PROFESSOR: OK, so let me go back to what we were doing. The plan for today is as follows. We're going to look at this unitary time evolution and calculate this operator u, given the Hamiltonian. That will be the first order of business today.

Then we will look at the Heisenberg picture of quantum mechanics. And the Heisenberg picture of quantum mechanics is one where the operators, the Schrodinger operators, acquire time dependence. And it's a pretty useful way of seeing things, a pretty useful way of calculating things as well, and makes the relation between classical mechanics and quantum mechanics more obvious. So it's a very important tool. So we'll discuss that. We'll find the Heisenberg equations of motion and solve them for a particular case today.

All this material is not so to be covered in the test. The only part-- of course, the first few things I will say today about solving for the unitary operator you've done in other ways, and I will do it again this time.

So going back to what we were saying last time, we postulated unitary time evolution. We said that psi at t was given by some operator U of t t0 psi t0. And then we found that this equation implied the Schrodinger equation with a Hamiltonian given by the following expression. ih dU dt of t t0 u dagger of t t0.

So that was our derivation of the Schrodinger equation. We start with the time evolution. We found that, whenever we declare that states evolve in time in that way, they satisfy a first order time differential equation of the Schrodinger form in which the Hamiltonian is given in terms of U by this equation.

And we talked about this operator. First we showed that it doesn't depend really on t0. Then we showed that it's Hermitian. It has units of energy. And as you may have seen already in the notes, there is a very clear correspondence between this operator and the way the dynamics follows with the ideas of Poisson brackets that are the precursors of commutators from classical mechanics. So that's in the notes. I will not go in detail in this. Many of you may have not heard of Poisson brackets. It's an interesting thing, and really that will be good enough.

So our goal today is to find U given H, because as we mentioned last time, for physics it is typically more easy to invent a quantum system by postulating a Hamiltonian and then solving it than postulating a time evolution operator. So our goal in general is to find U of t t0 given H of t. That's what we're supposed to do.

So the first thing I'm going to do is multiply this equation by u. By multiplying this equation by a u from the right, I will write first this term. ih dU dt of t t0 is equal to H of t U of t t0. So I multiplied this equation by u from the right. This operator is unitary, so u dagger u is one. That's why this equation cleaned up to this.

Now there's no confusion really here with derivatives, so I might this well write them with normal derivatives. So I'll write this equation as d dt of U t t0 is equal to H of t U of t t0. You should be able to look at that equation and say I see the Schrodinger equation there.

How? Imagine that you have a psi of t0 here, and you put it in. Then the right hand side becomes h and t acting on psi of t. And on the left hand side, this psi of t0 can be put inside the derivative because it doesn't depend on t. Therefore this becomes ih bar d dt of psi of t. So the Schrodinger equation is there.

OK so now let's solve this. We'll go through three cases. Case one, h is time independent. So we're doing this sort of quickly. So H of t is really H like that. No explicit time dependence there. So what do we have? ih bar. Let's write dU dt is equal H times U. And we tried to write a solution of the form U use equal to e to the minus iHt over h bar times U0.

Does that work? Well, we can think du dt and ih. So we get ih. When I take dU dt, I have to differentiate this exponential. And now in this exponential, this full operator

H is there. But we are differentiating with respect to time. And H doesn't depend on time, so this is not a very difficult situation. You could imagine the power series expansion. And H, as far as this derivative goes, is like if it would be even a number. It wouldn't make any difference if it's an operator.

So the derivative with respect to time of this thing is minus iH over h times the same exponential. Moreover, the position of this h could be here, or it could be to the right. It cannot be to the right of U0 though, because this is a matrix, a constant matrix that we've put in here as a possible thing for boundary condition.

So so far we've taken this derivative, and then i's cancel, the h bar cancels, and you get H. But this whole thing is, again, U. So the equation has been solved. So try this. And it works.

So having this solution we can write, for example, that U of t t0 is going to be e to the minus iHt over h bar, some constant matrix. When t is equal to t0, this matrix becomes the unit matrix. So this is e to the minus iHt0 over h bar times U0. And therefore from here, U0 is the inverse of this matrix, which is nothing else but e to the iHt0 over h bar. So I can substitute back here what U0 is and finally obtain U of t t0 is e to the minus iH over h bar t minus t0. And this is for h time independent. And that's our solution. There's very little to add to this.

