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PROFESSOR:

So today, let me remind you, for the convenience of also the people that weren't here last time, we don't need too much of what we did last time, except to know, more or less, what's going on. We were solving central potential problems in which you have a potential that just depends on r .

And at the end of the day, the wave functions were shown to take the form of a radial part and an angular part with the spherical harmonic here. The radial part was very conveniently presented as a U function divided by r . That's another function, but the differential equation for U is nice. It takes the form of a 1-dimensional Schrodinger equation for a particle under the influence of an effective potential.

This potential, effective potential, has the potential that you have in your Hamiltonian plus an extra term, a barrier. It's a potential that grows as r goes to 0, so it's a barrier that explodes at r equals 0. And this being the effective potential that enters into this 1-dimensional Schrodinger equation, we made some observations about this function U .

The normalization of this wave function is guaranteed if the integral of U squared over r is equal to 1. So that's a pretty nice thing. U squared meaning absolute value squared of U . And we also noticed that U must go like r to the l plus 1 near r going to 0.

So those were the general properties of U . I'm trying to catch up with notes. I hope to put some notes out today. But this material, in fact, you can find parts of it in almost any book. It will just be presented a little differently. But this is not very unusual stuff.

Now, the diagram that I wanted to emphasize to you last time was that if you're trying to discuss the spectrum of a central state potential, you do it with a diagram in which you list the energies as a function of l . And it's like a histogram in which for l equals 0, you have to solve some 1-dimensional Schrodinger equation.

This 1-dimensional Schrodinger equation will have a bound state spectrum that is non-degenerate. So for l equal 0, there will be one solution, two solutions, three. I don't know how many before the continuous spectrum sets in, or if there is a continuous spectrum. But there are some solutions.

For l equal 1, there will be some other solutions. For l equal 2, there might be some other solutions. And that depends on which problem you are solving. In general, there's no rhyme or reason in this diagram, except that the lowest energy state for each level goes up. And that's because the potential goes up and up as you increase l . Notice this is totally positive.

So whatever potential you have, it's just going up as you increase l . So the ground state should go up. The ground state energy should go up. So this diagram looks like this. We also emphasized that for every l , there are $2l + 1$ solutions obtained by varying M , because M goes from l to minus l .

Therefore, this bar here represents a single multiplet of l equals 1, therefore three states. This is a single multiplet of l equals 1, three more states. Here is five states, five states, but only one l equal 1 multiplet, one l equal 1 multiplet, 1, 1. There are no cases in which you have two multiplets because that would contradict our known statement that the spectrum of the potential of bound states in one dimensions is non-degenerate.

So that was one thing we did. And the other thing that we concluded that ties up with what I want to talk now was a discussion of the free particle, free particle. And in the case of a free particle you say, well, so what are you solving? Well, we're solving for solutions that have radial symmetry.

So they are functions of r [INAUDIBLE] angular distribution. So what do you find is

UEI of r is equal to rJ_l of kr , as we explained, where these were the spherical Bessel functions. And those are not as bad as the usual Bessel functions, not that complicated. They're finite series constructed with sines and cosines, so these are quite tractable. And that was for a free particle.

So we decided that we would solve the case of an infinite spherical well, which is a potential V of r , which is equal to 0 if r is less than a , and infinity if r is greater or equal than a . It's a small-- well, a is whatever size it is. It's a cavity, spherical cavity where you can live. And outside you can't be there.

This is the analog of the infinite square well in one dimension. But this is in three dimensions. An infinite spherical well should be imagined as some sort of hole in the material and electrons or particles can move inside and nothing can escape this. So this is a hollow thing.

So this is a classic problem. You would say this must be as simple to solve as the infinite square well. And no, it's more complicated. Not conceptually much more complicated, but mathematically more work. You will consider some aspects of the finite spherical well in the homework. The finite square well, you remember, is a bit more complicated. You can't solve it exactly. The finite spherical well, of course, you can't solve exactly either.

But you will look at some aspects of it, the most famous result of which is the statement that while any attractive potential in one dimension has a bound state in three dimensions. An attractive potential, so a finite spherical well, may not have a bound state, even a single bound state. So that's a very interesting thing that you will understand in the homework in several ways.

You will also understand some things about delta functions, that they're important. So we'll touch base with that. So that's as far as I got last time and just a review. If there are any questions, don't be shy if you weren't here and you have a question. Yes.

AUDIENCE: Is there any reason to expect [INAUDIBLE] intuitively should be like [INAUDIBLE]?

PROFESSOR: Well, the reason, intuitively the reason is basically the conspiracy between this U_{eff} , as I was saying, U_{eff} as r goes to 0 goes like r^{-l-1} . So first of all, this potential is very repulsive. Is that right? So that tends to ruin things. So you could say, oh, well, this thing is probably not going to get anything because near $r = 0$, you're being repelled.

But you can say, no, let's look at that $l = 0$. So you don't have that, so just V of r . But we take $l = 0$ -- I'm sorry, U here, U of U_{eff} has to go like that. So actually, U will vanish for $r = 0$. So the effective potential for the 1-dimensional problem may look like a finite square well, that is like that.

