f and the amplitude e-folds over about 10 days (by inspection). These small departures from pure inertial motion are indicative of wave-like dynamics considered in Part 2. **(c)** Acceleration estimated from the current meter data as $d\mathbf{V}'/dt + 2\boldsymbol{\Omega}\times\mathbf{V}'$, as if the measurements were made on a specific parcel. The large acceleration to the west northwest corresponds in time to the passage of Felix and the direction of the estimated acceleration is very roughly parallel to the wind direction (not shown here). Notice the much smaller oscillations of the acceleration having a period of about 0.5 inertial periods (especially the last several inertial periods). These are likely due to pressure gradients associated with the semidiurnal tide. This is a small part of the data described in detail by Zedler, S.E., T.D. Dickey, S.C. Doney, J.F. Price, X. Yu, and G.L. Mellor, 'Analysis and simulations of the upper ocean's response to Hurricane Felix at the Bermuda Testbed Mooring site: August 13-23, 1995', *J. Geophys. Res.*, **107**, (C12), **25**-1 - **25**-29, (2002), available online at http://www.opl.ucsb.edu/tommy/pubs/SarahFelixJGR.pdf.

Interestingly, there are two quite different flows that are consistent with a single parcel undergoing inertial motion given by Eqns. (114) and (115): 1) a *vortical inertial motion* associated with a steady, anticyclonic eddy (or vortex), and 2) a time-dependent but spatially quasi-homogeneous *inertial oscillation.* To treat either of these at a useful depth will require a more comprehensive two-dimensional *fluid* model that we will come to in Part 2.[32] For now, suffice it to say that vortical inertial motion is very rarely (never ?) observed in the ocean or atmosphere, while near-inertial oscillations are very widely observed in the upper ocean following a sudden shift in the wind speed or direction, (Fig. 17).

Observed near-inertial oscillations differ from pure inertial motion in that their frequency is usually slightly higher than f or 'blue shifted'. As we will see in Part 2, near-inertial oscillations may be thought of as the long wave length limit of gravity waves in the presence of rotation (inertial-gravity waves) and the slight blue shift is characteristic of the gravity wave dynamics. The amplitude of observed near-inertial oscillations also changes with time; in the case of Fig. (17), the current amplitude e-folds in about one week following the very strong, transient forcing caused by a passing hurricane. This decay is likely a consequence of energy dispersion in space by wave propagation, and almost certainly not the dissipation process modeled here as $-r\mathbf{V}$.

5.3 (Quasi-) Geostrophic motion

The long-term displacement of the parcel is associated with the time-independent part of the solution, Eqn. (110), which is the parcel equivalent of damped, geostrophic motion. Again it is helpful to take a short step back to the dimensional momentum equations (Sec. 4.3) and point out the subset that supports pure geostrophic motion, $r = 0$ and $d/dt = 0$, in which case the x-momentum equation vanishes term by term, and the y-component is algebraic,

$$\boxed{0 = -fu + g'\alpha} \tag{116}$$

where we have assumed reduced gravity and in this case $\alpha = \partial\eta/\partial y$. Thus pure geostrophic motion is in the x-direction only,

$$u = \frac{g'\alpha}{f},$$

[32] A preview. The $d(\ )/dt$ of Eqn. (111) is time rate of change following a given parcel and is thus Lagrangian. In order to discern the difference between a vortical inertial motion and an inertial oscillation we would need to compute trajectories of some additional, different parcels, but there is presently no clear motivation for proceeding that way. Analysis in an Eulerian frame is helpful: the time derivative is then $d(\ )/dt = \partial(\ )/\partial t + \mathbf{V}.\nabla(\ )$, a local time rate of change and an advective rate of change. If the balance is between the local time rate change and the Coriolis force, then the solution will be a spatially homogeneous *inertial oscillation*. If the balance is between the advective rate of change and the Coriolis force, then the solution will be a steady, spatially-dependent *vortical inertial motion*. A map of the velocity field would be completely different in these two flows, and yet the trajectory af a given parcel may be identical, Eqn. (114).

which is the geostrophic velocity scale, U_{geo}. In a more general vector form, good for any steady, horizontal force **G**,

$$\boxed{\mathbf{V}_{geo} = -\frac{1}{\rho_o f}\mathbf{k}\times\mathbf{G}} \tag{117}$$

where **k** is the vertical unit vector. In practice we usually reserve the distinction ’geostrophic’ for the case that the force is a horizontal pressure gradient, $\mathbf{G} = -\nabla P$ or equivalently a geopotential gradient, $\propto -g\nabla\eta$. If the force is the vertical divergence of a horizontal wind stress, $\mathbf{G} = \partial\tau/\partial z$, then the steady velocity is often termed an Ekman velocity.

