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PROFESSOR: Good afternoon. Let's get started. So we continue our discussion today about light atom interaction.

And just to sort of remind you where we are, we started last week, even before spring break, to talk about the matrix element which provides a coupling.

So now we have a coupling between the two states. And we want to understand what is the coupling doing to the system. What is a dynamical evolution of the system? How does the atomic wave function evolve when we couple two states using optical fields?

And well, as usual, we start with the basic phenomena. And this is we do perturbation theory with a dipole Hamiltonian. And we have done that on Monday. We've done perturbation theory.

And I made an important distinction between a monochromatic case and [INAUDIBLE] case. The monochromatic case gave us in perturbation theory this result. And if you stare at it for a while, you see, well, these are just Rabi oscillations.

It's not the formula of Rabi oscillations with the generalized Rabi frequency and power broadening because it's perturbation theory. So these are Rabi oscillations in the perturbative limit.

But also on Monday I showed you that for interactions with monochromatic light, we can just rewrite the Hamiltonian. That it not just looks like, it is exactly the Hamiltonian for spin 1/2 in a magnetic field. And we have already discussed the solution.

So, therefore, we know for monochromatic radiation, we can just go beyond perturbation theory. We can solve it exactly. OK. So monochromatic radiation we understand.

At least as long as we have just the coupling of the atom to a single mode of the electromagnetic field. And now we come back to the broadband case. I give a little bit more of a summary and an outline after I finish the broadband case.

So what we have right now is we have Rabi oscillations, which we derived for a single mode. And the broadband case is such that we now assume that we have a spectrum of frequencies. And we assume that to be flat and broad in a moment.

And what we have to do now is we have to substitute in this formula the Rabi frequency by the electric field. And the electric field becomes using the connection between electric field and energy density, eventually the Rabi frequency squared gets replaced by an integral over the spectral density.

It looks mathematically exact. But, of course, you all recognize that I've made a very important assumption here. Namely, that there is no correlation whatsoever between the different frequencies.

Because by integrating here, I'm just summing up the e squares assuming that there's no interference, no coherence, no correlation between the different frequencies.

OK, so with those assumptions I can now just formulate the mathematics exactly. I calculate now the probability to be in the excited state by assuming we have this kind of perturbative Rabi oscillations at every frequency component. But we integrate over all frequencies.

Any questions? So now-- and this is where I ended the last lecture. And there are actually now, this is really an interesting case.

I know I sometimes spend a lot of time on sort of where the math is simple, but the

physics is really interesting. So the probability to find an atom in the excited state is now this function which replaces, which are the Rabi oscillations, convoluted integrated over the spectral density.

And now we have to consider two cases. If this function is flat and is spectrally very broad, we can pull it out of the integral

However, this function here, which are your Rabi oscillations-- sine squared of an [? argument ?] divided by the [? argument. ?] I've plotted it here for you. This function is actually very peculiar. It has a height of t^2 . If you're doing Taylor Expansion for short times t .

But the width is t^{-1} . So as time goes by, it gets narrower, and taller, and taller. And if you take t^2 by t^{-1} , you get t . So the integral over it is just t .

So in other words, we have a function which goes as t^2 . But the integral is t . And if the width is so narrow that the width doesn't really matter, we can say this function has become a delta function.

But the integral over the function is t . So it becomes t times the delta function. And now we the two limiting cases. If very short times t -- this Rabi oscillation function is extremely broad.

And at infinitesimal times t , it is broader than any spectral bandwidths. So actually what we are doing is when we solve this integral at very short time, this is broader. And we pull this out of the integral. And this gives us a prefactor of t^2 .

So, therefore, the excitation probability is t^2 times the integral of the spectral function which is just the total intensity.

However, if you wait a little time, and eventually if the time is such that the time is longer than the inverse bandwidths of the spectral radiation, then the spectral radiation is broader than this function, f .

And we pull this out of the integral. And then what we have is we have simply the

integral here over the delta function which gives us a factor of t .

So we have two functions. And whichever is broader can be pulled out of the integral. And that means in one case, we have a probability which is t squared times the total intensity. And in the other case, we have a behavior which is the time times the spectral density at 0 detuning.

OK. So now let's-- just a side remark before I interpret this result, this function you would recover actually the Rabi oscillation. I plotted it in that way emphasizing the amplitude and the width.

But if you look for fixed detuning and you vary the time, this function is sort of spreading out and at a fixed detuning, you'll go up and down. And these are the Rabi oscillations we have discussed for monochromatic radiation before.

So these two limiting cases are actually very important. At very short times, the behavior is, you know, we have broad spectral radiation. But if the time is shorter than the inverse bandwidth of the radiation, even the broadband radiation is like monochromatic wave.

And the short time behavior is t squared times the intensity. That's exactly what we got for monochromatic light. In other words, if you're broadening, you have a broad spectral source, but if your inverse time is broader than the broadening of the spectral source, you're back to the monochromatic case.