We discussed that in recitation on Thursday. This unitary operator you've been seeing that from the beginning of the course in some sense, that you evolve energy eigenstate. If this acts on any energy eigenstate, h is an energy-- if you act here on an energy eigenstate, the energy eigenstate is an eigenstate precisely for H, you can put just the number here.

That is e to the, say, alpha h on a state psi n is equal to e to the alpha en psi n if h on psi n is equal to en on psi n. So the function of an operator acting on an eigenstate is just the function evaluated at the eigenvalue. So this is a rule that you've been using a really long time.

OK, so when h is time independent, that's what it is. How about when h has a little time dependence? What do I call a little time dependence? A little time dependence is an idea, the sign to make it possible for you to solve the equation, even though it has some time dependence. So you could have Hamiltonians that are time dependent, but still have a simplifying virtue.

So H of t is time dependent. But assume that H at t1 and h at t2 commute for all t1 and t2. So what could that be? For example, you know that the particle in a magnetic field, the spin in a magnetic field is minus gamma B dot the spin. And you could have a time dependent magnetic field, B of t times the spin. I'm not sure this is the constant gamma that they usually call gamma, but it may be.

Now then if the magnetic field is time dependent, but imagine its direction is not time dependent. So if its direction is not time dependent, then, for example, you would have here minus gamma Bz of t times Sz. And the Hamiltonian at different times commute because Sz commutes with itself, and the fact that it's time independent doesn't make it fail to commute.

So if you have a magnetic field that is fixed in one direction but change in time, you can have a situation where your Hamiltonian is time dependent, but still at different times it commutes. And you will discuss such case because it's interesting. But later on as we do nuclear magnetic resonance, we will have the more interesting case in which a magnetic field rotates and therefore it's not that simple.

So what happens if you have a time dependent Hamiltonian that actually commutes? Well, the claim is that U of t t0 is given by a natural extension of what we had before. You would want to put exponential of minus iHt, but the reason this worked was because the derivative with respect to time brought down an iH over h bar.

So one way to fix this is to put t t0 H of t prime dt prime. So this is an answer to try this. Look at this. If the Hamiltonian were to be time independent, you could take it out. And then you would get t minus t0. That brings you back to this case, so this looks reasonable.

So let me call this quantity R of t. And then you notice that R dot of t, the derivative of this quantity with respect to time. Well, when you differentiate an integral the upper argument, you get just the integrand evaluated at the time represented by the upper argument of the upper limit of integration. So this is H of t.

And now here comes a crucial point. You're trying to differentiate. This U is really e to the R. And you're trying to differentiate to see if the equation holds dU dt. So what is the dU dt? Would be d dt of 1 plus R plus RR plus 1 3 factor RRR.

And now what happens? You differentiate here, and the first term is R dot. Here You, would have one half R dot R plus R R dot. And then 1 over 3 factorial, but three factors. R dot RR plus R R dot R plus RR R dot.

But here is the claim R dot commutes with R. Claim R dot and R commute. Why is that? Well, R dot depends on H. And R is an integral of H as well, but the H at different times commute anyway, so this must be true. There's no place where you can get a contribution, because R dot is like an H, and here's an integral of H.

So since the Hamiltonians are assumed to commute, R dot commutes with R. And this becomes like a normal derivative of an exponential in which you can move the R dot to the left everywhere. And you're differentiating the usual thing. So this is R dot and times the exponential of R.

So actually that means that we've got pretty much our answer, because R dot is minus i over h bar H of t. And e to the R is U, so we got dU dt equals this, which is the same as this equation.

The only reason a derivative with respect to time will not give the usual thing is if R and R dot fail to commute, and they don't. So you could put the R dot here. You can put R dot on the other side, because it commutes with R, but it's better here. And therefore you've got this very nice solution. So the solution is not that bad.

Now finally, I want to discuss for a second the general case. So that's case-- there was a 1, a 2, a 3 H of t general. What can you do? Well, if H of t is general, there's not too much you can do. You can write something that will get you started doing things, but it's not obviously terribly useful. But it's interesting anyway that there's a way to write something that makes sense.

So here it is. U of t and t0. I'll write the answer and explain how it looks, and then you will see that it's OK. It's interesting. But it probably is not the most practical way you can solve this problem.

So here it is. There's an acronym for this thing. T it's called the time ordered exponential. This operator does something to the exponential function. So it's a definition. So I have to say what this time ordered exponential is, and it's the following. You take the exponential and just begin to expand. So 1 minus i over h bar-- or I'll put like this, plus minus i over h bar integral from t0 to t of dt1 H of t1. So far, so good. I've just expanded this.