Simple though (117) is, there are several important points to make regarding geostrophy:

1) Perhaps the key point is that when the Coriolis force is present along with a persistent applied force, there can exist (likely will exist) a steady velocity that is perpendicular to the applied force provided that the forcing persists for a sufficient time, several or more rotation times. Looking in the direction of the applied force, $\mathbf{V}_{geo}$ is to the right in the northern hemisphere, and to the left in the southern hemisphere.

2) For a given **G**, the geostrophic wind or current goes as $1/f$, and hence will be larger at a lower latitude. Clearly something beyond pure geostrophy will be important on or very near the equator where $f = 0$. With that important proviso, we can use Eqn. (117) to evaluate the surface geostrophic current that is expected to accompany the tilted sea surface of Fig. (1) outside of a near-equator zone, say ± 5 degrees of latitude.

3) A pure geostrophic balance is sometimes said to be degenerate, insofar as it gives no clue to either the origin of the motion or to the future evolution of the motion. Some other dynamics has to be added before these crucial aspects of the flow can be addressed. Nevertheless, geostrophy is a very important and widely used diagnostic relationship as noted above, and is the starting point for more comprehensive models.

4) An exact *instantaneous* geostrophic balance does not hold, in general, even in the idealized case, $E = 0$, because of nearly ubiquitous inertial motions. However, if we are able to time-average the motion over a long enough interval that the oscillating inertial motion may be averaged out, then the remaining, time-average velocity will be closer to geostrophic balance. Said a little differently, geostrophic balance may be present on time-average even if not instantaneously.

5) Because geostrophic motion may be present on long-term average (unlike inertial motion), the parcel displacements and transport associated with geostrophic motion may be very large. Thus, geostrophic motion makes up most of the circulation of the atmosphere and oceans.

An exact geostrophic balance is an idealization (albeit a very useful one) insofar as many processes can cause small departures, e.g., time dependence, advection, friction, and more. In the parcel on a slope experiments we can see that quasi-geostrophy, a phrase often used to mean near-geostrophy, will hold provided that the applied force varies slowly compared to the rotation time scale, $1/f$, and that the Ekman number is not too large, say $E \leq 0.1$, which commonly occurs. Aside from the startup transient, the former condition holds exactly in these experiments since the bottom slope is spatially uniform and unlimited in extent. The more realistic shallow water (fluid) model of Part 2 will supplant this latter condition with the requirement that the horizontal scale L of a layer thickness (mass) anomaly must exceed the rotation length scale, C/f, where C is the gravity wave speed dependent upon stratification. [33] Trajectories having larger E show a steeper descent of the slope, from Eqn. (110), $v_*/u_* = E$. It is important to note that friction is large or small depending upon the ratio r/f and not simply r alone. In other words, for a given r, frictional effects are greater at lower latitudes (smaller f). Very near the equator, E will thus be large for almost any r, and on that basis alone geostrophic motion would not be expected near the equator. Friction may be somewhat important in this regard, but a more comprehensive fluid model treated in Part 3 Sec. 3 shows that gravity wave dynamics is likely to be more important than is friction alone.

5.4 Energy balance

Energy balance makes a compact and sometimes useful diagnostic; it is compact since energy is a scalar vs. a vector momentum, and it is more or less useful depending mainly upon how well the dissipation processes may be evaluated. In this model problem, the energy source is the potential energy associated with the dense parcel sitting on a sloping bottom and we have the luxury of knowing the dissipation (bottom drag) exactly. As the parcel descends the slope, it will release potential energy and so generate kinetic energy and thus motion.

To find the energy balance equation, multiply the x-component momentum equation (105) by u_* and the y-component equation by v_* and add:

$$\frac{d(u_*^2+v_*^2)/2}{dt_*} - v_* = -E(u_*^2+v_*^2). \tag{118}$$

The term on the left is the time rate change of kinetic energy; the term on the right of (118) is the rate of work by bottom friction, always negative since bottom friction opposes the velocity. The second term on the left is the rate of work by the buoyancy force (in nondimensional units), which is also the rate of change of potential energy. The dimensional potential energy is just $PE = g'(z - Z_0) = -g'\alpha(y - Y_0)$

[33] A subtle distinction between the present single parcel model result and the usual idea of geostrophy is that the single parcel model is Lagrangian, i.e., the dependent variables are the position and velocity of a specific parcel, not the velocity at a point in space as is implicit in Fig. 1, say, which is an Eulerian representation. These two representations merge only in the highly idealized case that all fields are spatially uniform.

