PROFESSOR: All right. So that concludes the chapter, a big chapter in this course. It concludes time-dependent perturbation theory. The whole course is largely organized with approximation methods. We have done time-independent perturbation theory. We applied it to the hydrogen atom in detail. Then we did WKB, which is slowly varying things in space.

Then we did time-dependent perturbation theory. It had many parts to it. It had Fermi's golden rule, atoms. It had just transitions in general.

The next step would be adiabatic approximation, which is time-dependent changes but slow. This is the time analog of WKB. We're going to take a little break. And for half a lecture today and the full lecture next time, we're going to do particles in electromagnetic fields. That's an important subject.

It will involve Landau levels and things that you always hear about. And it will be useful for some of the things we've done. We've had a little exposure to it with the Pauli equation. You may remember the Dirac and Pauli equation that had this electromagnetic field.

So now we're going to take seriously the electromagnetic fields and study how they interact with quantum mechanics. And there's a lot of very interesting and beautiful ideas having to do with gauge invariance and why potentials are important.

Charged particles in electromagnetic fields. So charged particles in EM fields. So what is our useful problem? You have a particle with mass m, now charge q. And finally, there's an electric field and a magnetic field. And there's going to be interactions between them. So there's E and B.

And you've gone through this transition. Probably, you've been told in electromagnetism something along these lines. There is this E and B fields. And those are the physical fields. These are the things that do stuff on charges and particles.

And there is this mathematical entities that can help you, which are potentials, that allow you to rewrite these fields in terms of these quantities, potentials. And they're practical to solve the equation. But the potentials are less physical than the fields, because the potentials are ambiguous. You can change the value of the potential without changing the electric field. You can do things like that.

Well, that is somewhat misleading, put charitably. It's a pretty wrong viewpoint, actually. As we will discover and understand here, the potentials are really more important than the fields.

Quantum mechanics couples to the potentials. And as an afterthought, if couples to the fields, because it couples to the potentials. But even the definition of an electromagnetic field, at the end of the day-- nowadays, we really understand it as a definition of potentials. And I'll try to explain that.

This is pretty important if you try to describe anything that has topological content. Most of the times, if you're considering open space, open Minkowsky space, it doesn't make much difference whether you take the electric and magnetic field to be fundamental or the potentials. But as soon as you consider anything topological, a sample that lifts on a toroidal surface, that's a topological space, then, if you don't think about potentials, you could be absolutely wrong, completely wrong. Or if you have an extra dimension, and in many theories of physics, you could have extra dimensions-- and you think about the electric and magnetic fields then, if you don't think of potentials, you can be quite wrong.

So let's explain what's happening with these potentials. Basically, it all starts from the idea that you have a Maxwell's equation del dot B equal 0. And this is solved, it is said, by setting B equal to the curl of A. And indeed, if B is equal to the curl of A, this equation is satisfied.

The other equation that is relevant is this Faraday's law, which if B is already curl of A, you have that curl of E plus 1 over C da/dt is equal to 0. You see, I substitute B here equal curl of A. I commute the order of derivatives, the dt the curl. And this gives me this equation.

From this equation, we say that anything that has 0 curl is the gradient of something. And therefore, we say that E plus the 1 over c da/dt is the gradient of phi, from where we find the second equation, which says that E can be obtained as minus the gradient of phi minus 1 over c da/dt. I hope I have it right. Yes.

So this is the origin of the potentials. You've seen that. That's how they explain, probably, to you why you have potentials. And indeed, this is so far so good. But now, there is a freedom with this potentials which are called gauge transformations.

So what are the gauge transformations? It's the possibility of changing the potentials without changing the electromagnetic fields. So for example, since the curl of the grad of anything is 0-- it's almost like curl cross curl is 0. This is the curl of the grad of anything is 0. You can change A into a different A that we'll call A prime, given by A plus gradient of a function lambda.

And this will not change the value of the magnetic field B, because if you calculate the magnetic field B, the new magnetic field associated to the new vector potential is equal to curl of A prime. But that's equal to curl of A. And that's equal to the old B due to the old magnetic vector potential. So the vector potential has changed, but B the not change.

On the other hand, if you change the vector potential by something like this, now it can affect the E. But if simultaneously you change phi, E will be left unchanged. So what should you do to phi? So this is the first one for A.

If you want to keep E unchanged, you define phi prime equal to phi minus 1 over c d lambda/dt. And then when you compute the new E, you would do the gradient of the new phi, which will be the gradient of the old phi, but an extra term, which would be plus 1 over C d/dt of gradient of lambda. And it will cancel with the minus 1 over C d/dt of the gradient of lambda.