Now if I would continue expanding, I would get something that doesn't provide the solution. You see, this thing is the solution when the Hamiltonian at different times commute. So it's unlikely to be the solution when they don't commute. In fact, it's not the solution.

So what is the next term here? The next term is you think of the exponential as you would expand as usual. So you will have here plus one half of this thing squared. So I will put something and then erase it, so maybe don't copy. One half minus i over h bar squared. And you would say, well, t0 to t dt prime H of t prime. t0 to t dt double prime H of double prime.

Well, that would be just an exponential. So what is a time ordered exponential? You erase the one half. And then for notation call this t1 and t1. And then the next integral do it only up to time t1, and call this t2.

So t1 will always be greater than t2, because t2 is integrated from t0 to t1. And as you integrate here over the various t1's, you just integrate up to that value. So you're doing less of the full integral then you should be doing, and that's why the factor of one half has disappeared.

This can be continued. I can write the next one would be minus i over h bar cubed integral t0 to t H of t1 integral t0 to t1 dt2 H of t2. And then they next integral goes up to t2. So t0 to t2 dt3 H of t3.

Anyway, that's a time ordered exponential. And I leave it to you to take the time derivative, at least to see that the first few terms are working exactly the way they should. That is, if you take a time derivative of this, you will get H times that thing. So since it's a power series, you will differentiate the first term, and you will get the right thing. Then the second term and you will start getting everything that you need.

So it's a funny object. It's reassuring that something like this success, but in general, you would want to be able to do all these integrals and to sum them up. And in general, it's not that easy. So it's of limited usefulness. It's a nice thing that you can write it, and you can prove things about it and manipulate it.

But when you have a practical problem, generally that's not the way you solve it. In fact, when we will discuss the rotating magnetic fields for magnetic resonance, we will not solve it in this way. We will try to figure out the solution some other way. But in terms of completeness, it's kind of pretty in that you go from the exponential to the time ordered exponential. And I think you'll see more of this in 806.

So that's basically our solution for H and for the unitary operator U in terms of H. And what we're going to do now is turn to the Heisenberg picture of quantum mechanics. Yes, questions?

AUDIENCE: Why does R dot [INAUDIBLE]?

PROFESSOR: Because that's really a property of integrals. d dx integral up to x from x0 g of x prime dx prime is just equal to g of x. This is a constant here, so you're not varying the integral over in this limit. So if this limit would also be x dependent, you would get another contribution, but we only get the contribution from here. What's really happening is you're integrating up to x, then up to x plus epsilon subtracting, so you pick up the value of the function of the upper limit. Yes?

AUDIENCE: So what happens to the T that was pre factor?

PROFESSOR: What happens to this T?

AUDIENCE: Yeah, what happens?

PROFESSOR: That's just a symbol. It says time order the following exponential. So at this stage, this is a definition of what t on an exponential means.

AUDIENCE: OK.

PROFESSOR: It's not-- let me say T is not an operator in the usual sense of quantum mechanics or anything like that. It's an instruction. Whenever you have an exponential of this form, the time ordered exponential is this series that we've written down. It's just a definition. Yes?

AUDIENCE: So when we have operators in differential equations, do we still get [INAUDIBLE]?

PROFESSOR: If we have what?

AUDIENCE: If we have operators in differential equations do we still get unique [INAUDIBLE] solutions?

PROFESSOR: Yes, pretty much. Because at the end of the day, this is a first order matrix differential equation. So it's a collection of first order differential equations for every element of a matrix. It's pretty much the same as you have before. If you know the operator at any time, initial time, with the differential equation you know the operator at a little bit time later. So the operator is completely determined if you know it initially and the differential equation. So I think it's completely analogous. It's just that it's harder to solve. Nothing else. One last question.

AUDIENCE: So let's say that we can somehow fly in this unitary operator, and then we have a differential equation, and we somehow, let's say, get a wave function out of it. What is the interpretation of that wave function?

PROFESSOR: Well, it's not that we get the wave function out of this. What really is happening is that you have learned how to calculate this operator given H. And therefore now you're able to evolve any wave function. So you have solved the dynamical system.

If somebody tells you a time equals 0, your system is here, you can now calculate where it's going to be at the later time. So that's really all you have achieved. You now know the solution. When you're doing mechanics and they ask you for an orbit problem, they say at this time the planet is here. What are you supposed to find? x is a function of time. You now know how it's going to develop. You've solved equations of motion. Here it's the same. You know the wave function of time equals. If you know it at any time, you've solved problem.