So a lot of people get confused. I mean, I see that often in part 3 oral exams, that no matter what your spectral bandwidth is-- unless it's infinite. But let's discuss a pathological case.

For any broadband light, at short moments, the system evolves as t square. And t square is the hallmark of coherent time evolution.

The amplitude goes with t . The probability goes with t square. And only when the bandwidth of the spectral light dominates when we can replace this by the delta function. Then we are in the regime, which you all know is Fermi's Golden Rule,

where we [INAUDIBLE] equation.

And it all comes from this formula. It all comes from assuming Rabi oscillations. But then performing the spectral integral over the Rabi oscillation. This takes us from the narrowband case to the broadband case.

Any questions about that? Yes?

AUDIENCE: So what says the bandwidth here? Because we assume its narrowness arises just because of the short [INAUDIBLE].

PROFESSOR: I will use an LED which has a few nanometer bandwidths. Use sunlight which has a few hundred nanometer bandwidths. Whatever your light source is.

At this point, it's a general discussion. And I'm not going beyond the two limiting cases here. Of course, if you have a complicated spectral distribution, well, you're on your own to solve this integral.

But at short times, we have this behavior. At long times, we have this. And the two regimes are separated by the time where I have a crossover in one case. This is broader and I can pull it out in the other case.

This function is broader and I pull it out of the integral. And just where this happens, this is where we have the transition from t^2 to t behavior. From coherent evolution to rate equation.

OK. Yes. So let's now look at the situation of broadband light. Because later today I want to discuss with you Einstein's a and b coefficient. A very classical topic of atomic physics.

A very famous concept introduced by Einstein. And actually I want to use the perturbation theory for broadband light which we have now formulated to derive for you the b coefficient of Einstein's a and b coefficient theory.

OK. So for large time, we can now talk about rate. Which is the probability which increases linearly with time per unit time.

And this was the matrix element squared. $2 \epsilon_0 \hbar^2 \pi$. The 2 cancels. And then we have from the delta function, the spectral density at 0 detuning. Which is the spectral density at the resonance frequency.

So we have a rate equation now. That the rate equation is the b coefficient. Einstein's famous b coefficient times the spectral density. And the b coefficient is now the proportionality constant between in the equation above.

Which is πd^2 . But now in all the formula for the b coefficient, there's a factor of 3. Because the assumption is made that we have isotropy of space. The atoms are randomly oriented.

And, therefore, d_x^2 for given polarization, the dipole moment projected on a polarization of the light which is d_x^2 is just $1/3$ of the absolute value of the dipole moment squared.

So in other words, just to remind you what I have actually discussed is nothing else than Fermi's Golden Rule. And I could've reminded you of Fermi's Golden Rule where the rate is given. I just use a standard notation of textbooks.

You take the matrix element squared, you multiply by 2π , and then you have a delta function. And the delta function implies a delta function is always a reminder that it needs integration.

So whenever you have a delta function in Fermi's Golden Rule, you have to integrate. And there are two possibilities. You have to integrate over the spectrum of external fields. That's what we just did.

The other possibility is, which doesn't apply to what we just discussed, that you have to integrate over a continuum of final states.

This will be important when we use a Fermi's Golden Rule expression to talk about spontaneous emission where we have a continuum of final states. So anyway, I could have just said, let's start with Fermi's Golden Rule and let's jump to the final result.

But I really wanted to emphasize here the sort of intimate connection between Rabi oscillation, the t^2 dependence, and how this turns into a rate equation.

OK. Let's just summarize what we have done in a table. We have seen two different regime. In one case with the Rabi resonance, we are discussing a single final state of the atom.

A single mode of the electromagnetic field. All energy levels, all states are discrete. We are talking about unitary reversible time evolution. When we had rate equations, we are talking about many final states. We integrate over them.

Or and/or many modes of the external field. We are naturally dealing not with a discrete number, but with a continuum of states.

The time evolution has become irreversible. And is therefore no longer unitary evolution, but it's a dissipative evolution. And all this came about not because we have spontaneous emission.

I will tell you throughout this course that spontaneous emission is not as spontaneous as everybody assumes. Spontaneous emission is actually unitary time evolution. Unless you discard information.

But a lot of people think rate equation irreversibility comes from something which is genuinely spontaneous and irreproducible. I don't know anything in physics which is spontaneous and irreproducible. But we come to that later.

And this is an example where we obtain rate equation by simply driving a system. And the irreversibility comes by performing the integral over the spectral density.

So let me just write that down. Due to integration. Since we integrate over an infinite number of modes or states. Any questions?

OK, great. I wanted to make sure that this is very clear. OK, at this point, let me just summarize where we are in our discussion of atom light interaction. We've actually made a lot of progress.

We have discussed matrix elements. We have discussed the coupling of atoms to an external field at the level of the Schrodinger equation. And we have done perturbation theory.

And in perturbation theory, we found Rabi oscillations and we found rate equations. That's where we are right now.

So the feature now which is missing is, of course, damping spontaneous emission irreversibility-- another form of irreversibility. Right now our Rabi oscillations are undamped.