But the wave function has to vanish on this side. Even though you would say, it's a finite spherical well, why does it have to vanish? Here well, it's the unusual behavior of this U function. So the wave function that you can sort of imagine must vanish here. So in order to get a bound state, it has to have enough time to sort of curve so that it can fall, and it's sometimes difficult to do it.

So basically, it's the fact that the wave function has to vanish at the origin, the U wave function has to vanish. Now, the whole wave function doesn't vanish because it's divided by r . But the U does. So it's the reason why you don't have bound states in general.

And then there's also funny things like a delta function. You would say, well, a 3-dimensional delta function, how many bound states do you get, or what's going on? With a 1-dimensional delta function, you have one bound state, and that's it.

With a 3-dimensional delta function, as you will find, it's [INAUDIBLE] is rather singular, and you tend to get infinitely many bound states. And you cannot even calculate them because they fall off all the way through r and go to minus infinity energy. It's a rather strange situation. All right. Any other questions?

So let's do this infinite spherical well. Now, the reason we did the free particle first was that inside here, this is all free, so the solutions will be sort of simple. Nevertheless, we can begin with looking at the differential equation directly for

inside.

So $r < a$, you would have $-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dr^2} + V(r) \psi = E \psi$, where $V(r) = 0$ for $r < a$. And $k = \sqrt{2mE}/\hbar$ is $2mE/\hbar^2$ as usual.

So here I didn't say what k was. That was $2mE/\hbar^2$. And this doesn't quite look like the differential equation you have here. Well, $V(r) = 0$ for $r < a$, so you just have this term. The \hbar^2 's over $2m$ and the E have been rescaled by changing r to ρ . So the differential equation becomes simple and looking like this. So that was a manipulation that was done in detail last time, but you can redo it.

Now, this, as I mentioned, is not a simple differential equation. If you didn't have this, it would have a power solution. If you don't have this, it's just a sine or cosine. But if you have both, it's Bessel. So having a differential with two derivatives, $1/\rho^2$ and $1/\rho$, brings you into Bessel territory.

Anyway, this is the equation that, in fact, is solved by these functions because it's a free Schrodinger equation, and you can take it for $l = 0$. This is the only case we can do easily without looking up any Bessel functions or anything like that. You then have $\frac{d^2 \psi}{d\rho^2} = -k^2 \psi$.

And therefore, ψ goes like $A \sin(\rho) + B \cos(\rho)$. $\rho = kr$. ψ must behave like r^{l+1} , so ψ must behave like r . So for this thing to behave, ψ must behave like r . So it must behave like ρ as $\rho \rightarrow 0$. Therefore, this term cannot be there. The only solution is ψ is equal to $\sin(\rho)$, which is kr . So ψ of r must be of this form.

Then in order to have a solution of the 1-dimensional Schrodinger equation, it's true that the potential becomes infinite for $r = a$. So that is familiar. It's not the point $r = 0$ that is unusual. $r = a$, this must vanish.

So we need that $\psi(a) = 0$. So this requires $k = k_n$ so that $k_n a = n\pi$. So for $k = k_n$, where $k_n a = n\pi$, a multiple of

π , then the wave function will vanish at r equals a . So easy enough. We've found the values of k . This is quite analogous to the infinite square well.

And now the energies from this formula E_n will be equal to $h^2 k_n^2$ over $2m$. And it's convenient, of course, to divide by ma^2 so that you have $k_n a$ squared. So the energies are h^2 over $2ma^2$. Here we have $n \pi$ squared. I'll put them like this. $E_{n,0}$ for l equal 0, $E_{n,l}$'s energies.

Now, if you want to remember something about this, of course, all these constants are kind of irrelevant. But the good thing is that this carries the full units of energy. And you know in a system with length scale a , this is the typical energy. So the energies are essentially that typical energy times $n^2 \pi^2$.

So it's convenient to define, in general, $E_{n,l}$ to be $E_{n,l}$, for any l that you may be solving, divided by h^2 over $2ma^2$. So that this thing has no units. And it tells you for any level, the calligraphic E , roughly how much bigger it is than the natural energy scale of your problem.

So it's a nice definition. And in this way, we've learned that $E_{n,0}$ is equal to $n^2 \pi^2$. And a few values of this are $E_{1,0}$ about 9,869 [INAUDIBLE], $E_{2,0}$ equal 39,478, and $E_{3,0}$ is equal 88,826. Not very dramatic numbers, but they're still kind of interesting.

So what else about this problem? Well, we can do the general case. Let me erase a little here so that we can proceed. The general case is based on knowing the zeroes of this spherical Bessel function. So this is something that the first one you can do easily.

The zeroes of J_1 of ρ are points at which $\tan \rho$ is equal to ρ . That is a short calculation if you ever want to do it. That's not that difficult, of course, but you have to do it numerically. So the zeroes of the Bessel functions are known and are tabulated. You can find them on the web, little programs that do it on the web and give you [? directly those ?] zeroes.

So how are they defined? Basically, people define $Z_{n,l}$ to be the n -th zero with n

equals 1 like that of J_l . So more precisely, J_l of $Z_{n,l}$ is equal to 0. And all the Z and l 's are different from 0. There's a trivial zero at 0. And nevertheless, that is not counted. It's just too trivial for it to be interesting.