So by changing A and phi in this way, B and E are unchanged. So I can summarize this gauge invariance statement by saying that the new electric field due to due to phi prime and A prime is equal to the electric field due to phi and A. And the magnetic field new due to phi prime and A prime is equal to the magnetic field due to phi and A.

This is lifted to a principal. So we started describing electromagnetic fields with E and B. We can describe them with A and phi. And let's take that seriously, because we'll need it for quantum mechanics. And we'll add now the extra important physical assumption, which is that phi prime A prime is really equivalent, physically equivalent to phi A.

And an electromagnetic field configuration, as a mathematician will say, is the equivalent classes of potentials. So you want to describe an electromagnetic field? I ask one of you. And you tell me what is phi and what is A. I ask somebody else, they give me another phi and another A. And you say, they look different. But they may be the same. They are the same if they are related by some lambda. I'll write it. If there exists a lambda such that they are gauge transforms of each other.

So before you say, you tell me what the E is and what B is, and they have to agree. And you say, that's preferable, because if E and B are the same, they agree. That's it. Why do we go through A and phi? They don't have to agree. And they can be the same, because if they're related with some lambda, they are the same.

We seem to be going backwards into a lot more complicated situation. But that's what we need. And that's what the physics tells you is really going on. So some particular things that are curious can happen now. And they're very curious. And there are examples. And we'll see some of those examples.

So curious effects. 1, suppose phi and A and phi prime and A prime give the same E and B. So you and your peer editor came up with some phi and A and phi prime and A prime. And they give the same E and B. Are you and your peer editor on the hands of the same electromagnetic field configuration? Maybe but not obvious.

In classical physics, you would say, yes. If they give you the same E and B, these are equivalent. No, now, you would have to show that there is some lambda so that they are gauge equivalent to each other. You see, the fact that they are gauge equivalent guarantees they give the same B and E. But if they give the same B and E, it may happen that you still cannot find the lambda that relates them, because what you have to do is try to take your friend's A and your friend's phi, then this is yours and yours, and now you have to find the lambda. And you have to solve a differential equation.

What if you can't find the lambda? They give the same E and B, but they're not gauge equivalent to each other. That means that you have different field configurations. The fact that E and B are the same doesn't mean they are the same electromagnetic fields. And this can happen if you have a little bit of topology.

So suppose A and this give the same B field but are not related by a gauge transformation. Then these are inequivalent EM fields. They could have different quantum mechanics. And we will see that.

So one example is what people call Wilson loops. Wilson loops correspond to some closed curve in space that you cannot contract. Of course, in Minkowski space, in this room, I can contract anything. But sometimes, if you live on a torus or your sample is on a sphere or something, you may not be able to contract the curve.

Along that curve, you can put a vector potential that is pointing tangent to the curve all the time. And it's a constant, a constant vector potential. That constant vector potential has 0 B and 0 electric field. But if that constant is small or that constant is big, it's inequivalent. You won't find the gauge transformation that transforms each other. And therefore, those would be configurations of electromagnetic fields that give you the same E and B but are inequivalent. So that's the first curious effect.

There is another curious effect. And quantum mechanics is forcing you to do that. That's the main thing.

So the second curious effect is, given an E and B that satisfy Maxwell's equation, are these allowed fields? So somebody gives you an E field and a B field. And you say, OK, I'm going to check if these are possible physical electric and magnetic fields. And then you go and check Maxwell's equations.

And Maxwell's equations work out. They're all satisfied. You say, OK, this is good. Those electric and magnetic fields are fine.

Not so quick. To be sure those electric and magnetic fields are fine, you should give me a phi and an A that gives rise to them, because fundamentally, the electromagnetic fields are A and phi. And those you need anyway to do quantum mechanics with it. So if you have an electromagnetic field and you cannot do quantum mechanics with it, you would be suspicious. So you need to find phi and A.

Are these allowed? Only if you can find a phi and A. And sometimes that can fail. On a torus again-- we've described in this course as circle as a line x identified with x plus L. So here is L. And you identify.

A torus can be done by taking a piece of the plane and identifying this line with that and this line with this line. So you glue this, form a cylinder, and then you glue the other ends. On a torus, you can put a constant magnetic field. A constant magnetic fields satisfies Maxwell's equations.

So you say, oh, so you can put any constant magnetic field in a torus. Wrong. The vector potential has trouble existing. And only for particular values of the magnetic field, there's a consistent vector potential. You get the quantization of the flux of the magnetic field. So that's another example, an E and a B.

A constant magnetic field on a torus satisfies every Maxwell equation you may want to do. But still, it's not a valid electromagnetic field, because there's no A and phi that give rise to it.

So those are our lessons in why we need the vector potentials to describe physics in quantum mechanics, electromagnetic physics. And then let's write, therefore, the Schrodinger equation with the electromagnetic fields.