Whether we obtain them in perturbation theory or whether we use the spin formalism to get them in the resonant in the strong coupling case.

And here, for the rate equation, the way how we have solved it, the probability to be in the excited state just increases forever. The system will never reach equilibrium.

But that means in both cases, we have a missing element. And this is spontaneous emission.

So for the next hour or two, we'll talk about aspects of spontaneous emission. Spontaneous emission will actually eventually lead to damping of Rabi oscillation. And to a saturation of the excitation.

OK, so we're now discussing spontaneous emission. And we will discuss it in actually three levels. One is I will discuss Einstein's a and b coefficients. I sometimes hesitate.

Should I really discuss Einstein's a and b coefficients? It's sort of old fashioned. And I have already in perturbation theory given you a microscopic derivation of Einstein's b coefficient.

But everybody who is an atomic physicist knows about Einstein's a and b coefficient. It was really a stroke of a genius to do it. And it becomes sort of our language.

So what I'm doing here is I'm not beating it to death, but I give you sort of a short summary. It's also sort of I make a few comments which is actually amazing. How Einstein actually got results from the a and b coefficient which you can only get otherwise if you quantize electromagnetic field.

So it's also sort of historically interesting that Einstein actually developed the theory of the a and b coefficient before the Schroedinger equation. Before quantum mechanics was developed.

And often you call Schroedinger equation the first quantization and the field quantization, the second quantization. So in some sense, Einstein actually preempted or had already the results of second quantization before first quantization was developed.

Anyway, it's a landmark paper, how Einstein did it. And that's why I want to discuss it. But it's partially also in order to give you the historical context. But then, of course, we want to use the modern formalism of use of quantization of the electromagnetic field.

And we have already obtained just now the result for Einstein's b coefficient by just looking at the induced by the absorption rate or the stimulated rate. But then eventually having a microscopic quantization.

By having a quantization of the electromagnetic field, we can also do now microscopic fully quantum first principle calculation of the a coefficient.

So then we have already the b coefficient, we get the a coefficient out of microscopic calculation. So we don't really need Einstein's treatment of a and b at this point. But it's nice to see the connections.

So anyway. So this is the agenda. Einstein's a and b coefficient to pull out spontaneous emission without even putting it in. Then we'll talk about field quantization which automatically leads us to a treatment of spontaneous emission.

Any questions? So how was Einstein able to show that there is spontaneous

emission without sort of knowing the quantum character of fields?

Well, the point was he knew and understood that there would be thermal equilibrium. He said, I know what thermal equilibrium is. Thermal equilibrium is the Boltzmann coefficient. A Boltzmann probability for an atom to be in the excited state.

The probability to be in the excited state is just the Boltzmann factor and depends on temperature in the usual way. And he also knew that the spectrum of light would follow a plank distribution.

And if you put those things together, you go beyond that. Because you are in thermal equilibrium. This here what we derived so far does not have thermal equilibrium. And thermal equilibrium only comes through the damping of spontaneous emission.

So, therefore, by Einstein just using Boltzmann distribution and Planck law, he got spontaneous emission. And this is what I just want to show you. For most of you, it's a reminder.

OK. Einstein's a and b coefficients. I will post one of Einstein's papers on the website.

He was also the first to actually discuss mechanical forces of light. He realized that if you have a gas at a temperature which is different from the temperature in the womb, the gas has to equilibrate.

And the gas can only equilibrate loose, excess velocity by transferring its momentum to the photons. So some equations of laser cooling, the fact that light can exchange momentum with a particle.

And this is eventually what leads to equilibrium, was already in papers at the beginning of the 20th century. And it's just amazing if you read those papers. How modern the language is and how clear the language is.

But here, I'm not talking about the mechanical effects. But the mechanical effects of light which many people in this class use for a living, this is actually part of this

equation.

Because the equilibrium between-- again, I discuss here Einstein's a and b coefficient-- the equilibrium between the electronic structure, the ground and excited state with a photon field.

But Einstein also considered the equilibrium between the motional degree of the atom. And equilibrium between the motional degree of the atom and the radiation field requires the spontaneous force. The spontaneous radiation force.

I'm not discussing it here. But I'm discussing here is now an equilibrium between ground state and excited state.

So the probability to find an atom in the excited state is simply described by the Boltzmann factor.

Now it's traditional in the discussion of Einstein's a and b coefficient to allow for degeneracy factors at ground and excited state. I have to say I usually hate that I try not to talk about levels. I just talk about quantum states. Non-degenerate individual quantum states.

So in that sense, I try to characterize population in a quantum state, not in a level. But it is standard to follow Einstein's concept where we have degeneracies. I'm not emphasizing them here, but I will just write them down where they belong.

OK, so this takes care. We know what is a fraction of atoms in the excited state. So this is the equilibrium. The next thing we need is the light.

And Einstein assumed that it's a spectral density in a black-body cavity. So we need the energy density per frequency interval.

