PROFESSOR: We have to set up a little better the geometry of the calculation. And for that we have to think of various angles. We oriented there the electric field along the z direction. But that's not going to be too convenient for our calculation.

So these calculations are a bit of an art to do them. They're not that trivial. Not terribly difficult. But I think if you appreciate it, next time you ever have to do one of these things it will become clear.

So first your have to think physically. Is there an angle in this problem? Is there any important angle happening here? It's a question to you. I think if you figure out that you have a chance of doing a diagram that reflects this.

Is there a physical angle, you think, in this process? A relevant angle? Yes, Lou?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Perfect. Yeah. The relevant angle is you have a directional ready for the electric field. So if the electron comes off there's going to be an angle with respect to that electric field. So that's our physical angle in this question.

So let's try to draw this in a way. So I will draw it this way. And I brought colored chalk for this. I'll put the z-axis here, and I'll put the electron momentum in this direction. k is the lateral momentum.

Now the electric field is going to come at some angle with respect to the-- or the electron momentum is at some angle with respect to the electric field. So that angle will presumably stay there for the rest of the calculation. So let's do it with green.

So here is my electric field. And it's going to have an angle theta with respect to the direction of the electric field or the photon incident direction. Now, this is actually more like the polarization of the photon.

The electric field is the direction of the polarization of the photon. So one more vector, however, the position that we have to integrate over-- because we have the whole hydrogen atom and the whole of space to integrate. So x has a position.

So I'll use another color here. Unfortunately, the colors are not that different. Here is r, the vector r. So now I have several angles. I have theta for the electric field. But now that I've put all this axis I not only have theta, but I also have phi for the electric field. And for r, I will have theta prime and phi prime.

So r has a theta prime and a phi prime. We usually have theta and phi. But the answers, at the end of the day, are going to depend on theta. And if you have theta here, we want the theta to remain. So theta prime and phi prime are going to be our variables of integration because you integrate over r.

Finally, we have one more angle that we have to define. So we have theta phi, theta prime, phi prime, and the angle that was here-- this was the angle between r and the direction of the electric field. And we called it theta.

So we have to give it a new name here. And this angle, I'll call it gamma here. The angle between E and r. OK. So We have everything defined here. Maybe I should list it.

E has angles theta and phi, the direction of E. r has angles theta prime and phi prime. And gamma is the angle between E and r. And k along z hat. So this is our situation.

So what are we trying to calculate? Well, we want to use Fermi's golden rule. So we need to calculate final H prime initial. That's the matrix element of the Hamiltonian H prime between the final and initial states.

So what is this? Well, these are all wave functions that depend all over space, and this is a function of space. So let's do it. This is integral d cubed x. Let's put the final state first because it shows up there. So it's 1 over L cubed e to the minus i k dot r. So here is our final state, u final.

Then the Hamiltonian. That's simple. e E0 r cosine what? Cosine gamma. Is that right? This is what we had there. It's the angle between the electric field and r. It was theta to begin but now has become gamma.

And then the final state. So this is our H prime. And then the initial state is 1 over pi a0 cubed e to the minus r over a0. OK. This is our task. This is a matrix, and here it is. An integral of a plane wave against an electron wave function and an extra r dependence here.

This could range from undoable to difficult, basically. And happily, it's just a little difficult. But this is an integral-- we'll see what are the challenges on this integral. So let's take a few constants out.

So e, E0, pi, a0, that all will go out. So this is e E0 over square root of pi l cubed a0 cubed. So I took the l's out, the e, E, this thing. All right. Let's write the integral more explicitly.

This is r squared dr sine the volume element. Sine theta d theta d phi. But I'm integrating over x, which is integrating over r. And r is theta prime and phi prime. So these are all these ones. I'm integrating over all values of theta prime and phi prime.

Then what do I have? I have this exponent, k dot r. Well, my diagram shows how k dot r is easy. It involves a cosine of theta prime. So it's e to the minus ik, the magnitude of k, the magnitude of r cosine theta prime, because after all, k was along the z-axis. r is along the phi theta prime direction.

OK. We're progressing. This, this, that term, it's r cosine gamma. And the final term is not that difficult. e to the minus r over a0. That's what we have to do. An integral over phi prime, theta prime, and r. This is our challenge.

And the reason it's a challenge is the cosine gamma because this gamma is the angle. It depends on theta, depends on theta prime, depends on phi, phi prime. That's the problem. If we can solve that cosine gamma thing we can do this integral.