So these numbers, Z and l , are basically it. Why? Because what you need is, if you're looking for the l -th solution, you need $U_{E,l}$ of a equal 0. And $U_{E,a}$ of that equal 0 means that you need $k_{n,l}$ times a be equal to $Z_{n,l}$. So $k_{n,l}$ is the value of k .

And just like we quantized here, we had k_n , well, if you have various l 's, put the $k_{n,l}$. So for every value of l , you have $k_{n,l}$'s that are given by this. And the energy's like this. Let me copy what this would be. $E_{n,l}$ would be $E_{n,l}$ over this ratio.

And $E_{n,l}$ h squared, well, let me do it this way. I'm sorry. $E_{n,l}$ would be h squared $k_{n,l}$ over $2ma$ squared, over $2m$ like that. Then you multiply by a squared again. So you get $k_{n,l}$ a squared over $2ma$ squared.

So what you learn from this is that $E_{n,l}$, you divide this by that, is just $k_{n,l}$ times a , which is $Z_{n,l}$ squared. So that's the simple result. The $E_{n,l}$'s are just the squares of the zeroes of the Bessel function. So you divide it again by h squared over $2ma$, and that's all that was left.

So you need to know the zeroes of the Bessel function. And there's one, you might say, well, what for do I care about this? But it's kind of nice to see them. So $Z_{1,1}$ is equal to 4.49 $Z_{2,1}$ is equal to 7.72, and $Z_{3,1}$ is 10.90, numbers that may have no rhyme or reason.

Now, you've done here l equals 1. Of course, it continues down, down, down. You can continue with the first zero, first nontrivial zero, second nontrivial zero, third nontrivial zero, and it goes on. The energies the squares. So the squared goes like 20.19. This goes like 59.7. And this goes like 119 roughly.

Then you have the other zeroes. First zero for l equals 2, that is 5.76 roughly. Second zero for l equals to 2 is 9.1 roughly. And if you square those to see those other energies, you would get, by squaring, 33.21 and 82.72. And finally, let me do

one more. $Z_{1,3}$, the first zero of the l equal 3, and the $Z_{2,3}$, the second zero, are 6.99 and 10.4, which when squared give you 48.83 and 108.5.

OK. Why do you want to see those numbers? I think the reason you want to see them is to just look at the diagram of energies, which is kind of interesting. So let's do that. So here I'll plot energies, and here I put l .

And now I need a big diagram. Here I'll put the curly energies. And here is 10, 20, 30, 40, 50, 60-- and now I need the next blackboard, let's see, we're 60, let's see, more or less, here is about right-- 70, 80, 90, 100, 110. How far do I need? 120, ooh, OK, 120. There we go.

So just for the fun of it. Look at them to see how they look, if you can see any pattern.

So the first energy was 986, so that's roughly here. That's l equals 0 is the first state. Second is 39.47, so it's a little below here. Next is 88.82, so we are here, roughly.

Then we go l equals 1. What are the values? This one's 20.19. l equals 1, 20.19, so we're around here. Then 59.7 is almost 60. And then 119, so that's why we needed to go that high. So here we are.

And then l equals 3, you have 48.83, so that's 50. I'm sorry, l equals 2. 48.83. A little lower than that. No, I'm sorry. It's 33.21. I'm misreading that. 33 over here. And then 82.72, so we are here. And then l equals 3, we have 48.83, so that was the one I wanted, and 108.5.

That's it, and there's no pattern whatsoever. The zeroes never match. The only thing that is true is that 0, 1, 2, 3, they were ascending as we predicted. But no level matches with any other level.

If you were trying to say, OK, this potential is interesting, is special, it has magic to it, a spherical square well, it doesn't seem to have anything to it, in fact. It's totally random. I cannot prove for you, but it's probably true, and probably not impossible

to prove, that these zeroes are never the same. No l and l prime will have the same zero. No degeneracy ever occurs that needs an explanation. For example, this state could have ended up equal to this one or equal to this one, and it doesn't happen.

And that's OK, because at this level, we would not be able to predict why it happened. We actually, apart from the fact that this a round, nice box, what symmetries does it have, that box, except rotational symmetry? Nothing all that dramatic. So you would say, OK, let's look for a problem, which we'll deal now, that does have a more surprising structure, and let's try to figure it out.

Let's try the three dimensional harmonic oscillator. So 3D SHO. Isotropic. What is the potential? It's $\frac{1}{2} m \omega^2 (x^2 + y^2 + z^2)$, all the same constant. So it's $\frac{1}{2} m \omega^2 r^2$.

You would say, this potential may or may not be nicer than the spherical well, but actually, it is extraordinarily symmetric in a way that the spherical well is not. So we'll see why is that. Let's look at the states of this. Now, we're going to do it with numerology. Everything will be kind of numerology here because I don't want to calculate things for this problem.

So first thing, how you build the spectrum? H is equal to $\hbar \omega (N_1 + N_2 + N_3 + \frac{3}{2})$, where these are the three number operators, and 0. Now, just for you to realize, in the language of things that we've been doing, what is the state space? If we call H_1 the state space of a one dimensional SHO, what is the state space of the three dimensional SHO?

Well, conceptually, how do you build a three dimensional SHO? Well, you have the creation annihilation operators that you had for the x , y , and z . So you have the a_x , a_x^\dagger , the a_y , a_y^\dagger , and the a_z , a_z^\dagger , and you could act on the vacuum.