And this is nothing else than the occupation number of the mode times the energy of the photon. Times the density of states.

The photon number per mode is just given by the Bose-Einstein factor. Bose-Einstein statistics factor. The mode density is, as you know, in three dimension.

Ω^2 , π^2 over c^3 .

So, therefore, the spectral density of black-body radiation has-- and we need that, and Ω^3 dependence. And then it has this Bose-Einstein denominator in the well-known form. So this is now Planck's black-body spectrum in the units where we need it.

So all we need is now to find the famous Einstein a and b coefficient. We have to write down a rate equation for the atoms.

So the fact is we know already what equilibrium is. Excited state versus ground state population is the Boltzmann factor.

But now we write down a rate equation which involves a black-body field. And then we compare the solution of the rate equation to the solution we already know. And from that, we get Einstein's a and b coefficient.

OK, so the change in the population of the excited state has three different terms. One is the energy density of the black-body radiation can cause stimulated emission.

So, therefore, it's proportional to the number of atoms in the excited state. The energy density of the black-body radiation can cause absorption. This is proportional to the number of atoms in the ground state.

And then this equation as it stands would lead to contradiction when I compare the solution of this equation to the Boltzmann factor we already know. And the only way to fix it is to add an extra term. Which is spontaneous emission.

If spontaneous emission were not necessary, this a coefficient could in the end turn out to be 0. Or it can be undetermined. But as we see, it is necessary for consistency.

So this is pretty much the famous rate equation. And we are interested in the equilibrium solution. In equilibrium, all derivatives vanish.

And then by setting the derivatives to 0, I have one equation. And I will rewrite the equation by putting the spectral density of the light on one side. And everything else on the other side.

And what we have here is the a coefficient. The ground state population, the excited state population. The b coefficient.

So this is the spectral density of-- it's just an expression for the spectral density. We want to put in now that the excited state fraction is given by a Boltzmann factor.

So, therefore, N_e over N_g becomes the Boltzmann factor. And, yes, there are these degeneracy factors.

So I've pretty much divided the denominator and the numerator by the population in the excited state. And here I get B_{eg}

OK, so this is the result for the spectral density. But we know already that the spectral density has to be of the Planck form. So now we can simply compare what we know to what we obtained from the rate equation and make sure that it matches.

So it's good we have the exponential factor. And by bringing this expression to the form of the other expression, we actually have to fulfill two conditions. One is in order to make sure that kind of the total expression is OK, it gives us a ratio.

The Planck body spectrum is normalized. There is no unknown prefactor. So this determines the ratio of a and b .

And also since we have this functional form of the Bose-Einstein statistics which has this exponential factor minus 1, it gives us also a relation between the $2B$ coefficient. That one is the b coefficient for stimulated emission. And the other one is the b coefficient for force absorption.

OK. So with that, we have the relation between the a coefficient and the b coefficient. And we find that the B coefficient for absorption and emission are the same.

Well, we know it's the same coupling matrix. I mean, the Hamiltonian which connects ground to excited state, excited state to ground state. But if you really want to deal with degenerate states and not formulated for states, you have degeneracy factors.

OK So I could stop here. This is sort of the textbook result. But I want to rewrite the result that we recognize the quantization of the electromagnetic field.

So instead of just looking at the power in Planck spectrum, spectral density, and such. I want to bring in the photon number. I've already given you the Bose-Einstein distribution for the photon number in the mode.

So I take now equation a and multiply it with the average photon number in a mode of ω . This gives me on the left hand side-- I'm multiplying this with a photon number.

So on the left hand side, I have a times the photon number. On the right hand side, when I put in the photon number, the photon number with this expression just give me the Planck distribution, the spectral energy.

So yes. This gives me the spectral energy density times the b coefficient. Yes.

And this is nothing else than stimulated emission. So we realize that stimulated emission is nothing else than n times the photon number. The photon number n times spontaneous emission.

Similarly, we know that the rate for absorption becomes now, well, the same unless we have degeneracy factors. But just for the fundamental discussion, let's avoid the [? p ?] counting how many degenerate levels a level have. Let's just assume we have a situation that we just count every state individually.

Then I can summarize this result saw in the following. That the total rate for emission was proportional to n for stimulated emission. And then we have the extra 1 for spontaneous emission. Whereas the rate for absorption was n times the spontaneous emission.

So we find that this important formula that emission has an $n + 1$ factor. Absorption has an n factor. And it is, of course, this extra plus 1 which was absolutely crucial to establish thermal equilibrium. If it had been 0, no thermal equilibrium would have been reached.

So in other words, what is already in Einstein's treatment of the a and b coefficient is that if you understand absorption, which you can understand with the Schrodinger equation, and you understand and you write it in the fundamental way in photon numbers, then spontaneous emission is just the rate of absorption divided by n .

Spontaneous emission is like induced emission in its rate. But by just one single photon.

So as I pointed out, this is a result which is usually obtained with second quantization and it is already included in Einstein's a and b coefficient.