OK, so Heisenberg picture of quantum mechanics. Heisenberg picture. So basically the Heisenberg picture exists thanks to the existence of the Schrodinger picture. Heisenberg picture of quantum mechanics is not something that you necessarily invent from the beginning. The way we think of it is we assume there is a Schrodinger picture that we've developed in which we have operators like x, p, spin, Hamiltonians, and wave functions. And then we are going to define a new way of thinking about this, which is called the Heisenberg picture of the quantum mechanics.

So it all begins by considering a Schrodinger operator As hat, which is s is for Schrodinger. And the motivation comes from expectation values. Suppose you have time dependent states, in fact, matrix elements. One time dependent state alpha of t, one time dependent state beta of t. Two independent time dependent states.

So you could ask what is the matrix element of A between these two time dependent states, a matrix element. But then, armed with our unitary operator, we know that As is here, and this state beta at time t is equal to U of t comma 0 beta at time 0. And alpha t is equal to alpha at 0 U dagger of t0.

So the states have time dependence. But the time dependence has already been found, say, in principle, if you know U dagger. And then you can speak about the time dependent matrix elements of the operator As or the matrix element of this time dependent operator between the time equals 0 states.

And this operator is sufficiently important that this operator is called the Heisenberg version of the operator s. Has time dependence, and it's defined by this equation. So whenever you have Schrodinger operator, whether it be time dependent or time independent, whatever the Schrodinger operator is, I have now a definition of what I will call the Heisenberg operator. And it is obtained by acting with a unitary operator, U.

And operators always act on operators from the left and from the right. That's something that operators act on states from the left. They act on the state. But operators act on operator from the left and from the right, as you see them here, is the natural, ideal thing to happen. If you have an operator that's on another from the right only or from the left only, I think you have grounds to be suspicious that maybe you're not doing things right.

So this is the Heisenberg operator. And as you can imagine, there's a lot of things to be said about this operator. So let's begin with a remark. Are there questions about this Heisenberg operator. Yes?

AUDIENCE: Do we know anything about the Schrodinger operator?

PROFESSOR: You have to speak louder.

AUDIENCE: Is the Schrodinger operator related to the Hamiltonian [INAUDIBLE]?

PROFESSOR: Any Schrodinger operator, this could be the Hamiltonian, this could be x hat, it could be Sz, could be any of the operators you know. All the operators you know are Schrodinger operators.

So remarks, comments. OK, comments. One, at t equals 0 A Heisenberg becomes identical to A Schrodinger at t equals 0. So look why. Because when t is equal to 0, U of t-- of 0 0 is the operator propagates no state, so it's equal to the identity. So this is a wonderful relation that tell us you that time equals 0 the two operators are really the same.

And another simple remark. If you have the unit operator in the Schrodinger picture, what is the unit operator in the Heisenberg picture? Well, it would be U t 0 dagger 1 U t 0. But 1 doesn't matter. U dagger with U is 1. This is a 1 Schrodinger, and therefore it's the same operator. So the unit operator is the same. It just doesn't change whatsoever.

OK, so that's good. But now this is something interesting also happens. Suppose you have Schrodinger operator C that is equal to the product of A with B, two Schrodingers. If I try to figure out what is CH, I would put U dagger-- avoid all the letters, the t 0. It's supposed to be t 0. Cs U. But that's equal U dagger As Bs U.

But now, in between the two operators, you can put a U U dagger, which is equal to 1. So As U U dagger Bs U. And then you see why this is really nice. Because what do you get is that C Heisenberg is just A Heisenberg times B Heisenberg. So if you have C Schrodinger equals A Schrodinger, B Schrodinger, C Heisenberg is A Heisenberg B Heisenberg.

So there's a nice correspondence between those operators. Also you can do is for commutators. So you don't have to worry about this thing. So for example, if A Schrodinger with B Schrodinger is equal to C Schrodinger, then by doing exactly the same things, you see that A Heisenberg with B Heisenberg would be the commutator equal to C Heisenberg. Yes?

AUDIENCE: That argument for the identity operators being the same in both pictures. If the Hamiltonian is time independent, does that work for any operator that commutes with the Hamiltonian?

PROFESSOR: Hamiltonian is [INAUDIBLE].

AUDIENCE: Because then you can push the operator just through the exponential of the Hamiltonian.

