8.05 Quantum Physics II, Fall 2012
FINAL EXAM
Tuesday December 18, 1:30 pm -4:30 pm
You have 3 hours.

Answer all problems in the white books provided. Write YOUR NAME and YOUR SECTION on your white book(s).

THIS TEST MUST BE RETURNED WITH YOUR WHITE BOOK(S)

YOUR NAME:

The formula sheet need not be returned

There are six questions, totalling 100 points.

None of the problems requires extensive algebra.

No books, notes, or calculators allowed.

TIME MANAGEMENT:

One hour to answer questions 1,2, and 3.
30 minutes for problem 4.
45 minutes for each problem 5 and 6.
The last two questions in problem 5 and the last in problem 6 may be challenging

1. **True or false questions** [20 points] No explanations required. Just indicate T or F for true or false, respectively.

   (1) The wavefunction of a particle in a three-dimensional central potential must vanish at the origin.

   (2) The energy spectrum of a particle on a three-dimensional spherical infinite well ($V = 0$ for $r \leq a$, and $V = \infty$ otherwise) has a number of degeneracies that are not explained by angular momentum conservation.

   (3) $[H, L_z] = 0$ and $[H, L_x] = 0$ guarantee the full rotational invariance of the Hamiltonian.

   (4) When the spin angular momentum takes a half-integer value, the orbital angular momentum must also take a half-integer value in order that the total angular momentum takes an integer value.

   (5) $\mathbf{r} \times \mathbf{p} \;=\; -\mathbf{p} \times \mathbf{r}$.

   (6) $\mathbf{r} \times \mathbf{L} \;=\; -\mathbf{L} \times \mathbf{r}$.

   (7) If two Schrödinger operators commute, the corresponding Heisenberg operators may fail to commute if the Schrödinger operators have explicit time dependence.

   (8) The Heisenberg Hamiltonian is given by the Schrödinger Hamiltonian with each operator replaced by its Heisenberg version.

   (9) For a simple harmonic oscillator the Heisenberg number operator is identical to the Schrödinger number operator.

   (10) The knowledge of the time evolution operator $\mathcal{U}(t, t_0)$ does not suffice to construct the Hamiltonian of the system.

2. **Two short problems** [10 points]

   (a) A spin-1/2 particle with $\vec{\mu} = \gamma \vec{S}$ is in the up state along the z axis at $t = 0$. Give a time-independent magnetic field $\vec{B}$ that will make the spin trace a cone in which the spin points successively along the x, y, and z axis, in a motion with period T.

   (b) Consider the state built by combining two spin-1/2 particles as follows

$$\frac{1}{\sqrt{2}}\Big(|\vec{n};+\rangle|\vec{n};-\rangle - |\vec{n};-\rangle|\vec{n};+\rangle\Big)\,.$$

   Demonstrate that this is a singlet of the total angular momentum for any choice of the direction $\vec{n}$ by writing this state in terms of $|+\rangle$ and $|-\rangle$ states.

3. **Runge Lenz vector** [10 points]

(a) [4 pt] Is it true that $\mathbf{r}\cdot\mathbf{L} = 0$ and $\mathbf{p}\cdot\mathbf{L} = 0$? Explain your answer. (Their analogs are true in classical mechanics!).

(b) [6 pt] The quantum Runge-Lenz vector can be written as

$$\mathbf{R} \;=\; \frac{1}{me^2}\big(\,\mathbf{p}\times\mathbf{L} - i\hbar\,\mathbf{p}\big) - \frac{\mathbf{r}}{r}\,.$$

Prove that

$$\mathbf{R}\cdot\mathbf{L} \;=\; 0\,.$$

This identity was used to show that the two hidden angular momenta in hydrogen have the same magnitude.