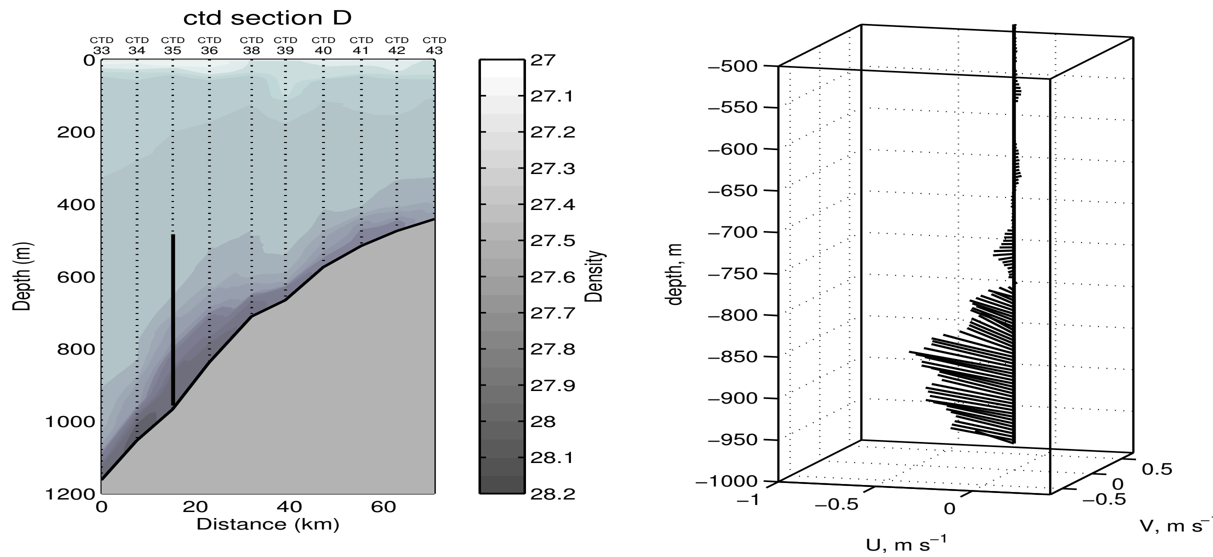

Figure 18: Observations of a dense bottom current, the Faroe Bank Channel Overflow, found on the southern flank of the Scotland-Iceland Ridge. **(left)** A section made across the current showing dense water that has come through the narrow Faroe Bank Channel (about 15 km width, at latitude 62 N and about 90 km to the northeast (upstream) of this site). This dense water will eventually settle into the deep North Atlantic where it makes up the Upper North Atlantic Deep Water. The units of density are kg m^{-3}, and 1000 has been subtracted away. By inspection of these data, the reduced gravity of the dense water is $g' = g\ \delta\rho/\rho_0 \approx g\ 0.5/1000 = 0.5\times10^{-2}$ m s^{-2}, and the bottom slope is roughly $\alpha = 1.3\times10^{-2}$. **(right)** A current profile measured at the thick vertical line shown on the density section. The density section was aligned normal to the isobaths and the current appeared to be flowing roughly along the isobaths. The core of the dense water has descended roughly 200 m between this site and the Faroe Bank Channel.

with Z_o the initial depth, and

$$v_* = \frac{vf}{g'\alpha} = -\frac{dz}{dt}\frac{f}{g'\alpha^2} = \frac{-dPE}{dt}\frac{1}{fU_{geo}^2} = \frac{-dPE_*}{dt_*},$$

the rate of change of potential energy in nondimensional units, fU_{geo}^2. It can be helpful to integrate (118) with time to compute the change in energy from the initial state:

$$\begin{array}{ccccccc} (u_*^2+v_*^2)/2 & - & \int_o^t v_* dt_* & = & - & \int_o^t E(u_*^2+v_*^2)dt_*, & \\ KE & + & PE & = & & FW, & \end{array} \tag{119}$$

where KE is the kinetic energy, PE is the change in potential energy as the parcel is displaced up and down the slope, and FW is the net frictional work done by the parcel, always a loss (Fig. 19).