PROFESSOR: Yeah, we'll see things like that. We could discuss that maybe a little later. But there are some cases, as we will see immediately, in which some operators are the same in the two pictures. So basically operators that commute with the Hamiltonian as you say, since U involves the Hamiltonian, and this is the Hamiltonian, if the operator commutes with the Hamiltonian and you can move them across, then they are the same. So I think it's definitely true.

So we will have an interesting question, in fact, whether the Heisenberg Hamiltonian is equal to the Schrodinger Hamiltonian, and we'll answer that very soon. So the one example that here I think you should keep in mind is this one. You know this is true. So what do you knowing the Heisenberg picture? That X Heisenberg of t times P Heisenberg of t commutator is equal to the Heisenberg version of this. But here was the unit operator. And therefore this is just ih bar times the unit operator again, because the units operator is the same in all pictures.

So these commutation relation is true for any Heisenberg operator. Whatever commutation relation you have of Schrodinger, it's true for Heisenberg as well.

OK, so then let's talk about Hamiltonians. Three, Hamiltonians. So Heisenberg Hamiltonian by definition would be equal to U dagger t 0 Schrodinger Hamiltonian times U of t 0. So if the Schrodinger Hamiltonian-- actually, if Hs at t1 commutes units with Hs at t2, the Schrodinger Hamiltonian is such that for all t1 and t2 they commute with each other.

Remember, if that is the case, the unitary operator is any way built by an exponential. It's this one. And the Schrodinger Hamiltonians commute. So as was asked in the question before, this thing commutes with that, and you get that they are the same. So if this is happening, the two Hamiltonians are identical. And we'll have the chance to check this today in a nice example.

So I will write in this as saying the Heisenberg Hamiltonian as a function of time then is equal to the Schrodinger Hamiltonian as a function of time. And this goes Hs of t1 and Hs of t2 commute. OK, now I want you to notice this thing. Suppose the Hs of t is some Hs of x,p, and t, for example.

OK, now you come and turn it into Heisenberg by putting a U dagger from the left and a U from the right. What will that do? It will put U dagger from the left, U dagger on the right. And then it will start working it's way inside, and any x that it will find will turn into a Heisenberg x. Any p will turn into Heisenberg p.

Imagine, for example, any Hamiltonian is some function of x. It has an x squared. Well the U dagger and U come and turn this into x Heisenberg squared. So what I claim here happens is that H Heisenberg of t is equal to U dagger H Schrodinger of x, p, t, U. And therefore this becomes H Schrodinger of x Heisenberg of t, P Heisenberg of t, and t.

So here is what the Heisenberg Hamiltonian is. It's the Schrodinger Hamiltonian where X's, and P's, or spins and everything has become Heisenberg. So the equality of the two Hamiltonians is a very funny condition on the Schrodinger Hamiltonian, because this is supposed to be equal to the Schrodinger Hamiltonian, which is of x, p, and t.

So you have a function of x, p, and t. And you put X Heisenberg P Heisenberg, and somehow the whole thing is the same. So this is something very useful and we'll need it.

One more comment, expectation values. So this is three. Comment number four on expectation values, which is something you've already-- it's sort of the way we began the discussion and wanted to make sure it's clear. So four, expectation values.

So we started with this with alpha and beta, two arbitrary states, matrix elements. Take them equal and to be equal to psi of t. So you would have psi t As psi t is, in fact, equal to psi 0 A Heisenberg psi 0. Now that is a key equation. You know you're doing expectation value at any given time of a Schrodinger operator, turn it into Heisenberg and work at time equals 0. It simplifies life tremendously.

Now this is the key identity. It's the way we motivated everything in a way. And it's written in a way that maybe it's a little too schematic, but we write it this way. We just say the expectation value of As is equal to the expectation value of AH.

And this, well, we save time like that, but you have to know what you mean. When you're computing the expectation value for a Schrodinger operator, you're using time dependent states. When you're computing the expectation value of the Heisenberg operator, you're using the time equals 0 version of the states, but they are the same. So we say that the Schrodinger expectation value is equal to the Heisenberg expectation value. We right it in the bottom, but we mean the top equation. And we use it that way.

So the Heisenberg operators, at this moment, are a little mysterious. They're supposed to be given by this formula, but we've seen that calculating U can be difficult. So calculating the Heisenberg operator can be difficult sometimes.