4. **Squeezing a Hamiltonian** [15 points]

The squeezing operator $S(\gamma)$ of the harmonic oscillator is unitary

$$S(\gamma) \;=\; \exp\Big(-\frac{\gamma}{2}(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a})\Big)\,, \quad \gamma\in\mathbb{R}\,.$$

Given the Hamiltonian

$$H(m,k) \;=\; \frac{\hat{p}^2}{2m} + \frac{1}{2}\,k\,\hat{x}^2\,, \quad \omega^2 \equiv \frac{k}{m}\,,$$

it is then natural to ask what is

$$S^\dagger(\gamma)\,H(m,k)\,S(\gamma) \;=\; \cdots \;\;?$$

(a) Show that the right-hand side above is $H(m',k')$ with some new parameters m' and k' that you should determine.

(b) How are the energy levels of $H(m,k)$ and $H(m',k')$ related ? Given an eigenstate $|E\rangle$ of $H(m,k)$, how would you build an energy eigenstate of $H(m',k')$?

(c) Would you say that the action of $S(\gamma)$ is a symmetry of the harmonic oscillator? Explain.

5. **A Three Dimensional Harmonic Oscillator** [25 points]

Consider a particle of mass m in a three-dimensional isotropic harmonic oscillator:

$$\hat{H} \;=\; \hbar\omega\big(\hat{N}_1 + \hat{N}_2 + \hat{N}_3 + \tfrac{3}{2}\big) \;=\; \hbar\omega\big(\hat{N} + \tfrac{3}{2}\big)\,,$$

$$\hat{N}_1 \;=\; \hat{a}_1^\dagger \hat{a}_1\,, \;\; \hat{N}_2 \;=\; \hat{a}_2^\dagger \hat{a}_2\,, \;\; \hat{N}_3 \;=\; \hat{a}_3^\dagger \hat{a}_3\,, \quad \hat{N} \;\equiv\; \hat{a}_i^\dagger \hat{a}_i\,, \;\; [\,\hat{a}_i\,,\,\hat{a}_j^\dagger] \;=\; \delta_{ij}\,, \quad i,j = 1,2,3\,.$$

As usual $\hat{a}_i^\dagger$ and $\hat{a}_i$ are raising and lowering operators, $\hat{N}_i$ are number operators and $\hat{N}$ is the *total* number. Repeated indices are summed over. Consider the following nine operators defined by

$$T_{ij} \;\equiv\; \hat{a}_i^\dagger\, \hat{a}_j\,, \qquad i,j = 1,2,3.$$

(a) Some T's are familiar. Write $\hat{H}$ and the number operators $\hat{N}_i$ in terms of T's.

(b) The T's form a closed algebra under commutation (commutators of T's give linear combinations of T's). Confirm this by evaluating the commutator $[T_{ij}\,, T_{pq}\,]$.

(c) Are the T operators symmetries? Answer this by computing $[\hat{H}, T_{pq}]$.

(d) Write the state $\hat{a}_1^\dagger(\hat{a}_3^\dagger)^2|0\rangle$ as the action of some T acting on the state $(\hat{a}_3^\dagger)^3|0\rangle$.

We consider redefined creation and annihilation operators

$$\hat{a}_L \equiv \frac{1}{\sqrt{2}}(\hat{a}_1 + i\hat{a}_2), \quad \hat{a}_R \equiv \frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2)\,, \quad [\hat{a}_L\,, \hat{a}_L^\dagger] \;=\; [\hat{a}_R\,, \hat{a}_R^\dagger] = 1\,,$$

$$\hat{N}_L = \hat{a}_L^\dagger \hat{a}_L, \quad \hat{N}_R = \hat{a}_R^\dagger \hat{a}_R\,, \quad \hat{N}_3 = \hat{a}_3^\dagger \hat{a}_3\,.$$