So we could stop here. We have already a major result which is usually obtained in field quantization. But there is one deficiency and we want clearly fix it and move on to the microscopic derivation.

And this is the following. Right now, we really assume black-body radiation. And this ratio $n + 1$ over n was only derived for average photon numbers in a spectrally broad field.

And what is left for microscopic treatment which I want to present now is even if you have just a single mode, the atom can only interact with a single mode.

We find that stimulated emission and absorption is proportional to n , the number of photons already present. And then there is plus 1 for spontaneous emission.

So in other words, we do it now sort of microscopically again. And what we get out of it is that everything we learned from Einstein's a and b coefficient is not just valid in thermal equilibrium. It's not just valid for average numbers. It's really valid for single mode physics.

OK, so the agenda is what is next. Is valid. $4n$.

So this expression is valid not only for an average over many modes, but for each single mode. Questions about Einstein's a and b coefficient?

OK. So we spend now the rest of today and parts of next Monday in a microscopic derivation of spontaneous emission using field quantization. But I just want to make you aware that we know already what it is.

We have a semi-classical derivation of the b coefficient. And Einstein's treatment gives us the ratio of a and b . So we know already at this point what the rate of spontaneous emission is.

But it is nice. I think also important for our education to obtain it in a microscopic way where we really show how we have to-- sum overall modes and such to obtain the expression.

Also I want to ask you questions. I want to ask you clicker questions afterwards. And one clicker question for you is what happens to spontaneous emission in one and two dimension?

Certain things will change. And it's much clearer what will change if you have a clear understanding how we sum up all of the modes. How all the possible modes contribute to spontaneous emission.

And, of course, in two dimension and one dimension, you have a different density of modes.

So with that motivation, we need a quantized electromagnetic field. Where we quantize the field for each mode.

And then we go back, we do the summation of all modes. And we've really understood in the most fundamental and microscopic way how photons and light interact.

OK, so our next chapter is the quantization of the radiation field. We do-- yes,

[INAUDIBLE]?

AUDIENCE: I just have a question. So when we compare the rate equation and the distribution of the photons, so there is a parameter, t , in both of them. So we just assume two t 's are the same because of their reaching thermal equilibrium.

PROFESSOR: Oh, yeah. Absolutely. I mean this is, of course, what Einstein assumed that the thermal equilibrium for atoms with the Boltzmann factor.

And the thermal equilibrium for photons described at the Planck distribution have to be reached at the same temperature. It was a thermodynamical argument assuming which is, of course, one of the tenets of statistical physics of thermodynamics.

If you have two systems and they interact with each other, they equilibrate at the same temperature. Yes, this is very important. This was a very important assumption.

Of course, as we know, when we have ultra cold atoms in a room temperature vacuum chamber, the atoms do not equilibrate. But if you would trap them for an infinite amount of time, they would equilibrate.

It's just that we lose the atoms from our trap. They're knocked out by [INAUDIBLE] gas collisions. A lot of other things happen.

But if you could isolate just ultra cold atoms in a trap, they would stay in this trap forever. There would be no other effect shortening our observation time.

Eventually, the atoms would just boil out of your trap because black-body radiation. Momentum transfer from black-body photons heats the atoms up to room temperature.

And this is, of course, one of the things which really amaze people when laser cooling came along. You know, everything at low temperature was cryogenic. If you want to keep a sample cold, you had to put liquid nitrogen shields, helium shields. You had to put multiple shields around-- if you had an optical [INAUDIBLE], you had

a window of liquid nitrogen temperature. One window at helium temperature. Just to make sure that the black-body radiation is absorbed and blocked.

Because it would've been absolutely detrimental if you had a sample at very low temperature and it would've been exposed to black-body radiation.

So it's really a unique feature of the atoms that they are, and this is what you will calculate in this week's problem set, that the atoms are almost completely transparent to the black-body radiation.

They only react if the hyperfine frequency or they react far, far, far, far, far off in the tail of the black-body radiation with an electronic transition.

But nevertheless, as Einstein has taught us, and as we know from general principles, this will not mean that the atoms stay cold and are decoupled.

It just means that it takes maybe the age of the universe. I've never calculated the number. It would really take forever until the atoms in the atom trap reach the ambient temperature.

So Einstein's argument was an idealized argument which in practice would never happen. But if you exclude all other processes, you have a consistent system by saying I only have atoms with their kinetic energy.

I have black-body radiation. And everything has to equilibrate. And as I said before, the argument for Einstein's a and b coefficient simply assumes that the ground and excited state population equilibrium.

But you can carry the argument even further and say even the Maxwell-Boltzmann velocity distribution of the atoms has to equilibrate at the ambient temperature. Beautiful argument.

And what you find from this argument is it's really amazing. You find the photon recall is $\hbar k$. Einstein pulled it out simply by making this assumption. I will post the paper on that.

OK, field quantization. We discuss the quantization of the electromagnetic field really from first principles. From vector potential, radiation field, Coulomb gauge, transverse vector potential, in 8422.