So what we try to do in order to simplify that is find an equation that is satisfied by the Heisenberg operator, a time derivative equation. So let's try to find an equation that is satisfied by the Heisenberg operator rather than a formula. You'll say, well, this is better. But the fact is that seldom you know U. And even if you know U, you have to do this simplification, which is hard.

So finding a differential equation for the operator is useful. So differential equation for Heisenberg operators. So what do we want to do? We want to calculate ih bar d dt of the Heisenberg operator.

And so what do we get? Well, we have several things. Remember, the Schrodinger operator can have a bit of time dependence. The time dependence would be an explicit time dependence. So let's take the time derivative of all this. So you would have three terms. ih bar dU dagger dt As U plus U dagger As dU dt plus-- with an ih bar-- U dagger ih bar dAs minus dt. dAs dt and U.

Well, you have these equations. Those were the Schrodinger equations we started with today. The derivatives of U, or the derivatives of U dagger. so what did we have? Well, we have that ih bar dU dt was HU-- H Schrodinger times U. And therefore ih bar dU dagger dt. I take the dagger of this. I would get a minus sign. I would put it on the other side. Is equal to U dagger Hs with a minus here. And all the U's are U's of t and t0. I ran out of this thick chalk. So we'll continue with thin chalk.

All right, so we are here. We wrote the time derivative, and we have three terms to work out. So what are they? Well we have this thing, ih bar this. So let's write it. ih bar d d dt of As-- of A Heisenberg, I'm sorry-- Is equal to that term is minus U dagger Hs A Schrodinger U.

The next term plus ih bar dU dt on the right. So we have plus U dagger As Hs dU dt, so U. Well, that's not bad. It's actually quite nice. And then the last term, which I have very little to say, because in general, this is a derivative of a time dependent operator. Partial with respect to time, it would be 0 if As depends, just say, on X, on P, on Sx, or any of those things, has to have a particular t.

So I will just leave this as plus ih bar dAs dt Heisenberg. The Heisenberg version of this operator using the definition that anything, any operator that we have a U dagger in front, a U to the right, is the Heisenberg version of the operator.

So I think I'm doing all right with this equation. So what did we have? Here it is. ih bar d dt of A Heisenberg of t. And now comes the nice thing, of course. This thing, look at it. U dagger U. This turns everything here into Heisenberg. H Heisenberg, A Heisenberg. Here you have A Heisenberg H Heisenberg, and what you got is the commutator between them. So this thing is A Heisenberg commutator with H Heisenberg. That whole thing. And then you have plus ih bar dAs dt Heisenberg.

So that is the Heisenberg equation of motion. That is how you can calculate a Heisenberg operator if you want. You tried to solve this differential equation, and many times that's the simplest way to calculate the Heisenberg operator. So there you go. It's a pretty important equation.

So let's consider particular cases immediately to just get some intuition. So remarks. Suppose As has no explicit time dependence. So basically, there's no explicit t, and therefore this derivative goes away. So the equation becomes ih bar dAh, of course, dt is equal to Ah Heisenberg sub h of t.

And you know the Heisenberg operator is supposed to be simpler. Simple. If the Schrodinger operator is time independent, it's identical to the Schrodinger

Hamiltonian. Even if the Schrodinger operator has time dependence, but they commute, this will become the Schrodinger Hamiltonian. But we can leave it like that. It's a nice thing anyway.

Time dependence of expectation value. So let me do a little remark on time dependence of expectation values. So suppose you have the usual thing that you want to compute. How does the expectation value of a Schrodinger operator depend on time?

You're faced with that expectation value of As, and it changes in time, and you want to know how you can compute that. Well, you first say, OK, ih bar d dt. But this thing is nothing but psi 0 A Heisenberg of t psi 0.

Now I can let the derivative go in. So this becomes psi 0 ih bar dAh dt psi 0. And using this, assuming that A is still no time dependence, A has no explicit time dependence, then you can use just this equation, which give you psi 0 Ah Hh psi 0.

So all in all, what have you gotten? You've gotten a rather simple thing, the time derivative of the expectation values. So ih bar d dt. And now I write the left hand side as just expectation value of H Heisenberg of t.

And on the left hand side has to the A Schrodinger expectation value, but we call those expectation values the same thing as a Heisenberg expectation value. So this thing becomes the right hand side is the expectation value of A Heisenberg H Heisenberg like that.

And just the way we say that Heisenberg expectation values are the same as Schrodinger expectation values, you could as well write, if you prefer, as d dt of A Schrodinger is equal to the expectation value of A Schrodinger with H Schrodinger.

