

**PROFESSOR:** We are going to recap some of the last results we had on scattering and then push them to the end and complete our discussion of this integral equation and how we approximate scattering with it. It's a pretty powerful method of thinking about scattering and gives you a direct solution, which is in a sense more intuitive perhaps than when you're dealing with partial waves. So let's see where we were last time.

So we set up our solution of the Schrodinger equation in a funny way. We had an incoming wave plus a term that still had an integral over all of space of our Green's function integrated against a scaled version of the potential. Remember that  $v$  was defined to be equal to the potential up to some constants,  $\hbar^2$  over  $m$ .

And we had this integral equation where one could show that  $\phi$  satisfies the correct Schrodinger equation. Now this function  $G$  plus was our Green function that we also discussed. And it was given by the following formula. It just depends on the magnitude of this vector argument of the Green's function, so  $1$  over  $4\pi e$  to the  $ik$ , magnitude of that divided by magnitude of that vector.

So in most applications, we look at the wave function. This should be  $\phi$  of  $r$ . We look at the wave function far away. And if the potential is localized, this integral, since it involves the potential, can only be non-zero, have non-zero contributions to the integral on the region where the potential exists. So  $r'$ , whenever you're doing this integral, must be small compared to  $r$ .

This we can draw here. So assume we have a potential. And you have an observation point,  $r$ . We'll call the unit vector in this direction  $n$ . This is the vector  $r$ . And an integration point is a point,  $r'$  here. And there is this distance, this vector, in between  $r$  minus  $r'$ . And finally, on this graph, we could also have a direction that we could call the incident direction, perhaps the  $z$ -direction. And that's the direction which your wave is incident. This is the wave that were represented here.

So we did a little approximation of this Green's function for the case where  $r'$  where you're looking is much smaller than  $r$ . In that case, the distance,  $r$  minus  $r'$ , is approximately equal to  $r$  minus-- so that's the distance, the vectors. When you don't put an arrow, we mean the distance. So this is  $r$  minus  $n \cdot r'$ . To 0-th approximation, this is

the most important term.

To next approximation comes this term because that term is suppressed with respect to the original term by a factor of  $r' / r$ . So if you factored, for example, the  $r$  here, you would say  $1 - n r' / r$ . So you see that this ratio enters here, which says that the second term is smaller by this factor.

We could even include more terms here. But those would not be relevant unless  $k$  is extremely high energy. So when  $k$  is very large, it would make a difference. But it has to be extremely large. And in general, this approximation isn't enough for any  $k$  for sufficiently far away. So this is really good enough.

And the approximation is done on the Green's function at two levels. You need to be mildly accurate with respect to this quantity. But you need to be more accurate with respect to this phase because the phase can change rather fast depending on the wavelength of the object. So if you make an error in the total distance of 1%, that's not big deal. But if you make an error here, such that  $k \cdot$  times this distance is comparable to 1, the phase could be totally wrong. So the phase is always a more delicate thing.

And therefore, with this approximation, your  $G$  plus of  $r$  minus  $r'$  becomes  $\frac{1}{4\pi r} e^{i k r}$ . So we replace this term by just  $r$ . And in this exponent, the absolute value is replaced by two terms. The first one is  $e^{i k r}$ , is here. The second one is  $e^{-i k n \cdot r'}$ .

We also have here the incident vector. You can write  $e^{i k z}$  as  $e^{i k \text{ incident} \cdot r}$  with  $r$  if  $k \text{ incident}$  is a vector,  $k$ , times the unit vector in the  $z$ -direction. So that's a matter of just notation. Might as well use a vector for  $k \text{ incident}$  because that is really the vector momentum that is coming in. And it's sometimes useful to have the flexibility, perhaps some moment. You don't necessarily want the wave to come in the  $z$ -direction.

There's another vector that is kind of interesting. You're observing the wave in the direction of the vector  $r$ , the direction of the unit vector  $n$ . So it's interesting to call  $k$  an incident. I think people write it with just  $i$ , so  $k_i$  for incident,  $k$ . And then, well, we're looking at the direction  $r$ . So we call the scattering momentum or the scattered momentum,  $k_n$ . You're looking at that distance. So might as well call it that way.

So with these things, we can rewrite the top equation making use of all the things that we've

learned. And it's going to be an important equation,  $\psi$  of  $r$  is equal to  $e$  to the  $ikz$  plus  $f$  of  $\theta$   $\phi$   $e$  to the  $ikr$  over  $r$ . So we have to replace the  $e$  to the  $ikz$  by this.

Then we have the Green's function that has to be integrated. And you have to see, we've decided already to approximate the Green's function. So it's integral over  $r'$ . So the only term here that has an  $r'$  is this exponential. So all of this can go out. So the way I want to make it go out, we will have the minus 1 over  $4\pi$  here and the integral  $d^3r'$ . And I'm going to have the rest of the Green's function placed in the standard position we've used for scattering.

So we're still redoing the top integral, the  $G$  plus, which includes some of  $G$  plus. We've taken care of all of this part. And now we have that other factor. So it's  $e$  to the minus  $ikn$ . I could call it the scattering one. But I will still not do that. Then we have the  $u$  of  $r'$  and the  $\psi$  of  $r'$ . So this is our simplified expression when we make use of the fact that we're looking far enough and the Green's function has simplified.