So the way you can think of the state space of the one dimensional oscillator is this is one dimensional oscillator and I have all these things. Here is the other one dimensional, here is the last one dimensional. But if I want to build a state of the three dimensional oscillator, I have to say how many a_x 's, how many a_y 's, how

many az 's. So you're really multiplying the states in the sense of tensor products. So the H , for a 3D SHO, is the tensor product of three H_1 's, the H_1 x , the $1y$, and the z . You're really multiplying all the states together. Yes?

AUDIENCE: So this is generalized to when you have a wave function that's separable into products of different coordinates. Can you express those as tensor products of the different states, basically?

BARTON
ZWIEBACH: You see, the separable into different coordinates, it's yet another thing because it would be the question of whether the state is separable or is entangled. If you choose, for example, one term like that, a_1 , x , ax dagger, ay dagger, az dagger, with two of those here, the wave function is the product of an x wave function, a y wave function, and a z wave function. But if you add to this ax dagger squared plus ay plus az , it will also be factorable, but the sum is not factorable. So you get the entanglement kind of thing.

So this is the general thing, and the basis vectors of this tensor product are basis vectors of one times basis vectors of the other basis vector. So basically, one basis vector here, you pick some number of ax 's, some number of ay 's, some number of az 's.

So this shows, and it's a point I want to emphasize at this moment, it's very important, that even though we started thinking of tensor products of two particles, here, there are no two particles in the three dimensional harmonic oscillator, no three particles. There's just one particle where there's one kind of attribute that that's doing in the x direction, one kind of attribute that it's doing in the y , one kind of attribute that it's doing in the z .

And therefore, you need data for each one, and the right data is the tensor product. You're just combining them together. We mentioned that the basis vectors of a tensor product are the products of those basis vectors, of each one, so that's exactly how you build states here.

So I think, actually, you probably have this intuition. I just wanted to make it a little

more explicit for you. So you don't need to have two particles to get a tensor product. It can happen in simpler cases.

So here's the thing that I want to do with you. I would like to find that diagram for the three dimensional SHO. That's our goal that we're going to spend the next 15 minutes probably doing. How does that diagram look?

So I'll put it somewhere here maybe, or maybe here. I won't need such a big diagram. So I'll have it here. Here is l , and here are the energies. So ground states.

The ground state, you can think of it as a state like that. How should I write it? A state like that. No oscillators acting on it whatsoever, so the N 's are N_1 equals N_2 equals N_3 equals 0, and you get E equals $\hbar \omega$ times $3/2$. So $3/2 \hbar \omega$.

So actually, we got one state, and it's the lowest energy state. Energy lowest possible. So let me write here, energy equals $3/2 \hbar \omega$. We got one state over here.

Now, can it be an l equals 1 state or an l equals 2 state or an l equals 3 state? How much is l for that state? You see, if it's a spherically symmetric problem, it has to give you a table like that. It's guaranteed by angular momentum, so we must find. My question is whether it's l equals 0, 1, 2, 3, or whatever. Anybody would like to say what do they think it is? Kevin?

AUDIENCE: It's 0, right?

BARTON 0. And why?

ZWIEBACH:

AUDIENCE: Because we wrote the operator for l in terms of a_x , a_y , and a_z , and you need one to be non-zero. You need a difference between them to generate a rotation.

BARTON OK, that's true. It's a good answer. It's very explicit. Let me say it some other way,

ZWIEBACH: why it couldn't possibly be l equals 1. Yes?

AUDIENCE: Because the ground state decreases for l decreasing.

BARTON The ground state energy does what?

ZWIEBACH:

AUDIENCE: It's smaller for smaller l , and so for l equals 0, you have to have a smaller ground state than for l equals 1.

BARTON That's true. Absolutely true. The energy increases so it cannot be l equals 1,

ZWIEBACH: because then there will be something below which doesn't exist. But there may be a more plain answer.

AUDIENCE: The state is non-degenerative.

BARTON Yes. There's just one state here. We built one state. If it would be l equals 1, there

ZWIEBACH: should be three states because l equals 1 comes with m equals 1, 0, and minus 1. So unless there are three states, you cannot have that.

All right. So then we go to the next level. So I can build a state with a_x dagger on the vacuum, a state with a_y dagger on the vacuum, and a state with a_z dagger on the vacuum using one oscillator. Here, the N 's are 1, different ones, and the energy is $\hbar \omega (1 + 3/2)$, so $5/2$.

And I got three states. What can that be? Well, could it be three states of l equals 0? No. We said there's never a degeneracy here. There's always one thing, so there would be one state here, one state here, one state here maybe. We don't know, but they would not have the same energy, so it cannot be l equals 0.

Now, you probably remember that l equals 1 has three states. So without doing any computation, I think I can argue that this must be l equals 1. That cannot be any other thing. It cannot be l equals 2 because you need five states. Cannot be anything with l equals 0. So it must be l equals 1. So here is l equals 0, and here is l equals 1, and there's no state here, but there's one at $5/2 \hbar \omega$. So we obtain one state here.