It's really the same equation. This equation we derived a couple of lectures ago. And now we know that the expectation values of Schrodinger operators are the same as the expectation value of their Heisenberg counterparts, except that the states are taking up time equals 0.

So you can use either form of this equation. The bottom one is one that you've already seen. The top one now looks almost obvious from the bottom one, but it really took quite a bit to get it.

One last comment on these operators. How about conserved operators? What are those things? A time independent As is set to be conserved if it commutes with a Schrodinger Hamiltonian. If As commutes with As equals 0.

Now you know that if As with Hs is 0, Ah with Hh is 0, because the map between Heisenberg and Schrodinger pictures is a commutator that is valued at the Schrodinger picture is valued in the Heisenberg picture by putting H's. So what you realize from this is that this thing, this implies Ah commutes with Hh. And therefore by point 1, by 1, you have to dAh dt is equal to 0.

And this is nice, actually. The Heisenberg operator is actually time independent. It just doesn't depend on time. So a Schrodinger operator, it's a funny operator. It doesn't have time in there. It has X's, P's, spins, and you don't know in general, if it's time independent in the sense of conserve of expectation values.

But whenever As commutes with Hs, well, the expectation values don't change in time. But as you know, this d dt can be brought in, because the states are not time dependent. So the fact that this is 0 means the operator, Heisenberg operator, is really time independent.

Whenever you have a Schrodinger operator, has no t, the Heisenberg one can have a lot of t. But if the operator is conserved, then the Heisenberg operator will have no t's after all. It will really be conserved.

So let's use our last 10 minutes to do an example and illustrate much of this. In the notes, there will be three examples. I will do just one in lecture. You can do the other ones in recitation next week. There's no need really that you study these things at this moment. Just try to get whatever you can now from the lecture, and next week you'll go back to this.

So the example is the harmonic oscillator. And it will illustrate the ideas very nicely, I think. The Schrodinger Hamiltonian is p squared over 2m plus 1/2 m omega squared x hat squared.

OK, I could put x Schrodinger and p Schrodinger, but that would be just far too much. x and p are the operators you've always known. They are Schrodinger operators. So we leave them like that.

Now I have to write the Heisenberg Hamiltonian. Well, what is the Heisenberg Hamiltonian? yes?

AUDIENCE: It's identical.

PROFESSOR: Sorry?

AUDIENCE: It's identical.

PROFESSOR: Identical, yes. But I will leave that for a little later. I will just assume, well, I'm supposed to do U dagger U. As you said, this is a time independent Hamiltonian. It better be the same, but it will be clearer if we now write what it should be in general. Have a U dagger and a U from the right. They come here, and they turn this into P Heisenberg over 2m plus 1/2 m omega squared x Heisenberg. OK, that's your Heisenberg Hamiltonian.

And we will check, in fact, that it's time independent. So how about the operators X Heisenberg and P Heisenberg. What are they? Well, I don't know how to get them, unless I do this sort of U thing. That doesn't look too bad but certainly would be messy. You would have to do an exponential of e to the minus iHt over t with the x operator and another exponential. Sounds a little complicated. So let's do it the way the equations of the Heisenberg operators tell you.

Well, X and P are time independent Schrodinger operators, so that equation that I boxed holds. So ih dx Heisenberg dt is nothing else than X Heisenberg commuted with H Heisenberg. OK, can we do that commutator? Yes. Because X Heisenberg, as you remember, just commutes with P Heisenberg. So instead of the Hamiltonian, you can put this. This is X Heisenberg P Heisenberg squared over 2m.

OK well, X Heisenberg P Heisenberg is like you had X and P. So what is this commutator? You probably know it by now. You act with this and these two p. So it acts on one, acts on the other, gives the same on each. So you get P Heisenberg times the commutator of X and P, which is ih bar times a factor of 2. So we could put hats. Better maybe.

And then what do we get? The ih there and ih cancels. And we get some nice equation that says dX Heisenberg dt is 1 over m P Heisenberg. Well, it actually looks like an equation in classical mechanics. dx dt is P over m. So that's a good thing about the Heisenberg equations of motion. They look like ordinary equations for dynamical variables.

Well, we've got this one. Let's get P. Well, we didn't get the operator still, but we got an equation. So how about P dP dt. So ih dP Heisenberg dt would be P Heisenberg with H Heisenberg. And this time only the potential term in here matters. So it's P Heisenberg with 1/2 m omega squared X Heisenberg squared.