So we dedicate one or two classes to just discuss all the steps to have full quantization of the electromagnetic field with all the bells and whistles.

So sometimes when I teach this course, I say, well, you've heard about field quantization, I can refer to that. Or I can refer you to 8422.

But in the end I thought, why don't I just give you a 10 minute derivation. Just sort of focusing on the essential because this makes this cause more self-contained and more complete.

So I give you now a ten minute quantization of the electromagnetic field. Pretty much going straight to showing you electromagnetic field isn't harmonic oscillator.

And now let's use the quantum description of the harmonic oscillator. And then we have a quantum description of the electromagnetic field.

So this is not rigorous, but it is logically complete. So we focus in the discussion of the quantization of the electromagnetic field, we focus just on a single mode of the electromagnetic field.

Each mode will be an harmonic oscillator. And then we have many harmonic oscillators comprising the electromagnetic field.

So a even single mode, we assume that we have plain waves with a polarization, with an amplitude.

The electric field is the derivative of the vector potential. And the shortest way to show you an analogy with the harmonic oscillator is to remind you that the total energy-- which is actually if you're wondering about a factor of 2, the electric and the magnetic part, the total energy is quadratic in the amplitude of the vector potential.

By the way, there is a factor of $1/2$ because if you have a sinusoidal variation, you take the time average, cosine squared, which is $1/2$.

Well, if the total energy is quadratic in the amplitude, this immediately allows us to draw analogies to an harmonic oscillator.

And we can use the vector potential of the single mode of the electromagnetic field to define two quantities, q and p . Let me write it in that way.

Ωq , plus ip , is related to a in the following way. And yes, I was just ranting about this.

v is the volume. We assume everything happens in a finite volume of space. Value would say I have two new quantities, q and p . So I need two equations. And the two equations involved a and a complex conjugate.

So now we had an expression for the total energy in terms of the amplitude of the vector potential.

So now I can rewrite it the amplitude square of the vector potential is a times a^* . And with that I get the total energy to be proportional to q square plus p square.

And that should remind you, and if everything was set up to remind you, that this looks like a harmonic oscillator where q is the position variable. And p is the momentum variable.

So now, I mean all this is classical. All this is just clever definitions. But now we have to do a leap to quantum physics. We cannot logically derive it. We have to make a leap.

And the leap is that we postulate that this should now be described as a quantum harmonic oscillator. And this transition is done by simply postulating that the two quantities we have defined fulfill the canonical commutator for position and momentum.

So we've started with the vector potential, expressed the energy as a vector

potential, and now we say we recognize through those definitions that this is an harmonic oscillator with variables q and p which are defined in terms of the vector potential.

So then you know if you have the quantized harmonic oscillator, you immediately introduce creation and annihilation operators. Which are linear superpositions of q and p in the following form.

And a dagger has a minus sign here. And all the prefactors were cleverly set up in such a way that the commutator of a and a^\dagger is 1.

And now we can do all the substitutions. We can express p and q by a and a^\dagger . But p and q were related to the vector potential, a . And the vector potential a square defined the energy.

So now we have an expression for the energy which is no longer involving a or p and q . It involves a and a^\dagger . And surprise, surprise, we find that our total energy because we have operators now has become a Hamiltonian, Has this well-known result with the photon number operator $a^\dagger a$, $a^\dagger a$ plus $1/2$.

So this is sort of the quickest way which takes us in a few minutes to the quantized electromagnetic field. Of course, all I need is to come back to spontaneous emission stimulated emission are the matrix elements of this operator's a and a^\dagger .

And this is where, of course, stimulated and spontaneous emission-- all that comes in. The non-vanishing matrix elements in this description of the electromagnetic field are the ones where a annihilates a photon and the matrix element is square root n .

Or where a^\dagger creates a photon, adds a photon to n photons already present. And then the matrix element is n plus 1.

OK, so we went from a to q . And p . And we went to a and a^\dagger . But a is also related to the electric field by taking the time derivative of the vector potential.

So now, of course, we can go from our expressions of a and a^\dagger all the way back. Just substitute, substitute, substitute. And find an expression for the electric

field in terms of a and a^\dagger .

The result is that we have a and here we have a^\dagger . We have a polarization vector. We have the plain wave vector. And the complex conjugate.

And the complex conjugate of a is a^\dagger so the electric field is a superposition of a and a^\dagger . The electric field becomes an operator which is the sum of creation-annihilation operator.

So with that, we can go back to our Hamiltonian. Our Hamiltonian for the interaction between light and atoms in the simplest possible case was the dipole Hamiltonian.

Which involves a dipole matrix element. The charge of the electron is negative. That's why the minus sign has disappeared.

And now all we do is from our treatment before in the Schrodinger equation where the electric field was an external field. Now the electric field becomes the operator acting on the quantum state of the electromagnetic field.

So by the way, this prefactor here is because the rest of it is just dimensionless. This prefactor has, so we have the matrix element here, this prefactor is an electric field.

