PROFESSOR: So we continue today our study of electromagnetic fields, and quantum mechanics, and particles in those electromagnetic fields.

So last time, we described what we must do in order to couple a particle to an electromagnetic field. And the rule was oddly simple. You get the Schrodinger equation, and the Hamiltonian is now changed into this form.

If you had a free particle, you're accustomed to have p squared over 2m. That has changed, and you have a new term, with p, minus q over c, A squared.

Thank you.

So now our task is to understand how this is compatible, and what are the implications of these changes. They're pretty significant changes. This expression, p, minus q over c, A, is what used to be just p. And that used to be just the kinetic energy. And now it looks a little more strange, in fact.

So in terms of electromagnetic potentials, we know that the potentials are not unique. There are this gauge transformations that establish that these are completely equivalent potentials-- so two potentials, A and phi, and A prime and phi prime, are physically equivalent. They define the same electromagnetic field if they are related in this way.

We then mention that therefore you have an issue with the Schrodinger equation. You would want the physics to remain invariant in their gauge transformations. So you could write the Schrodinger equation for the new gauge potentials, with A prime and phi prime. And you could compare with the Schrodinger equation with the old potentials, A and phi.

And you could ask, OK, is it solved by the same wave function? And the answer is no. The wave function must change. But there is a way to get one way function from the other. Here is the formula. Psi prime and psi are related by this factor, this function, U, which involves exponential of the gauge parameter, multiplied by a couple of physical constants, including the charge, q, of the particle.

Now we're going to use the notation, q, all the time. Many books put E in there. And that's the charge of the electron. And then you always wonder, is that E, or minus E, or which sign is it. So here, q is the charge of the particle. If you have an electron, q is minus E. But let's leave it q open there, so you can work with arbitrary charges in any circumstance.

So this Hamiltonian is what we have to understand. They want to make a couple more comments about what this Hamiltonian is. And we must think a little about the gauge invariance as well. So there it is. A Hamiltonian contains this term squared. So let's just ask ourselves, what does that term really imply. Well, there's 1 over 2m. So for the Hamiltonian, 1 over 2m. And we have this factor squared.

So following the careful things we're accustomed to do in quantum mechanics, I would put P squared here, minus q over c, P dot A, minus q over c, A dot P, plus q squared over c squared, A squared, plus q phi. That's how the Hamiltonian looks.

But you have to be careful. Here, this term we kind of understand what it is. It's vector potential squared-- no problem. Here is p squared. We're accustomed to that. That's a Lapacian. We have two terms that could be a little strange. A dot P, probably is all clear. A is dotted with a vector momentum, which is an operator. It's the gradient. So that's going to differentiate whatever is to the right.

The question is, what this P dot A? And P dot A must be thought in the operator way, that P acts on everything to the right, including A. So if this Hamiltonian is to act in a wave function, that doesn't mean that the P operator just acts on A. It acts on A and everything to the right.

So P dot A, if you think of it as a derivative, is h bar over i, divergence of A, plus A dot P. You see, this P is h bar over i gradient, but it's acting on everything. So you could put up psi to the right of here, and it would act on everything there.

So maybe in order to justify this equation, you should just write h bar over i gradient acting on A times a wave function, psi. And go through this and see that at the end of the day, this P dot with A means act on A, but then you still have to act on everything to the right. So that's this term here.

So this term is pretty important. Sometimes your A satisfies what is called the Coulomb gauge, in which delta of A is zero, but in general it doesn't.

So this helps clarify what the Hamiltonian really is. It would be a mistake to say that these two terms are the same. And it would be also a mistake to say that this thing is just the divergence of A. Both are wrong things.

So the whole Hamiltonian, now, is P squared over 2m. Then you have twice-- well, let's have this term. So you have plus, the i goes to the numerator, as i h bar q, over 2m c, divergence of A. And that's just the divergence of A, the function. It's not any more a differential operator. It's just acted on it. Then you have this term twice. So minus q over mc, A dot P, plus this term, q squared over 2mc squared, A squared, plus q phi.

So that's the Hamiltonian. Let me just make sure I got it right-- P squared over 2m, plus [INAUDIBLE], minus qAP-- OK, so that's right.

OK, so this is our Hamiltonian. If you just want to write it very explicitly. It's not generally all that useful to have the explicit form. But in some examples-- there will be at least one example where it is nice to know. And in fact, the H formula on the top of the blackboard hides, in a nice way, a little bit the complexity of this whole coupling.

Yes.

AUDIENCE: Is this for scalar particles?

PROFESSOR: Yes, this is for a scalar particle.

So we're taking a particle, at this moment, without spin. For spin particles, there would be a little extra term sometimes.