So what do we? We get 1/2 m omega squared. Then we get again a factor of 2. Then we get one left over Xh. And then a P with Xh, which is a minus ih bar. So the ih bars cancel, and we get dPh dt is equal to-- the h bar cancelled-- m omega squared Xh. Minus m.

All right, so these are our Heisenberg equations of motion. So how do we solve for them now? Well, you sort of have to try the kind of things that you would do classically. Take a second derivative of this equation. d second Xh dt squared would be 1 over m dPh dt. And the dPh dt would be [INAUDIBLE] 1 over m times minus m omega squared Xh.

So d second Xh dt squared is equal to minus omega squared Xh, exactly the equation of motion of a harmonic oscillator. It's really absolutely nice that you recover those equations that you had before, except that now you're talking operators. And it's going to simplify your life quite dramatically when you try to use these operators, because, in a sense, solving for the time dependent Heisenberg operators is the same as finding the time evolution of all states. This time the operators change, and you will know what they change like.

So you have this. And then you write Xh is equal to A cosine omega t plus B sine omega t where A and B are some time independent operators. So Xh of t, well, that's a solution.

How about what is P? Ph of t would be m dX m dx dt. So you get minus m omega sine omega tA plus m omega cosine omega tB. OK, so that's it. That is the most general solution.

But it still doesn't look like what you would want, does it? No, because you haven't used the time equals 0 conditions. At time equals 0, the Heisenberg operators are identical they to the Schrodinger operators. So at t equals 0, Xh of t becomes A, but that must be X hat, the Schrodinger operator. And at t equals 0, Ph of t becomes equal to this is 0 m omega B. And that must be equal to the P hat operator.

So actually we have already now A and B. So B from here is P hat over m omega. And therefore Xh of t is equal to A, which is X hat cosine omega t plus B, which is P hat over m omega sine omega t. An Ph of t is here. A is-- Ph of t is m omega B, which is [INAUDIBLE] P. So it's P hat cosine omega t minus m omega X hat sine omega t.

So let's see. I hope I didn't make mistakes. P hat minus m omega X hat sine omega t. Yep, this is correct. This is your whole solution for the Heisenberg operators. So any expectation value of any power of X and P that you will want to find its time dependence, just put those Heisenberg operators, and you will calculate things with states at time equals 0. It will become very easy.

So the last thing I want to do is complete the promise that we had about what is the Heisenberg Hamiltonian. Well, we had the Heisenberg Hamiltonian there. And now we know the Heisenberg operators in terms of the Schrodinger one.

So Hh of t is equal to Ph-- 1/2m Ph squared. So I have P hat cosine omega t minus m omega X hat sine omega t squared plus 1/2 m omega squared Xh squared. So X hat cosine omega t plus P hat over m omega sine omega t squared.

So that's what the Heisenberg Hamiltonian is. So let's simplify this. Well, let's square these things. You have 1/2m cosine squared omega t P hat squared. Let's do the square of this one. You would have plus 1/2m m squared omega squared sine squared omega t X squared.

And then we have the cross product, which would be plus-- or actually minus 1/2m. The product of these two things. m omega sine omega t cosine omega t. And you have Px plus XP.

OK, I squared the first terms. Now the second one. Well, let's square the P squared here. What do we have? 1/2 m omega squared over m squared omega squared sine squared of omega t P squared. The x plus 1/2 m omega squared cosine squared omega t X squared.

And the cross term. Plus 1/2 m omega squared over m omega times cosine omega t sine omega t XP plus PX. A little bit of work, but what do we get? Well, 1/2 m. And here we must have 1/2 m, correct. 1/2 m. Sine squared omega t P squared. So this is equal 1/2 m P squared.

These one's, hall here you have 1/2 m omega squared. So it's 1/2 m omega squared cosine and sine squared is X hat squared. And then here we have all being over 2. And here omega over 2, same factors, same factors, opposite signs. Very good. Schrodinger Hamiltonian.

So you confirm that this theoretical expectation is absolutely correct. And what's the meaning? You have the Heisenberg Hamiltonian written in terms of the Heisenberg variables. But by the time you substitute these Heisenberg variables in, it just becomes identical to the Schrodinger Hamiltonian.

All right, so that's all for today. I hope to see in office hours in the coming days. Be here Wednesday 12:30, maybe 12:25 would be better, and we'll see you then.

[APPLAUSE]