And it's something you should always know. This electric field is actually the electric field of a single photon. This is the correct normalization.

If you want to factor out the volume, the frequency, and all that, you combine these factors in such a way that it's electric field of a single photon. Then we have the dipole moment.

And then we have an expression with creation and annihilation operator over here. I assume now that the atom sits at r equals 0. So why should I carry forward an e to the ikr term.

We conveniently place the atoms at r equals 0. But I have to say a word or two about the e to the $i\omega t$ factor. I have been deliberately cavalier about my

formulation in quantum physics, of quantum mechanics, whether I use the Schroedinger or the Heisenberg picture.

And you know in the Schroedinger picture the wave function is time dependent, not the operator. In the Heisenberg picture, it's the other way around.

And I have to tell you every time I do a calculation and look at it, I'm getting confused about the two pictures.

So anyway, trust me that in this case when I want to discuss the Schroedinger picture, the time dependent factor should not be present. But you really have to look at the derivation and carefully realize the two are connected with a unitary transformation.

You really have to figure out in which [INAUDIBLE] presentation you are. But I want to not focus on the formality here. But I'm not carrying forward this factor because I want to discuss the Schroedinger picture.

OK. Yes. So now we can look at the matrix elements of our interaction Hamiltonian. And just to be clear, we have written down this Hamiltonian for just a single mode of the radiation field.

Depending on what we are interested in, we may have to sum over many, many modes.

So we are looking at transitions from an initial state which may be an excited state. To a final state which may be a ground state. And since we have quantized the magnetic fields, we also have to specify the state of the quantum field.

And we assume that the uncoupled Hamiltonian, of course, has simply number states as eigenstates [INAUDIBLE] photons and prime photons.

So the only non-vanishing matrix elements are the following. e is a charge. \hat{e} is the polarization. ϵ_1 is the electric field of a single photon.

And, of course, we only have a coupling by the fully quantized Hamiltonian when we

have a dipole matrix element connecting state a and b . I mean these are all sort of things we have already discussed in another context.

But now the a 's and a daggers which only act on the photon field, give rise value to two couplings. One is absorption and one is emission.

Absorption takes place when we look at the matrix element when the final state has one more photon. And emission takes the other way around. When the final state has one more, which way do we go? Let me just write it down. And then read it off.

I think I've inverted, but anyway, initial and final state can differ by plus one photon or minus one photon. In one case it's absorption, the other case it's emission.

And the matrix element is n or $n + 1$. So one is absorption. And one is emission.

So finally if we ask, what are the rates of absorption and emission when we assume we have a situation where-- and we have now discussed the matrix element and this matrix element could become the basis of Fermi's Golden Rule.

We just have to specify time dependent perturbation theory. But in any case, whatever we do when we talk about the rate, it will involve the matrix elements squared.

So now we can ask what happens when we couple ground and excited states. And let's assume we have an excited state. And we sum over all possible photon occupation numbers of the ground state.

Well, when we go from the excited state to the ground state, there will be only one term contributing to the sum. Where we have one photon more because it has been emitted.

So, therefore, because of the square root n and $n + 1$ dependence of the matrix element, we find that for the processes where photon is emitted, where the atomic system gives away a photon, the sum of all the possible rates becomes simply $n + 1$. And in the case of absorption it becomes n .

So in other words, we have now done the field quantization what Einstein pulled out of a thermodynamic equilibrium argument. Namely that if you have a system that the rate of emission versus the rate of absorption is $n + 1$ over n .

But we did not assume any spectral distribution. We know this $n + 1$ over n applies to every single mode of the electromagnetic field. Questions about that?

I also want to tell you, just as a side remark, a lot of people think that when emission is $n + 1$, the plus 1 is different from n . That this plus 1 is sort of a spontaneously emitted photon which has maybe some random phase.

And the n which is stimulated photons, they go in the same mode as they joined sort of the identical to the photons already present. I don't see any of that in that treatment.

So a spontaneously emitted photon is identical to the photon which would be emitted in a stimulated way. You just have $n + 1$. This is the matrix element for coupling to this mode.

At some point, spontaneous emission can happen in many modes. And if it goes to many modes, then there is some integral or some summation involved. And this can cause a certain randomness.

But at the level of a single mode, I do not see any difference between the one photon and the n photons at this level of discussion. Just keep that in mind.

And actually, we'll discuss micromasers. You can have put an excited atom in the cavity. And you have a fully reversible exchange. You spontaneously emit, you absorb. You spontaneously emit, you absorb.

You have Rabi oscillations which involve a single photon. And they involve spontaneous emission. Fully reversible. Completely [INAUDIBLE] evolution.

So we have 10 minutes left. Yes, I think this is just enough to derive for you. Now to derive for you using the fully quantized picture.

To derive from first principles microscopically an expression for Einstein's coefficient.

So in other words, what I'm doing now is I really directly calculate for you the rate of spontaneous emission. And I'm not getting it through the back door by treating absorption and then saying, well, there's n and $n + 1$. Or borrowing some argument from Einstein.