The Coriolis force does no work on the parcel since it is perpendicular to the velocity, and hence does not appear directly in the energy balance. Rotation nevertheless has a profound effect on the

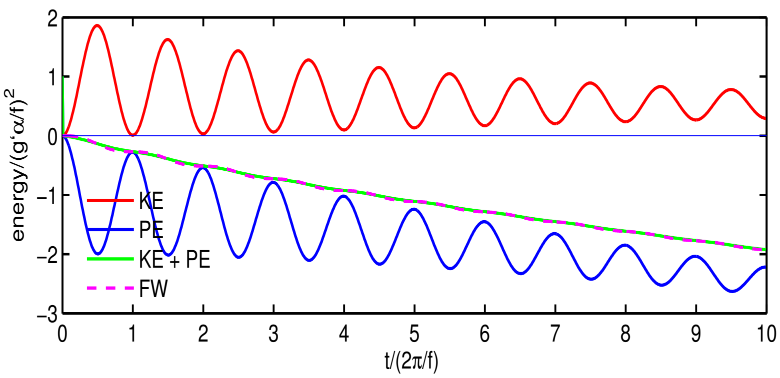

Figure 19: The energy balance for the trajectory of Fig. (16) having $E = 0.2$. These data are plotted in a nondimensional form in which the energy or work is normalized by the square of the velocity scale, $U_{geo} = g'\alpha/f$, and time is nondimensionalized by the inertial period, $2\pi/f$. Potential energy was assigned a zero at the initial depth of the parcel. Note the complementary inertial oscillations of PE and KE, and that the decrease of total energy was due to work against bottom friction (the solid green and dashed red lines that overlay one another).

energy balance. The inertial oscillations that carry the parcel up and down the slope show up in the energy balance as a reversible (aside from friction) interchange of kinetic and potential energy, exactly analogous to a simple pendulum. The most profund consequence of rotation is that it inhibits the release of potential energy. In the important limit that $E \rightarrow 0$, and aside from inertial motion, the parcel velocity will be perpendicular to the buoyancy force, as in Eqn. (117), and the parcel will coast along an isobath in steady, energy-conserving geostrophic motion. If there is some friction, as there is in the case shown, then the cross-isobath component of the motion carries the parcel to greater bottom depth and thus releases potential energy at a rate that is proportional to the Ekman number, Eqn. (107), $v_*/u_* = E = r/f$. Whether friction or rotation is dominant, and thus whether the motion is rapidly dissipated or long-lived, depends solely upon the Ekman number in this simplified system (Fig. 16b).

5.5 Problems

(1) Draw the vector force balance for inertial oscillations (include the acceleration) with and without bottom friction as in Fig. (16, right).

(2) What value of r is required to mimic the observed decay of near-inertial oscillations of Fig. (17)? Does the same model solution account also for the small, super-inertial frequency shift noted in the field data?

(3) Write the non-dimensional form of the pure inertial motion model and solution, Eqn. (114). This model is so reduced that there is, admittedly, not much to gain by nondimensionalizing Eqn. (111).

(4) The parcel displacement, Eq. (114), $\delta = U_o/f$ associated with an inertial motion goes as $1/f$, and hence $\delta \rightarrow \infty$ as $f \rightarrow 0$, i.e., as the latitude approaches the equator. We can be pretty sure that something will intervene to preclude infinite displacements. One possibility is that the north-south variation of f around the equator will become relevant as the displacement becomes large, i.e., the $f-$plane assumption that $\delta \ll R_E$ noted with Eqn. (87) will break down. Suppose that we keep the first order term in $f(y)$, and assume $f = \beta y$, i.e., an equatorial beta-plane. Describe the equatorial inertial oscillations of a parcel initially on the equator, and given an impulse U_o directed

toward the northeast. How about an impulse directed toward the northwest? You should find that these two cases will yield quite different trajectories. This is an example, of which we will see more in Part 2, of the anisotropy that arises from rotation and Earth's spherical shape.

(5) In Sec. 5.1 it was noted that dimensional analysis may be somewhat arbitrary, as there are usually several possible ways to nondimensionalize any given model. For example, in this parcel on a slope problem the time scale $1/r$ could be used to nondimensionalize (that is, to scale or measure) the time. How would this change the solution, Eqn. (108) and the family of trajectories?