And this corresponds to a degeneracy. This must correspond to l equals 1 because

it's three states. And that degeneracy is totally explained by angular momentum's central potential. It has to group in that way. Of course, if my oscillator had not been isotopic, it would not group that way. So we've got that one and we're, I think, reasonably happy.

Now, let's list the various l 's. l equals 0, l equals 1, l equals 2, l equals 3, l equals 4, l equals 5. How many states? 1, 3, 5, 7, 9, 11. OK, good enough. So we succeeded, so let's proceed to one more level. Let's see how we do.

Here, I would have a_x dagger squared on the vacuum, a_y dagger squared on the vacuum, a_z dagger squared on the vacuum. Three states, but then I have three more, $a_x a_y$, both dagger on the vacuum, $a_x a_z$, and $a_y a_z$, for a total of six states.

So at N equals 2, the next level, let's call N equals N_1 plus N_2 plus N_3 . So this is N equals 2. This is N equals 1. You've got six states. They must organize themselves into representations of angular momentum, so they must be billed by these things.

So I cannot have l equals 3. I don't have that many states. I could have two l equals 1 states, three and three. That would give six states, or a five and a one. So what are we looking at? Let's see what we could have.

Well, we're trying to figure out the next level, which is $7/2 \hbar \omega$. If I say this is built by two l equals 1's, I would have to put two things here, and that's wrong. There cannot be two multiplets at the same energy. So even though it looks like you could build it with two l equal 1's, you cannot.

So it must be an l equals 2 and an l equals 0. So l equals 2 plus l equals 0, this one giving you five states and this giving you one state. So at the next level, this cannot be, but what you get instead, l equals 2. You get one state here and one state there.

This is already something a little strange and unexpected. For the first time, you've got things in different columns that are matching together. Why would these ones match with these ones? That requires an explanation. You will see that explanation

a little later in the course, and that's something we need to understand.

So far, so good. We seem to be making good progress. Let's do one more. In fact, we need to do maybe a couple more to see the whole pattern.

Let's do the next one, N total equals 3. And now you have-- I'll be very brief-- ax cubed, ay cubed, az cubed, ax squared times ay or az , ay squared times ax or az , and az squared times ay or ax , and ax ay az , all different. And that builds for three states here, two states here, two states here, two states here, and one state here. So that's 10 states. Yes?

AUDIENCE: [INAUDIBLE]?

BARTON No. It's just laziness. I just should have put ax squared ay dagger or ax squared az
ZWIEBACH: squared.

AUDIENCE: [INAUDIBLE]?

BARTON No, this is a sum. This is what we used to call the direct sum of vector spaces. This
ZWIEBACH: is not the product. That's pretty important. Here, it's a sum. We're saying 6 is 5 plus 1, basically. Six states are five vectors plus one vector.

Now, it can seem a little confusing because-- well, it's not confusing. If it would be a product, it would be 1 times 5, which is 5. So here, it's 6. It's a direct sum. It's saying the space of states at this level is six dimensional. This is a five dimensional vector space, this is a one dimensional vector space. This is a direct sum, something we defined a month ago or two months ago, direct sums. So this is funny how this is happening. This tensor product is giving you direct sums of states.

Anyway, 10 states here. And now it does look like we finally have an ambiguity. We could have l equals 4, which is nine states, plus l equals 0. You cannot use any one more than once. We've learned that for any energy level, we cannot have some l appear more than once because it would imply degeneracy.

So I cannot build this with 10 singlets, or three l equal 1's and one l equals 0. I have to build it with different things, but I can build it as 9 plus 1, or I can build it as l

equals 3 plus l equals 1. And the question is, which one is it?

AUDIENCE: [INAUDIBLE].

BARTON 3 and 1, is that right? How would you see that?

ZWIEBACH:

AUDIENCE: Because the lowest energy with $l=3$ has to be lower than the lowest energy with $l=4$.

BARTON Yes. Indeed, it would be very strange. It shouldn't happen. The energies are sort of

ZWIEBACH: in units, so here is $l=3$ and here is $l=4$. If $l=4$ would be here, where could be $l=3$? It cannot be at a lower energy. We've accounted for all of those. This is terribly unlikely, and it must be this.

And therefore, you found here next level, $9/2 \hbar \omega$, you got $l=3$, $l=1$. It's possible to count. You start to get bored counting these things. So if you had to count, for example, the number of states with 4, how would you count them a little easier?

Well, you say, I need a_x dagger to the n_x , a_y dagger to the n_y , and a_z dagger to the n_z . That's the state. And you must have n_x plus n_y plus n_z equals 4. And you can plot this, make a little diagram like this, in which you put n_x , n_y , and n_z .

And you say, well, this can be as far as 4, this can be as high as 4, this can be as high as 4, so you have triangle, but you only have the integer solutions. n_x plus n_y plus n_z equals 4 is that whole hyperplane, but only integers and positive one. So you have here, for example, a solution. This line is when n_z plus n_y is equal to 4.

So here's n_z equals 4, n_z equals 3, 2, 1, 0. These are solutions. Here, you have just one solution. Then you would have two solutions here, three solutions here, four here, and five there. So the number of states is actually 1 plus 2 plus 3 plus 4 plus 5. The number of states is 1 plus 2 plus 3 plus 4 plus 5, which is 15. And you don't have to write them.