Any operator in $(\hat{a}_L, \hat{a}_L^\dagger)$ commutes with any operator in $(\hat{a}_R, \hat{a}_R^\dagger)$. With these operators the Hamiltonian and the (honest!) angular momentum operators are given by

$$\begin{aligned}
\hat{H} \;&=\; \hbar\omega\big(\hat{N}_L + \hat{N}_R + \hat{N}_3 + \tfrac{3}{2}\big)\,,\\
\hat{L}_z \;&=\; \hbar\;(\hat{a}_R^\dagger \hat{a}_R - \hat{a}_L^\dagger \hat{a}_L) \;=\; \hbar(\hat{N}_R - \hat{N}_L)\,,\\
\hat{L}_+ \;&=\; \sqrt{2}\,\hbar\,(\,\hat{a}_3^\dagger\,\hat{a}_L - \hat{a}_R^\dagger\,\hat{a}_3\,)\,,\\
\hat{L}_- \;&=\; \sqrt{2}\,\hbar\,(\,\hat{a}_L^\dagger\,\hat{a}_3 - \hat{a}_3^\dagger\,\hat{a}_R\,)\,.
\end{aligned}$$

This basis of operators is best to analyze the angular momentum content of states.

(e) Consider the set of degenerate states of total number $\hat{N} = n$, with n some positive integer. These states organize into representations of angular momentum with various values of ℓ. What is the highest value ℓ_* of ℓ? Explain. What are the other values of ℓ that arise ? You need *not* explain this (we derived it in class).

For the $\ell = \ell_*$ multiplet construct explicitly the *three* states with largest value of L_z. For the multiplet with next highest value ℓ_{**} of ℓ, construct the state with maximal L_z. Don't bother to normalize any of the states.

(f) Find an operator K that is T-like (namely, involves the product of one creation and one annihilation operator), commutes with the Hamiltonian, and acting on $|\ell_{**}, \ell_{**}\rangle$ gives $|\ell_*, \ell_*\rangle$. This operator "explains" the degeneracy of the ℓ_* and ℓ_{**} multiplets.

6. **Conservation of Angular Momentum in Particle Decays.** [20 points]

This problem deals with the decay of an unstable particle X to two daughter particles A and B. In the decay process $X \to AB$, total angular momentum is conserved. In the X rest frame, the total angular momentum $\mathbf{J} = \mathbf{S}_X$ is given just by the spin of particle X. After the decay, the total angular momentum receives three contributions:

$$\mathbf{J} = \mathbf{S}_A + \mathbf{S}_B + \mathbf{L},$$

where $\mathbf{S}_A$ is the spin of particle A, $\mathbf{S}_B$ is the spin of particle B, and $\mathbf{L}$ is the relative orbital angular momentum between A and B. Because angular momentum is conserved in this decay, if the initial state is an eigenstate of $\mathbf{J}^2$ and $\mathbf{J}_z$, then the final state is also an eigenstate with the same eigenvalues.

(a) Consider the case where X is spin-0 and both A and B are spin-1/2 (i.e. $s_X = 0$, $s_A = 1/2$, $s_B = 1/2$). What values of the orbital angular momentum ℓ are allowed, consistent with angular momentum conservation?

(b) Repeat the above problem for $s_X = 3/2$, $s_A = 1/2$, and $s_B = 1$.

(c) There are certain processes for which a two-body decay is forbidden. Explain why a neutron n *cannot* decay to a proton p and an electron e^- via the decay $n \to p\, e^-$, despite the fact that this decay is consistent with energy and charge conservation. Recall that both the proton and the neutron are spin-1/2 particles. [Note: the three-body decay $n \to p\, e^- \,\overline{\nu}$ is allowed, where $\overline{\nu}$ is a spin-1/2 antineutrino.]

(d) A mystery particle X of unknown spin s_X is polarized such that $m_X = +s_X$. It decays via $X \to AB$, where $s_A = 1/2$, $s_B = 0$, but the relative orbital angular momentum ℓ is also unknown. (For simplicity, you may assume the decay yields a single value of ℓ.) After the decay, the z-component m_A of the spin of particle A is measured, and it is found to have probabilities:

$$P(m_A = 1/2) = 1/5, \qquad P(m_A = -1/2) = 4/5\,.$$

What is s_X and what is ℓ? [Hint: you may want to consider the action of $\mathbf{J}_+$ on both the initial and final states.]

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