So let's keep that for a moment and try to think how we could solve this integral equation. Now there's no very simple methods of solving them exactly. You have to do interesting things here. It's fairly nontrivial. But finding approximate solutions of integral equations is something that we can do. And it's relatively simple.

So we will come back to this because in a sense, this is nice. But you must feel that somehow the story is not complete. We want to know the wave function far away. This has the form of this function that when we wrote  $\psi$  is equal to  $e$  to the  $ikz$  plus  $f$  of  $\theta$   $\phi$   $e$  to the  $ikr$  over  $r$ . If we know this  $f$  of  $\theta$   $\phi$ , we know everything about the scattering. But it almost looks like this is  $f$  of  $\theta$  and  $\phi$ . But we don't know  $\psi$ . So have we made any progress?

Well, the Born approximation is the way in which we see that we could turn this into something quite simple. So let's do Born approximation. So our first step is to rewrite once more for us this equation,  $\psi$  of  $r$  is equal to  $e$  to the  $ikr$  plus integral  $d^3q'$   $G$  plus of  $r$  minus  $r'$   $u$  of  $r'$   $\psi$  of  $r'$ .

OK. We wrote it. Good. Now let's rewrite it with instead of  $r$ , putting an  $r'$  here. You'd say, why? Well, bear with me a second. Rewrite with  $r$  replaced by  $r'$ . If we have an  $r'$ , replace it with  $r''$  and so on.

So let's rewrite this. So we have  $\psi$  of  $r'$  is equal to  $e$  to the  $ikr'$  plus integral of  $d$

cubed  $r$  double prime  $G$  plus of  $r$  prime minus  $r$  double prime  $u$  of  $r$  double prime  $\psi$  of  $r$  double prime. So I shifted. Whenever I had an  $r$ , put an  $r$  prime. Whenever I had an  $r$  prime, put an  $r$  double prime.

The reason we do this is that technically, I can now substitute this quantity here into here and see what I get. And I get something quite interesting. I get the beginning of an approximation because I now get  $\psi$  of  $r$  is equal to  $e$  to the  $ikr$  plus  $d$  cubed  $r$  prime  $G$  plus of  $r$  minus  $r$  prime-- I'm still copying the first equation--  $u$  of  $r$  prime.

And here is where the first change happens. Instead of  $\psi$  of  $r$  prime, I'm going to put this whole other thing. So I'll write two terms. The first is just having here,  $e$  to the  $ikr$  prime. The second, I'll write this another line. So integral  $d$  cubed  $r$  prime  $G$  plus of  $r$  minus  $r$  prime  $u$  of  $r$  prime. And then the second term would be another integral,  $d$  cubed  $r$  double prime  $G$  plus of  $r$  prime minus  $r$  double prime  $u$  of  $r$  double prime  $\psi$  of  $r$  double prime.

So look at it and see what has happened. You have postponed the fact that you had an integral equation to a next term. If I would drop here this term, this would not be an integral equation anymore.  $\psi$  is given to this function that we know. Green's function we know. Potential we know. Function we know. It's a solution.

But the equation is still not really solved. We have this term here. And now you could go on with this procedure and replace this  $\psi$  of  $r$  double prime. You could replace it by an  $e$  to the  $ikr$  double prime plus another integral  $d$   $r$  triple prime of  $G$ , like that. And then here you would have an  $e$  to the  $ikr$  double prime and then more terms and more terms.

So I can keep doing this forever until I have this integral,  $G$  plus  $u$  incident wave. And here I have  $G$  plus  $u$   $G$  plus  $u$  incident wave. And then I will have  $G$  plus  $u$ ,  $G$  plus  $u$ ,  $G$  plus  $u$ , incident wave. And it would go on and on and on. So this is called the Born approximation if you stop at some stage and you ignore the last term.

So for example, we could ignore this term, in which we still have an unknown  $\psi$ , and say, OK, we take it this way. And this would be the first Born approximation. You just go up to here. The second Born approximation would be to include this term with  $\psi$  replaced by the incident wave. The third Born approximation would be the next term that would come here.

So let me write the whole Born approximation schematically, whole Born approximation schematically. I could write three terms explicitly or four terms if you wish. But I think the

pattern is more or less clear now. What do you have?  $\Psi(r) = e^{-ikr} + \dots$  and now it becomes schematic-- integral-- so I put just integral--  $G(u) e^{-iku}$ . Don't put labels or anything. But that's the first integral.

And if you had to put back the labels, you would know how to put them. You'd put an integral over some variable. Cannot be  $r$  because that's where you're looking at. So you do an  $r'$ , the  $G(r - r')$ , the  $u$  at  $r'$ , and this function at  $r'$ . That's how this integral would make sense. The next term would be integral  $G(u)$  of another integral of  $G(u) e^{-iku}$ .

That's the next term. And the next term would be just  $G(u) G(u) e^{-iku}$ . And it goes on forever. That's the Born series.