So let's say a couple more words about this thing, in particular the gauge invariance.

So we now have seen what the Hamiltonian is. Let's think of the gauge invariance. How could you establish this gauge invariance?

So an identity that is very useful takes the following form. This differential operator-- h bar over i gradient, the p, minus q over c, A prime, times U-- and you could put the psi here if you wish-- is in fact equal to U, h bar over i gradient, minus q over c, A, psi.

So I call this a very remarkable identity. And look what's happening here. This is useful for this kind of equation. If you have a psi prime which is U psi, the factor U interacts very nicely with this operator. In fact, as you move U from the right to the left, the gauge potential goes from A prime to A.

A rather nice thing about this derivative-- it's as if this derivative had a very special symmetry property that U can be moved across, almost as if U was a constant. But of course it's not. But when you move it across, the only effect is to change the A prime to A. So a gauge transformation on the A.

That is the reason this derivative, equipped with an extra term, is sometimes called a gauge covariant derivative. That is, it transforms nicely under gauge transformations. It does a nice job.

I write this equation because, in part of the exercises, you will be asked to show that this statement about gauge transformations is correct, that if this equation-- top-- holds, the bottom equation holds with the replacements indicated here. And this identity makes the task of proving the equation-- the gauge invariance-- very simple. Because you can imagine this psi prime-- put U psi-- and then getting the U out, so the psi prime just will become psi, and the U going through these factors and simplifying very nicely.

So this is a simple equation to show. For example, I can take the left-hand side and see what happens. So the first term, you have a gradient acting on the product of two functions. And that's just the gradient acting on each. So h over i, gradient of U, times psi, plus U, h over i, gradient of psi. That's the first term.

The second term is minus q over c, A, times U psi-- because A prime is A plus gradient of lambda. So you're going to have minus q over c, gradient of lambda, U psi. So I wrote the first line.

Now we can take the gradient of U. U is here. When we take the gradient of U, what do we get? h bar over i, now gradient of U-- you differentiate the exponentials, so you take the gradient of what is in the exponent. So it's iq over h bar c, gradient of lambda, times U itself, because you're differentiating an exponential, times psi.

And then you have all these other terms. I'll couple this term. These two terms here are plus U h bar over i, gradient of psi, minus q over c. U and A commute, because A is a function of x and t, and U is a function of lambda, which is also a function of x and t. So there's no momentum here. These two things commute, so U can be moved to the left-- A psi minus this term.

And if we've done the arithmetic right, the first and last term should cancel, and they do. That i cancels, the h bar cancels, there's q over c times that, and there's minus that term, so these two cancel. And this is precisely the right-hand side. So this covariant derivative is very nice.

There's something about the Schrodinger equation of course that, in a sense, it's all made of covariant derivatives. Let's look at that.

So any version of the Schrodinger equation-- here is the typical version of the Schrodinger equation. So recall that the vector potential in general can be thought of as a four-vector. You may or may not have seen this in the literal dynamics. But it's just like time and x form a four-vector. The scalar potential and the vector potential form a four-vector. So the four-vector with index mu, 0, 1, 2, 3 form this.

And then, if you look at the Schrodinger equation, what do we have? We have i h bar, d psi, dt, minus q phi-- I'm bringing the last term on the right-hand side to the left. So it's minus q times phi, which is minus A0 psi is equal to 1 over 2m, this i h bar over i gradient, minus q over c, A, squared, and psi.

And now look at this. It can be written as follows. Minus c times h over i, d over d of ct, minus q over c, A0, psi, equal the same thing on the right-hand side.

So I've rewritten the left-hand side in a slightly different way, all the terms. So I put an extra c, this d, dct. That canceled the c. The h bar, the i went to the denominator.

And now this all looks like this covariant derivative. Look at this covariant derivative. h over i, d dx, minus q over c, A. And here it is, h over i, d dx0-- because 0 is component of the [INAUDIBLE], minus q over c, A0. So the whole Schrodinger equation is built with this funny derivatives-- d dx minus the vector potential added in the net. These are the covariant derivatives. These are nice operators.

You see, the operator P is always called the canonical momentum-- canonical momentum. And this canonical momentum is a momentum such that x with P, if you put the hat, is i h bar.

But this canonical momentum, P, is not mass times velocity, not at all. This canonical momentum is a little unintuitive. It's the one that generates translation. The one that is mass times velocity is really this whole combination, is mass times velocity. Because if it's mass times velocity, this term, 1 over 2m, the mass squared times velocity squared, that gives you kinetic energy. So we have to be aware that the canonical momentum is not necessarily the simplest, most intuitive object.