And I think even if you were doing it numerically, that cosine gamma there is a little bit of a headache. You don't want to do an integral that you can really do like this numerically. So you really want to do it.

So what we need is to calculate. So we can begin by saying, I'm going to calculate what cosine gamma is. And here is the way you can calculate cosine gamma. When you have two unit vectors, cosine of the angle between two unit vectors is just the dot product of those two unit vectors.

So for gamma, we can consider a unit vector along e and a unit vector along r and take the dot product. And remember, for an arbitrary unit vector, it's sine theta cos phi sine theta sine phi cos theta. This is the theta phi decomposition of an arbitrary unit vector.

So cosine gamma is the dot product of a vector n along the e times a vector n along r. And vector n along e and r are just with theta in one case and phi and theta prime and phi prime.

So when I make the dot product, I get sine theta, cos phi, sine theta prime, cos phi prime. That is the product of the x components. Plus sine theta, sine phi prime times sine theta and sine phi. Sine theta prime sine phi prime, plus cos theta, cos theta prime.

OK. Doesn't look much easier, but at least it's explicit. But it's actually much easier. And why is that? Because there's a lot of factors in common here. In fact, sine theta and sine theta prime are in both.

So what you get here is cosine gamma is equal to sine theta sine theta prime. And then you have cos phi, cos phi prime, plus sine phi, sine phi prime. And that's cosine of phi minus phi prime plus cos theta cos theta prime. OK. So that's cosine gamma.

Now suppose you were to put this whole thing in here. It's a big mess but all quantities that we know. But there's one nice thing, though-- a very nice thing happening. Think of the interval over d phi prime. This does not depend on phi prime. This does not depend on phi prime. But cosine gamma can depend on phi prime.

But here it has this thing, cosine of phi minus phi prime. So when you try to do the integral of this phi prime with this term, this term will give, eventually, the integral of d phi prime times cosine of phi minus phi prime. There is a lot of messy things.

But this integral is 0 because you are averaging over a full term. So happily, all this term will not contribute. And that's what makes the integral doable.

So what do we have then? Our whole matrix element, f H prime i, has become e E0 over square root of pi l cube a0 cube. And now you just have to integrate. This is the only term that contributes.

And when you put this term, for sure you can now do the d phi prime integral because that term is phi dependent. So that integral gives you just a factor of 2 pi from the integral of d phi.

And from this thing, cosine theta is the angle between the electric field and k. So it's a constant for your integral. You're integrating over theta prime and phi prime. So cosine theta also goes out.

And here is the integral that remains. r cube dr. It was r squared, but there was an extra r from the perturbation. r cubed dr into the minus a over r-- r over a0. And this thing you can pass two cosine variables. It's pretty useful.

So this goes minus 1 to 1 d cos theta prime, cos theta prime e to the minus ikr cos theta prime. This cos theta prime came from here. And that's it.

So this is a nice result. It looks still difficult, but we've made great progress. And in fact, we've dealt with a really difficult part of this problem, which is orienting yourself of how you're going to approach the matrix element.

So to finish up I'll just give you a little more of the answer. We'll complete the discussion. We need probably 15 more minutes to finish it up. So the only thing I'm going to say now is that if you look at the notes, every integral here is easily doable.

So basically, there's two integrals. And the way to do them is first do the r integral. You will have to have these terms. And then do the theta integral. And they are kind of simple, both of them.

You have to keep up a lot of constants. But here is the answer for the matrix element. So that part of the integral I think you all can do. But you have to take your time. i 32 square root of pi e E0 a0. I did a lot of work here, actually, in writing it in a comprehensible way because it's pretty messy.

I cube a0 cube, 1 over 1 plus k squared a0 squared cube cosine theta. By now, it starts to simplify a little. OK. That was actually plenty of work to get it to write it this way. I feel pretty happy about that writing.

Why? First, OK, there is these numbers. Nothing I can do about it. But there's a multitude of constants that I have simplified and done all kinds of things. But it was not worth it.

First, ka0, that is unit free. This is unit free. This is unit free. k is 1 over length. ka0 to the 4 is the length cube, and here is the length cube. No units here either.

Here, nice units. This is units of energy. Why? Electric field times distance is potential, times electric charge is energy. Energy. So this is how this should be. The matrix element of an energy between normalizable states should be an energy. And that has become clear here.

The next steps that we have to do, which we'll do next time, is to integrate over states, and put in the density of states, and do a little simplification. But now it's all trivial. We don't really have to integrate anymore because Fermi's golden rule did the job for us.