PROFESSOR: All right, it is time to get started. Thanks for coming for this cold and rainy Wednesday before Thanksgiving. Today we're supposed to talk about the radial equation. That's our main subject today. We discussed last time the states of angular momentum from the abstract viewpoint, and now we make contact with some important problems, and differential equations, and things like that.

And there's a few concepts I want to emphasize today. And basically, the main concept is that I want you to just become familiar with what we would call the diagram, the key diagram for the states of a theory, of a particle in a three dimensional potential. I think you have to have a good understanding of what it looks, and what is special about it, and when it shows particular properties.

So to begin with, I'll have to do a little aside on a object that is covered in many courses. I don't know to what that it's covered, but it's the subject of spherical harmonics. So we'll talk about spherical harmonics for about 15 minutes. And then we'll do the radial equation. And for the radial equation, after we discuss it, we'll do three examples. And that will be the end of today's lecture.

Next time, as you come back from the holiday next week, we are doing the addition of angular momentum basically. And then the last week, more examples and a few more things for emphasis to understand it all well.

All right, so in terms of spherical harmonics, I wanted to emphasize that our algebraic analysis led to states that we called jm, but today I will call lm, because they will refer to orbital angular momentum. And as you've seen in one of your problems, orbital angular momentum has to do with values of j, which are integers.

So half integers values of j cannot be realized for orbital angular momentum. It's a very interesting thing. So spin states don't have wave functions in the usual way. It's only states of integer angular momentum that have wave functions. And those are the spherical harmonics. So I will talk about lm, and l, as usual, will go from 0 to infinity. And m goes from l to minus l.

And you had these states, and we said that algebraically you would have L squared equals h squared l times l plus 1 lm. And Lz lm equal hm lm.

Now basically, the spherical harmonics are going to be wave functions for these states. And the way we can approach it is that we did a little bit of work already with constructing the l squared operator. And in last lecture we derived, starting from the fact that L is r cross p and using x,y, and z, px, py, pz, and passing through spherical coordinates that L squared is the operator minus h squared 1 over sine theta d d theta sine theta d d theta again plus 1 over sine squared theta d second d phi squared.

And we didn't do it, but Lz, which you know is h bar over i x d dy minus y d dx can also be translated into angular variables. And it has a very simple form. Also purely angular. And you can interpret it Lz is rotations around the z-axis, so they change phi. So it will not surprise you, if you do this exercise, that this is h over i d d phi. And you should really check it.

There's another one that is a bit more laborious. L plus minus, remember, is Lx plus minus i Lz. We have a big attendance today. Is equal to-- more people. h bar e to the plus minus i phi i cosine theta over sine theta d d phi plus minus d d theta.

And that takes a bit of algebra. You could do it. It's done in many books. It's probably there in Griffith's. And these are the representations of these operators as differential operators that act and function on theta and phi and don't care about radius.

So in mathematical physics, people study these things and invent these things called spherical harmonics Ylm's of theta and phi. And the way you could see their done is in fact, such that this L squared viewed as this operator, differential operator, acting on Ylm is indeed equal to h squared l times l plus 1 Ylm. And Lz thought also as a differential operator, the one that we've written there. On the Ylm is h bar m Ylm.

So they are constructed in this way, satisfying these equations. These are important equations in mathematical physics, and these functions were invented to satisfy those equations. Well, these are the properties of those states over there. So we can think of these functions as the wave functions associated with those states.

So that's interpretation that is natural in quantum mechanics. And we want to think of them like that. We want to think of the Ylm's as the wave functions associated to the states lm. So lm. And here you would put a position state theta phi.

This is analogous to the thing that we usually call the wave function being a position state times the state side. So we want to think of the Ylm's in this way as pretty much the wave functions associated to those states.

Now there is a little bit of identities that come once you accept that this is what you think of the Ylm's. And then the compatibility of these equations. Top here with these ones makes in this identification natural.

Now in order to manipulate and learn things about those spherical harmonics the way we do things in quantum mechanics, we think of the completeness relation. If we have d cube x x x, this is a completeness relation for position states.

And I want to derive or suggest a completeness relation for these theta phi states. For that, I would pass this integral to do it in spherical coordinates. So I would do dr rd theta r sine theta d phi. And I would put r theta phi position states for these things. And position states r theta phi. Still being equal to 1.

And we can try to split this thing. It's natural for us to think of just theta phi, because these wave functions have nothing to do with r, so I will simply do the integrals this way. d theta sine theta d phi. And think just like a position state in x, y, z. It's a position state in x, in y, and in z multiplied. We'll just split these things without trying to be too rigorous about it. Theta and phi like this.

And you would have the integral dr r squared r r equal 1. And at this point, I want to think of this as the natural way of setting a completeness relation for theta and phi. And this doesn't talk to this one, so I will think of this that in the space of theta and phi, objects that just depend on theta and phi, this acts as a complete thing. And if objects depend also in r, this will act as a complete thing.

So I will-- I don't know. Maybe the right way to say is postulate that we'll have a completeness relation of this form. d theta sine theta d phi theta phi theta phi equals 1. And then with this we can do all kinds of things. First, this integral is better written. This integral really represents 0 to pi d theta sine theta 0 to 2 pi d phi. Now this is minus d cosine theta. And when theta is equal to 0, cosine theta is 1 to minus 1 integral d phi 0 to 2 pi.

So this integral, really d theta sine theta d phi this is really the integral from minus 1 to 1. Change that order of d cos theta integral d phi from 0 to 2 pi. And this is called the integral over solid angle. That's a definition. So we could write the completeness relation in the space theta phi as integral over solid angle theta phi theta phi equals 1.