So 15 states, what could it be? Well, you go through the numerology and there

seem to be several options, but not too many that make sense. You could have something with l equals 5, but by the same argument, it's unlikely. But you could have something with l equals 4 and begin with it. So it must be an l equals 4, which gives me already nine states, and there are left with six states. But you know that with six states, pretty much the only thing you can do is l equals 2 and l equals 0, so that must be it.

The next state here, l equals 4, is here. This was $11/2 \hbar \omega$, and then it goes 4, 2, 0. Enough to see the pattern, I think. You could do the next one. Now it's quick because you just need to add 6 here. It adds one more, so it's 21 states, and you can see what can you build.

But it does look like you have this, this, and that you jump by two units. So you have 0, then 1, and nothing. Then 2, and you jump the next to 0. And then 3 is the next one, and then you jump 2, and that's it. And here, jump 2 and jump 2. So in jumps of 2, you go to the angular momentum that you need.

So how can you understand a little more of what's going on here? Why these things?

Well, as you may recall, we used to have this discussion in which you have an a_x and a_y . You could trade for a right and a left. And with those, the angular momentum in the z -direction was $\hbar N_{\text{right}} - N_{\text{left}}$. This is for a two-dimensional oscillator, but the x and y of the three-dimensional oscillator works exactly the same way.

So L_z is nicely written in terms of these variables. And it takes a little more work to get the other-- the L_x and L_y , but they can be calculated. And they correspond to the true angular momentum of this particle. It's the real angular momentum. It's not the angular momentum that you found for the two-dimensional harmonic oscillator. It's the real one. So here we go with a little analysis. How would you build now states in this language?

You can understand things better in this case because, for example, for N equals 1,

you could have a state a right dagger on the vacuum, a z dagger, a left dagger on the vacuum. And then you can say, what is the L_z of this state?

Well, a right dagger on the vacuum has L_z equal \hbar . This has 0 because L_z doesn't depend on the z-component of the oscillator. And this has minus \hbar . So here you see actually, the whole structure of the L equal 1 multiplet. We said that we have at this level L equals 1. And indeed, for L equals 1, you expect the state with L_z equal plus 1, 0, and minus 1. So you see the whole thing.

For n equals 2, what do you get? Well, you see a state, for example, of a right dagger a right dagger on the vacuum. And that has L_z equals $2\hbar$. And therefore, you must have-- since you cannot have states with higher L_z , you cannot have a state, for example, here with L_z equal 3. So you cannot have an L equal 3. In fact, for any N that you build states, you can only get states with whatever N is is the maximum value that L_z can have, which is something I want to illustrate just generically for a second.

So in order to show that, let me go back here and exhibit for you a little of the general structure. So suppose you're building now with N equal n . The total number is N . So you have a state with a right dagger to the n on the vacuum. And this is the state with highest possible L_z because all the oscillators are aR dagger. So L_z is the highest. And highest L_z is, in fact, $n\hbar$.

Now, let's try to build a state with a little bit less L_z . You see, if this is a multiplet, this has to be a multiplet with some amount of angular momentum. So it's going to go from L_z equal n , n minus 1, up to minus n . There are going to be $2n$ plus 1 states of this much angular momentum because this has to be a multiplet. So here you have a state with one unit less of angular momentum, a right dagger to the n minus 1, times an az dagger.

I claim that's the only state that you can build with one unit less of angular momentum in the z-direction because I've traded this aR for an az. So this must be the second state in the multiplet. This multiplet with highest value of L , which is equal to n , corresponds to an angular momentum l , little l , equals n .

And then, it must have this $2n + 1$ states. And here is the second state. So this is L_z equals $n\hbar$. And here, $(n - 1)\hbar$. And I don't think there's any other state at that level. Let's lower the angular momentum once more. So what do we get?

a right dagger $(n - 2)\hbar$. That's another state with one less angular momentum than this. This, in fact, has $(n - 2)\hbar$.

Now, is that the unique state that I can have with two units less of angular momentum? No. What is the other one?

AUDIENCE: aR to the $(n - 1)$ a l?

PROFESSOR: Correct, that lowers it. third so here you have aR to the $(n - 1)$ a left on the vacuum. That's another state with two units less of angular momentum. So in this situation, a funny thing has happened. And here's why you understand the jump of 2.

This state, you actually-- if you're trying to build this multiplet, now you have two states that have the same value of L_z . And you actually don't know whether the next state in the multiplet is this, or that, or some linear combination. It better be some linear combination.

But the fact is that at this level, you found another state. So this multiplet will go on and it will be some linear combination. Maybe this diagram doesn't illustrate that. But then you will have another state here. So some other linear combination that builds another multiplet.

And this multiplet has two units less of angular momentum. And that explains why this diagram always jumps. It always jumps 2. And you could do that here.

If you tried to write the next things here, you will find two states that you can write. But if you go one lower, you will find three states. Which means that at the next level, you built another-- you need another state with two units less of angular momentum each time.

So pretty much that's it. That illustrates how this diagram happens. The only thing we haven't answered, and you will see that in an exercise, how could I have understood from the beginning that this would have happened rather than building it this way that there's this thing?

