

PROFESSOR: Our subject today then is Fermi's golden rule. So that's what we're going to develop. Fermi's golden rule. Fermi's golden rule.

And this has to do with the study of transitions. And typically, the interesting and sophisticated thing about this subject is that you have a transition from some initial state to a state that is part of a continuum of states. That's what makes it complicated.

The transition from one discrete state to another discrete state with a perturbation is kind of a simple matter to do. But when you can go into a continuum, you have to integrate over the set of final states, and that makes it a lot more interesting.

So we go from a discrete state into a continuum. And that makes it somewhat challenging. So we will consider this in two forms. It's worth considering the case of what we call constant perturbations.

And you might say, well, aren't we doing time dependent perturbation theory? Yes, we are. But this kind of perturbation, you will think of it as saying that H is H_0 plus a V that is time independent.

But the way we think of it is that here is time, here is time 0, and here is V . V turns on at time equals 0. So there's a little bit of time dependence. There was no V before time equals 0, and there is a V after time equals 0.

So it's almost like the Hamiltonian changes. And we want to see what transitions we get. Because always the subject is the subject of transitions. And we will see an application of this later today.

The other case is what is called a harmonic perturbation, in which H is equal to H_0 plus δH . And δH is going to be harmonic. $2H \text{ prime}$. That's conventions to put a 2 in there. It's convenient. $\text{Times cosine } \omega t$. And this will be for t between 0 and t_0 .

Sorry if some people can't see this. Too far to the right. $2H \text{ prime cosine of } \omega t$ when t is between 0 and t_0 . And it's 0 otherwise. And $H \text{ prime}$, of course, is time independent as well. Time independent.

So these are the two main cases when we will consider transitions. And we'll do the first one in

a lot of detail today. The subject is trying to get the transitions understood for this case. And after that, next time, we will do the case of the harmonic perturbation that brings in a few new issues.

But then, all of the rest is the same. So once you've understood how to do the constant transition, the harmonic perturbation is going to be easier. So constant perturbations will be useful to understand, for example, the phenomenon of auto-ionization. Some atoms sometimes ionize spontaneously, and this has to do with this subject.

Harmonic perturbations has to do with even a more popular subject, which is atoms interacting with radiation. You send in an electromagnetic field. That is a harmonically varying perturbation, and that's going to allow you to calculate transitions. And by the time we're done with this, you will be experts in calculating atomic transitions.

So one subject we need since we're going to be doing transitions between a discrete state or continuum is to describe the continuum. And for that, for the continuum, we use the concept of a density of states in the continuum.

Now, we will be considering momentum eigenstates. And the momentum, you know very well the momentum of a free particle is a continuum. Takes absolutely continuous values. So there is no way you can count them or you can tell how many there are per momentum range. It's like saying how many numbers are there from 0 to 1.

The way you have to do that always is by adding an extra parameter you wish you didn't have to add. And that's kind of things that we have to do in physics sometimes. We have to add parameters.

You consider when we discussed last time the delta function perturbation, we broadened it, and we were able to calculate. And then we saw that the broadening size didn't matter for the result. Therefore, good.

So here, we'll do the same. You know that the momentum eigenstates of a particle in an interval or in a circle are quantized. And then you can count quantized states. So we will put the whole world in a box. A big box. Size of the galaxy, size of the Earth, size of the laboratory maybe is big enough.

And we'll put this parameter L there for the size of the box. And we will get the density of states because now the states can be counted. Afterwards, by the time we're all done with the

transition rates of the Fermi's golden rule, we hope that length is going to disappear.

So that's something we will see that it happens in calculations. And there will be good reasons why it happens. So we'll put the world in a big box. It's not that big the way I draw it, but it's supposed to be very big. Length L .

And you think of the quantization of momentum by considering a wave function that would be normalized and has-- it's a momentum eigenstate. It has momentum in the x direction, y direction, and z direction. And that wave function, if all the sides of this world, this cubical world with length L , that wave function is properly normalized.

You square it, and the norm is equal to 1 times this factor squared. You integrate it. It's all good. So this is a nice wave function in which the integral $\psi^2 V$ cubed x over the box is 1.

And then what else do we do? We decide that we have to consider periodic boundary conditions. Well, think of it as a torus, properly speaking, in which each direction is a circle. So the wave function repeats itself after x increases by L , after y increases by L , or after z increases by L .