(6) Assuming small Ekman number, how long does it take for a geostrophic balance to arise after a parcel is released? Are the time-averaged solutions of the single parcel model the solutions of the time-averaged model equations? Suppose the model equations were not linear, say that friction is $\propto U^2$, then what?

(7) Inertial oscillations do not contribute to the long-term displacement of the parcel, though they can dominate the instantaneous velocity. Can you find an initial condition on the parcel velocity that prevents these pesky inertial oscillations? You can test your ideas against solutions from partslope.m (Section 7).

(8) Explain in words why a geostrophic balance (or a near geostrophic balance) is expected in this problem, given only small enough E and sufficient space and time.

(9) Make a semi-quantitative test of geostrophic balance for the westerly wind belt seen in Fig. (2). Sample (by eye) the sea surface height of Fig. (1) along an east-west section at 33 ^{o}N, including at least a few points in the western boundary region. Then estimate the east-west profile of the inferred geostrophic current (and note that the buoyancy of the sea surface is effectively the full g since the density difference is between water and air). What is the current direction? Using this result as a guide, sketch the (approximate) large-scale pattern of surface geostrophic current over the subpolar gyre and lower subtropics on Fig. (1). You can check your result against observed surface currents, http://oceancurrents.rsmas.miami.edu/atlantic/florida.html

(10) Assuming that the descent of the dense water from Faroe Bank Channel to the site observed in (Fig. 18) was due mainly to bottom friction, which trajectory of Fig. (16) is analogous to this current? Said a little differently, what is the approximate Ekman number of this current?

(11) An important goal of this essay has been to understand geostrophic balance, the characteristic feature of many large scale geophysical flows. However, it has also been noted that pure geostrophy is a dead end insofar as it gives no clue to the origin or the evolution with time. To predict the evolution of a flow we have to understand what are usually small departures from pure geostrophy, here limited to time-dependence, e.g., inertial motion, and friction. With that in mind, compare the relative importance of friction in the time-average momentum balance, Fig. (16), right, and in the energy balance, Fig. (19).

(12) The Coriolis parameter, f, changes signs crossing the equator, and thus a northern hemisphere inertial motion turns clockwise in time, and a southern hemisphere inertial motion turns anticlockwise. You are in very good company if you are wondering ... 'What's it do right on the equator?' (see S. Adams, *It's Obvious You Won't Survive by Your Wits Alone*, p. 107, Andrews and McNeil Press, Kansas City, Kansas, 1995). By symmetry, the horizontal component of the Coriolis force must vanish right on the equator and hence the contrast between mid-latitude phenomena and equatorial phenomena is of great interest and taken up in the Part 3 essay. To start on this, how would you characterize the near-equatorial (± 10 deg of latitude) wind and height

relationship of the 500 hPa winds evident in Fig. 2? You may find it helpful to see an animation of these charts available online at http://www.nrlmry.navy.mil/metoc/nogaps/.

6 Summary and Closing Remarks

6.1 What is the Coriolis force?

The winds and currents of Earth's atmosphere and oceans are necessarily observed and analyzed from the perspective of an Earth-attached and thus a rotating, non-inertial reference system. The inertial frame equation of motion transformed to a general rotating frame includes two terms due to the rotation, a centrifugal force term and a Coriolis term, $-2\mathbf{\Omega}\times\mathbf{V}'M$ (Section 2). There is nothing *ad hoc* or discretionary about the appearance of these terms in a rotating frame equation of motion. In the case of an Earth-attached frame, the centrifugal force is cancelled by the aspherical gravitational mass attraction associated with the slightly out of round shape of the Earth (Section 4). The Coriolis force remains, and is of first importance for large scale, low frequency winds and currents.

It is debatable what the Coriolis term should be called, whether a force as done here, or an acceleration. The latter is sensible insofar as the Coriolis force on a parcel is exactly proportional to the mass of the parcel, regardless of what the mass may be. This is a property shared with gravitational mass attraction, but not with central forces that arise from the physical interaction of objects. Nevertheless, we chose the Coriolis 'force' label, since we are especially concerned with the consequences of the Coriolis term.

Because the atmosphere and the oceans are thin when viewed in the large and also stably stratified, the horizontal component of winds and currents is generally much, much larger than is the vertical component. In place of the full three-dimensional Coriolis, it is usually sufficient to consider only the horizontal component acting upon the horizontal wind or ocean currents,

$$-2\mathbf{\Omega}\times\mathbf{V}' \approx -\mathbf{f}\times\mathbf{V}' = fv'\mathbf{e_x} - fu'\mathbf{e_y}$$

where $\mathbf{f} = f\mathbf{e_z}$, and

$$f = 2\Omega\sin(latitude)$$

is the Coriolis parameter, which will arise very, very often in a study of geophysical fluid dynamics.