And what you will find is that there's some operators that commute with the Hamiltonian that move you from here to here. And that explains it all. Because if you have an operator that commutes with the Hamiltonian, it cannot change the energy. And if it changes the value of L , it explains why it happened. So that's something that you need to discover, what are these operators.

I can give you a hint. An operator for the type a_x dagger a_y destroys a y oscillator, creates an x one. It doesn't change the energy because it adds one thing and loses one. So this kind of thing must commute with the Hamiltonian. And these are the kind of objects-- there are lots of them. So surprising new things that commute with the Hamiltonian. And there's a whole hidden symmetry in here that is generated by operators of this form. So it's something you will see.

Now, the last 15 minutes I want to show you what happens with hydrogen. There's a similar phenomenon there that we're going to try to explain in detail. So a couple of things about hydrogen. So hydrogen H is equal to p squared over $2m$ minus e squared over r .

There's a natural length scale that people many times struggle to find it, the Bohr radius. This does it come from here? How do you see immediately what is the Bohr radius?

Well, the Bohr radius can be estimated by saying, well, this is an energy but it must be similar to this energy. So p is like h over a distance. So let's call it a_0 . So that's p squared. m is the reduced mass of the proton electron system roughly equal to the electron mass. And then you set it equal to this one because you're just doing units. You want to find the quantity that has units of length and there you got it. That's the famous Bohr radius. p is h bar over a distance, therefore this thing must be an

energy. It must be equal to this. And from this one, you can solve for a_0 .

It's $\frac{h^2}{m e^2}$. The $\frac{1}{e^2}$ is very famous and important. It reflects the fact that if you had the interaction between the electron and the proton go to 0, the radius would be infinite. As it becomes weaker and weaker the interaction, the radius of the hydrogen atom blows up. So this is about 0.529 angstroms, where an angstrom is 10^{-10} meters. And what is the energy scale?

Well, $\frac{e^2}{a_0}$ is the energy scale, roughly. And in fact, $\frac{e^2}{2a_0}$, if you wish, is a famous number. It's about 13.6 eV. So how about the spectrum? And how do you find that?

Well, there's one nice way of doing this, which you will see in the problem, to find at least the ground state. And it's a very elegant way based on factorization. Let me mention it.

It is called Hamiltonian. It can be written as a constant γ plus $\frac{1}{2m} \sum_k (p_k + i\beta \frac{x_k}{r})^2$. It's a factorized version of the Hamiltonian of a hydrogen atom. Apparently, not a well-known result. Professor [? Jackiw ?] mentioned it to me. I don't think I've seen it in any book.

So there's a constant β and a constant γ for which this becomes exactly that one. So γ and β to be determined.

And you have to be a little careful here when you expand that this term and this term don't commute. And this and this don't commute. But after you're done, it comes out.

And then, the ground state wave function is-- since this is an operator and here it's dagger, the ground state wave function is-- the lowest possible energy wave function is one in which this would kill the wave function. So $(p_k - i\beta \frac{x_k}{r})$ should kill the ground state wave function. And then the energy, E_{ground} , would be equal to precisely this constant γ .

And you will show, in fact, that yes, this has a solution. And that's the ground state energy of the oscillator of the hydrogen atom, of course. So this looks like three equations, pk with k equals 1 to 3. But it reduces to 1 if the state is spherically symmetric. So it's a nice thing and it gives you the answer.

Now, the whole spectrum of the hydrogen atom is as interestingly degenerate as one of the three-dimensional harmonic oscillator. And a reminder of it is that-- should I go here? Yes.

You have here energies and here l 's. l equals 0 you have one state. l equals 1 you have another state that's here. But actually, l equals 0 will have another state. And then it goes on like that, another state here, state here, state here for l equals 2. And the first state is here. The first state of this one aligns with this one. The first state of that aligns with that. So they go like that, the states just continue to go exactly with this symmetry.

So let me use label that is common, to call this the state n equals 0 for l equals 0. n equals 1. n equals 2. n equals 3. This is the first state with l equals 1 is here. So we'll call it n equals 0. n equals 1. n equals 2. The first state here is n equals 0, n equals 1.

And then the energies. You define n to be n plus l plus 1. Therefore, this corresponds to n equals 1. This corresponds to n equals 2. That corresponds to n can be 1 and l equals 0 or n can be 0 and l equals 1. This is n equal 3, and things like that.

And then the energies of those states, n is, in fact, minus z squared. Well, forget the z squared. e squared over $2 a_0$ 1 over n squared. So the only thing that happens is that there's a degeneracy, complete degeneracy. Very powerful degeneracy.

And then, l can only run up to-- from 0, 1, up to n minus 1 in these variables. So this is the picture of hydrogen.

So you've seen several pictures already-- the square well, the three-dimensional

harmonic oscillator, and the hydrogen one. Each one has a different picture.

Now, in order to understand this one-- this one is not that difficult. But the one of the hydrogen is really more interesting. It all originates with the idea of what is called the Runge-Lenz vector, which I'm going to use the last five minutes to introduce. And think about it a little.

So it comes from classical mechanics. So we have an elliptical orbit, orbits, and people figured out there was something very funny about characterizing elliptical orbit. So consider a Hamiltonian, which is $p^2/2m$ plus $v(r)$, a potential. The force, classically, would be minus the gradient of the potential, which is minus the derivative of the potential with respect to r times the r unit vector.