You could have chosen a box with a finite big wall at the end. It would make no difference for the counting at the end of the day. So the conditions are that k_x multiplied by L should be a multiple of $n_x 2\pi$ times n_x , k_y times L , 2π times n_y , k_z times L , 2π times n_z .

And therefore, the total number of states can be calculated by taking a little differential of this thing. We say, well, if we let k_x vary by a little bit, dk_x from this equation times L is $2\pi dn_x$ dk_y times L , 2π times dn_y , and dk_z times L is 2π times dn_z .

So if L were to consider a little interval of momentum defined by dk_x , dk_y , dk_z , a little cube in momentum space, the number of states in this cube would be dn , which is dn_x , dn_y , dn_z . And this is equal to L^3 over $(2\pi)^3 d^3k$, which is dk_x , dk_y , dk_z .

I say, if in this momentum range the quantum number n_x contains this number of values, dn_x , the quantum number n_y can take a set of values, and the same for the quantum number in the z direction. The total set of quantum numbers is the product of them.

So there is that many states in this little cube of momentum space-- that is, momentum space that is between some k and a little more, called dk . Well, that's a famous formula. And the n is

equal to $\frac{1}{2\pi^3} \frac{d^3k}{k^3}$. Some people know this by heart.

But now we want to write this in the terms of a density of states as a function of energy. So if this is the number of states in some interval in momentum space, I can try to convert this into saying, well, all these states, because the momentum is varying within some little bounds, the energy is varying within some little bounds. So let's figure out how much the dE is. And then the total number of states will be given in this range by the number of states per unit energy multiplied by this thing.

So this is the density of states, which is states per unit energy. And many times, this is the quantity we really want. We have here an energy E as well. So we think of this $V^3 k^3$ as a little cube in momentum space. If momentum space has an origin here, you can imagine states with some momentum, and then a little cube in here saying how much the momentum varies.

So all these states have some momentum and vary a little bit in the momentum. Therefore, they all basically have some energy up to some little variation. So how do we connect these two things?

Well, you remember, E is equal to $\frac{\hbar^2 k^2}{2m}$. And how about d^3k ? We'll try to think of d^3k physically as all the states that have momentum k . And now, this is a space diagram now.

This is x , y , and z . So here are the states with momentum k . They point in this direction. So this is the direction the states are pointing. And now we think of them as having some possible angle here. $d\Omega$.

So this is the direction. So let me not say they're x , y , and z . Let me use angles here. θ and ϕ , representing the azimuthal and polar angle of this direction. So we have a little range here and a little range of magnitude of momentum.

So let me see if I can draw it kind of nicer. You can imagine a little cone as it grows, and the last part is a little thick piece of the cone here of thickness dk . So d^3k is the volume of that little pillbox here. And it's $k^2 d\Omega dk$. So that's d^3k .

And now to relate it to the energy we have here, this equation. So we take a differential. And it's $dE = \frac{\hbar^2 k dk}{m}$. So d^3k here is $k^2 dk d\Omega dk$, and

this is the same $k dk$ here.

So we can write it as $k d\omega$. And for $k dk$ we have m over h squared d energy. So we're almost done with our computation.

We go back to this equation. And we have ρ of $E dE$, or dn , is equal to l over 2π cubed d cubed k . But now we have what d cubed k is. It's m over h squared $k d\omega dE$. So this is our formula.

And this is ρ of $E dE$. And here, for example, you have ρ of E . Actually, if it were ρ of E , I should have here k expressed in terms of E . Just makes the formula a little more messy. But if you think of it as a function of the energy, you have to say the words properly.

What is ρ of E ? ρ of E is the density of states per unit energy, but when you're only counting states that have a momentum within an angle $d\omega$. So this is the density of states per unit energy, because this gives you states. So this must be states per unit energy. But you're only looking at the states that are within an angle $d\omega$.

If you were to look at the density of all states that have energy E , you would have to integrate over ω . But many times, we want to make a transition. You have an atom and you're sending an electromagnetic wave. And you want to make a transition, and you want to see how many electrons, for example, are kicked out in some direction.

So you need the density of states within some solid angle. So this is a relatively useful thing to have, not to integrate, so that you can keep control over your states and orient them at a given angle.

So we've done a little bit of basic preparation. We've said what the Fermi golden rule aims to do, and it requires transition to the continuum where we'll have to use density of states. And here is an example of how you calculate the density of states.