It's such an important quantity, we should just hit the system with a Hamiltonian and out comes a spontaneous emission rate. And this is what we're doing.

So the starting point is what we have discussed at the beginning of the class. We want to discuss Fermi's Golden Rule. We want to use the rate. And to remind you the rate for process is the matrix element squared by h plus square.

And then we have to multiply with the density of states. So this is the density of states. Pair polarization.

Actually, I made a few corrections to my notes because I realize I have to be very, very careful in telling you what the states are. Because this is what this exercise is about.

And we are writing it down for one mode by mode. So the density of state is now pair polarization. We take care of polarizations later. Per unit frequency interval. Yes.

So this rate, but now I have to add one caveat. I was just thinking how I should express it. This rate, if it's all of spontaneous emission, is the Einstein coefficient.

But there is one caveat. And this is the emission of an atom which has a dipole moment is not isotropic. So I have to be a little bit more careful with the solid angle.

I cannot just calculate a rate and assume everything is isotropic. If I would do that, I would save a few minutes. But I would have really swept something under the rug.

So what I'm calculating first is the rate in a given unit angle. And then I do an integration over the unit angle. And eventually I will integrate over the dipole

[INAUDIBLE].

So, therefore, the density of the photon states is sort of photons with their k vectors go into all space, but I wanted to have the density of states per unit angle.

And this quantity is ω^2 , $8\pi^3$, c^3 , times v . And, of course, if you multiply this by 4π , you get your normal density of states. Because the density of states is isotropic.

But the rate which we calculate will not be isotropic because of the dipole matrix element. And the dipole pattern. So, therefore, we start with a differential formulation. Spontaneous emission per solid angle.

And then when we do, when we integrate over the solid angle, we have to take care of a sine square factor because of the dipole pattern. Good.

Now Fermi's Golden Rule takes us from an excited state to ground state. And since we use the fully quantized treatment, our product states of atomic states and photon states.

And so we assume we start with an atom in the excited states. And all modes, mode 1, mode 2, mode 3, are empty. And the final state, well, one photon is emitted and it can appear in any of the modes. And we have to do an integral of all possibilities.

Good. So we did all the work with quantizing the electromagnetic field. Because we want to calculate those matrix elements. Let me just carry over the prefactors, the electric field of a single photon.

Here we have the dipole moment between the polarization and the atomic dipole matrix element. And now since we are talking about an emission problem, we have from the matrix elements squared as we just discussed, an $n + 1$ factor.

But the population we start with 0 photons. So, therefore, it's just one.

So all the work we did on quantization of the electromagnetic field is that even without any photon present, we have a coupling. You can say it's a coupling caused

by the vacuum which is like the coupling we would have gotten if we have exactly one photon per mode.

So let me just write that out. Yes. I think we can finish that.

OK, we have now taken care of the matrix element. So we insert the matrix element now in our Fermi's Golden Rule expression for the ad omega.

Let me just keep track of all the factors. [INAUDIBLE] omega in the matrix element. This comes from the electric field of a single photon.

We have a matrix element. Dipole matrix element times polarization. And the density of states gives us omega square. So see already we'll get a spontaneous emission omega cube expression.

One omega comes because the electric field of a single photon, the electric field squared of a single photon is proportional to omega. And an omega square factor comes from the density of states.

It's really important to keep that apart. The omega³ dependence has two different sources.

It's always nice to see that we assumed an [INAUDIBLE] volume and it cancels out. OK. Let me just write it down and then we do the final step.

So if everything were isotropic, I could just multiply with 4 pi. And the last factor would be dropped. But if you want to go from the spontaneous emission per solid angle to the total spontaneous emission, we have to average it.

And what has angular factors is actually the projection between the atomic dipole moment and the polarization.

So this is the relevant term. And now we have to distinguish. There are two polarizations. One polarization, the polarization when we have a dipole moment.

And you have light which is polarized in such a way that the light goes here.

This is a dipole moment, the light goes here. And now the light which propagates here can have a polarization like this. Which has a projection of sine theta, with the dipole moment.

And if the light goes there and the polarization is like this, it's orthogonal to the dipole moment. So the scalar product is 0.

So for one polarization, we have a sine theta factor. For the second polarization, the scalar product for the dipole moment is 0.

So, therefore, and that's the last conclusion I want to draw today. Is this integration over the solid angle boils down to that we can pull the matrix element out of the integral.

And what is left is the projection factor, sine square theta. We have to integrate over the whole solid angle. And this gives us 2/3.

So, therefore, our final result is that the microscopic expression for the a coefficient has its factor of 3 in the denominator. And this factor of 3 only comes because I correct the average over the dipole pattern.

Well, then we have $4\pi\epsilon_0$. I mentioned the important dependence on the frequency cubed. OK, so this is our final result for today. And I will discuss next week what are its units? How big is it? What is the quality factor of the atomic oscillator?

But we can start next week with this result. All right.