6.2 Consequences of the Coriolis force for the circulation of the atmosphere and ocean

Here in Part 1 we have made a small start toward understanding the profound consequences of the Coriolis force with an analysis of a dense parcel released onto a slope (Section 5). This revealed two kinds of motion that depend directly upon the Coriolis force. There is a free oscillation, usually called an inertial oscillation, in which an otherwise unforced current rotates at the inertial frequency, f. These inertial oscillations are a prominent phenomenon of the upper ocean current following the passage of a storm. A crucial, qualitative effect of rotation is that it makes possible a steady motion for which an external force (wind stress or geopotential gradient) is balanced by the Coriolis force acting upon a geostrophic current,

$$\mathbf{V}_{geo} = -\frac{g}{\rho_o f}\mathbf{k} \times \nabla \eta.$$

A characteristic of geostrophic motion is that the velocity is perpendicular to the applied force; in the northern hemisphere, high SSH is to the right of a geostrophic current (Fig.1). It would be easy to over-interpret the results from our little single parcel model, but, a correct inference is that Earth's rotation — by way of the Coriolis force — is a key to understanding the persistent, large scale circulation of both the atmosphere and the ocean (outside of equatorial regions).

6.3 What comes next?

This introduction to the Coriolis force continues in Parts 2, 3, 4 and 5 with an emphasis on the consequences of rotation for the atmosphere and ocean. Specific goals are to understand

Part 2: a rotating shallow water model, potential vorticity balance and geostrophic adjustment. What circumstances lead to a near geostrophic balance? As we have noted throughout this essay, a near geostrophic balance is almost inevitable for large scale, low frequency motions of the atmosphere or ocean. The essential piece of this is to define large scale and low frequency, which depends upon the stratification and the Coriolis parameter, f, and so varies substantially with latitude.

The plan/method of Part 2 is to construct a model of a single fluid layer, often called the shallow water model, and to conduct a sequence of initial value experiments that may or may not evolve into geostrophic balance. The potential vorticity of the initial state is a key diagnostic. The tools and methods introduced in Part 2 are thus a very considerable advance over those employed here in Part 1, and are more likely to be directly useful in your own research. Everything that you have learned in Part 1 regarding the Coriolis force acting on a single parcel will be essential background for understanding the more comprehensive models and experiments introduced in Part 2 and on.

Part 3: β-effects, and westward propagation. The single new feature introduced in Part 3 is the explicit recognition that the Coriolis parameter varies with latitude, in the beta-plane approximation, $f = f_o + \beta y$ with y the north coordinate. The resulting beta-effects includes some of the most interesting and important phenomena of geophysical flows — a pronounced east-west asymmetry in many large scale phenomenona, and westward propagation of eddies in the ocean and long waves in the westerlies (Fig. 2).

Part 4: wind-driven ocean circulation and the Sverdrup relation. The goals are to learn how the

Sverdrup relation is established following the onset of a wind field applied to an ocean basin, and to understand some important properties of the wind-driven, upper-ocean gyres, e.g., the marked east-west asymmetry.

Part 5: on the seasonally-varying wind-driven circulation of the Arabian Sea. This capstone project that brings together many of the results from the previous essays applied to the Arabian Sea. As the title implies, this is mostly oceanographic, though some details of the South Asia Monsoon winds are highly significant.

6.4 Supplementary material

Matlab scripts that will help you do the homework problems may be downloaded from https://www2.whoi.edu/staff/jprice/aCt-codes which includes the following:

rotation_1.m solves for the three-dimensional motion of a parcel as seen from an inertial and from a rotating reference frame. Used to make Fig. 11.

rotation_2.m solves for the inertial oscillation of a parcel moving freely on a rotating, parabolic surface so that centrifugal force is cancelled, and as seen from an inertial and from a rotating reference frame.

partslope.m solves for the motion of a single dense parcel on a slope and subject to buoyancy, bottom friction and Coriolis forces as in Section 5. It is easy to specify a new experiment by changing the initial velocity, bottom slope, etc.

sphere_check.m used to check the spherical system equations of motion, and useful as an introduction to spherical coordinates (Sec. 3.4).

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Resource: Topics in Fluid Dynamics
James Price