Now classically-- this all begins classically. Except for spin 1/2 systems, classical physics really tells you a lot of what's going on. So classically, dp/dt is the force and it's minus $v'(r)$ over r times the r vector over r . And dL/dt , the angular momentum, it's a central potential. The angular momentum is 0. It's rate of change is 0. There's no torque on the particle, so this should be 0.

Now, the interesting thing that happens is that this doesn't exhaust the kind of things that are, in fact, conserved. So there is something more that is conserved. And it's a very surprising quantity. It's so surprising that people have a hard time imagining what it is. I will write it down and show you how it sort of happens.

Well, you have to begin with $p \times L$. Why you would think of $p \times L$ is a little bit of a mystery, but it's an interesting thing.

Now, here is a computation that will be in the notes that you can try doing. And it takes a little bit of work, but it's algebra. If you compute this and do a fair amount of work, like five, six lines-- I would suspect it's fairly non-trivial to do it if you don't see how it's being done, but it will be in the notes-- you get the following thing.

Just by manipulating the time derivative of $p \times L$, you get this. Which is equal to m times the potential differentiator times r^2 times the time derivative of this. So time derivative, time derivative. You can get the conservation if this is a constant.

So when is this a constant?

If this is some constant, say, e^2 , you would get a conservation. But what is v prime equals e^2 over r^2 ?

It would give you that v of r is essentially minus e^2 over r . That's the potential of hydrogen. Or the $1/r$ potential, $1/r^2$ force field. So in $1/r$ potentials, this is a number. And then you get an incredible conservation law, d/dt of $\mathbf{p} \times \mathbf{L}$ minus $m e^2 \hat{\mathbf{r}}/r$ is equal to 0.

So something fairly unexpected that something like this could be conserved. So actually, you can try to figure out what this is. And there's two neat-- first, one thing that people do, which is convenient, is to make this into unit-free vector. So define \mathbf{R} to be $\mathbf{p} \times \mathbf{L}$ over $m e^2$ minus $\hat{\mathbf{r}}/r$. This has no units. And it's supposed to be conserved.

Now, one thing you will check in the homework is that this is conserved quantum mechanically as well. That is, this is an operator that commutes with a Hamiltonian. Very interesting calculation.

This is a Hermitian operator, so you will have to Hermiticize the $\mathbf{p} \times \mathbf{L}$ to do that. But it will commute with the Hamiltonian. But what I want to finish now is with your intuition as to what this is.

And this was a very interesting discovery, this vector. In fact, people didn't appreciate what was the role of this vector for quite some time. So apparently, it was discovered and forgotten, and discovered and forgotten like two or three times. And for us, it's going to be quite crucial because I said to you that this operator commutes with the Hamiltonian.

So actually, you will get conservation laws and will help us explain the degeneracy of the hydrogen atom. So it will be very important for us. Now, how does this look? First of all, if you had a circular orbit, how does it work?

Have a circular orbit. Let's see, \mathbf{p} is here, \mathbf{L} is out of the board. $\mathbf{p} \times \mathbf{L}$ is here

over $m e^2$. And the radial vector is here, the hat vector. So the sum of these two vectors $\mathbf{p} \times \mathbf{L}$ and the radial vector must be conserved. But how could it be?

If they don't cancel, it either points in or points out. And then it would just rotate and it would not be conserved. So actually, for a circular orbit, you can calculate it. See the notes. Actually, it's an easy calculation. And you can verify that this vector is, in fact, precisely opposite this. And it's 0. So you say, great. You discover something that is conserved, but it's 0.

No. The thing is that this thing is not 0 for an elliptical orbit. So how can you see that?

Well here at this point, \mathbf{p} is up here. \mathbf{L} is out. And $\mathbf{p} \times \mathbf{L}$, just like before, is out and $\hat{\mathbf{r}}$ is in. And you say, well, OK. Now the same problem. If they don't cancel, it's going to be a vector and going to rotate. But it has to be conserved. So actually, let's look at it here.

Here, the main thing of an ellipse, if you have the focus here, is that this line is not-- the tangent is not horizontal. So the momentum is here. \mathbf{L} is out of the blackboard, but $\mathbf{p} \times \mathbf{L}$ now is like that. And the radial vector is here. And they don't cancel.

So the only thing that can happen-- since this is vertical, this is vertical. It's a little bit to the left-- is that the \mathbf{r} vector must be a little vector horizontal here. Because the sum of this vector and this vector-- it has to be horizontal. Here we don't know if they can cancel. But if they don't cancel, it's definitely horizontal. We know it's conserved, so it must be horizontal here. So it points in. So the Runge-Lenz vector \mathbf{r} points in. And it's, in fact, that.

So here you go, this is a vector that is conserved. And its properties that is really about the axis of the ellipse. It tells you where the axis is.

In Einstein's theory of gravity, the potential is not $1/r$ and the ellipsis [precession?] and the Runge vector is not conserved. But in $1/r$ potentials, it is conserved.

The final thing-- sorry for taking so long-- is that the magnitude of \mathbf{r} is precisely the

eccentricity of the orbit. So it's a really nice way of characterizing the orbits and we'll be using it in the next lecture. See you on Wednesday.