Then the key property of the spherical harmonics, or the lm states, is that they are orthogonal. So delta l, l prime, delta m, m prime. So the orthogonality are of this state is guaranteed because Hermitian operators, different eigenvalues, they have to be orthogonal. Eigenstates of Hermitian. Operators with different eigenvalues.

Here, you introduce a complete set of states of theta phi. So you put l prime m prime theta phi theta phi lm. And this is the integral over solid angle of Yl prime m prime of theta phi star. This is in the wrong position. And here Ylm of theta phi being equal delta l l prime delta m m prime.

So this is orthogonality of the spherical harmonics. And this is pretty much all we need. Now there's the standard ways of constructing these things from the quantum mechanical sort of intuition. Basically, you can try to first build Yll, which corresponds to the state ll.

Now the kind of differential equations this Yll satisfies are kind of simple. But in particular, the most important one is that L plus kills this state. So basically you use the condition that L plus kills this state to find a differential equation for this, which can be solved easily. Not a hard differential equation. Then you find Yll. And then you can find Yll minus 1 and all the other ones by applying the operator L minus. The lowering operator of m.

So in principle, if you have enough patience, you can calculate all the spherical harmonics that way. There's no obstruction. But the form is a little messy, and if you want to find the normalizations so that these things work out correctly, well, it takes some work at the end of the day. So we're not going to do that here. We'll just leave it at that, and if we ever need some special harmonics, we'll just hold the answers. And they are in most textbooks. So if you do need them, well, you'll have to do with complicated normalizations.

So that's really all I wanted to say about spherical harmonics, and we can turn then to the real subject, which is the radial equation. So the radial equation. So we have a Hamiltonian H equals p squared vector over 2m plus v of r. And we've seen that this is equal to h over 2m 1 over r d second dr squared r plus 1 over 2mr squared L squared plus v of r.

So this is what we're trying to solve. And the way we attempt to solve this is by separation of variables. So we'll try to write the wave function, psi, characterized by three things. Its energy, the value of l, and the value of m. And it's a function of position, because we're trying to solve H psi equal E psi. And that's the energy that we want to consider.

So I will write here to begin with something that will not turn out to be exactly right, but it's important to do it first this way. A function of art r that has labels E, l, and m. Because it certainly could depend on E, could depend on l, and could depend on m, that radial function. And then the angular function will be the Ylm's of theta and phi.

So this is the [INAUDIBLE] sets for the equation. If we have that, we can plug into the Schrodinger equation, and see what we get. Well, this operator will act on this f.

This will have the operator L squared, but L squared over Ylm, you know what it is. And v of r is multiplicative, so it's no big problem.

So what do we have? We have minus h squared over 2m 1 over r. Now I can talk normal derivatives. d r squared r times fElm plus 1 over 2mr squared. And now have L squared acting on this, but L squared acting on the Ylm is just this factor. So we have h squared l times l plus 1 times the fElm.

Now I didn't put the Ylm in the first term because I'm going to cancel it throughout. So we have this term here plus v of r fElm equals E fElm. That is substituting into the equation h psi equal E psi. So first term here. Second term, it acted on the spherical harmonic. v of r is multiplicative. E on that.

But then what you see immediately is that this differential equation doesn't depend on m. It was L squared, but no Lz in the Hamiltonian. So no m dependent. So actually we were overly proven in thinking that f was a function of m. What we really have is that psi Elm is equal to a function of E and l or r Ylm of theta phi.

And then the differential equation is minus h squared over 2m. Let's multiply all by r. d second dr squared of r fEl. Plus look here. The r that I'm multiplying is going to go into the f. Here it's going to go into the f. Here it's going to go into the f. It's an overall thing. But here we keep h squared l times l plus 1 over 2mr squared rfEl plus v of r fEl rfEl equal e times rfEl.

So what you see here is that this function is quite natural. So it suggests the definition of uEl to be rfEl. So that the differential equation now finally becomes minus h squared over 2m d second dr squared of uEl plus there's the u here, the u here, and this potential that has two terms. So this will be v of r plus h squared l times l plus 1 over 2mr squared uEl equals E times eEl.

And this is the famous radial equation. It's an equation for you. And here, this whole thing is sometimes called the effective potential. So look what we've got. This f, if you wish here, is now of the form uEl of r Ylm over r theta phi. f is u over r.

So this is the way we've written the solution, and u satisfies this equation, which is a one dimensional Schrodinger equation for the radius r. One dimensional equation with an effective potential that depends on L.

So actually the first thing you have to notice is that the central potential problem has turned into an infinite collection of one dimensional problems. One for each value of l. For different values of l, you have a different potential. Now they're not all that different. They have different intensity of this term. For l equals 0, well you have some solutions. And for l equal 1, the answer could be quite different. For l equal 2, still different. And you have to solve an infinite number of one dimensional problems. That's what the Schrodinger equation has turned into.

So we filled all these blackboards. Let's see, are there questions? Anything so far? Yes?

AUDIENCE: You might get to this later, but what does it mean in our wave equations, in our wave function there, psi of Elm is equal to fEl, and the spherical harmonic of that one mean that one has an independence and the other doesn't. Can they be separated on the basis of m?

PROFESSOR: So it is just a fact that the radial solution is independent of n, so it's an important property. n is fairly simple. The various state, the states with angular momentum l, but different m's just differ in their angular dependence, not in the radial dependence. And practically, it means that you have an infinite set of one dimensional problems labeled by l, and not labeled by m, which conceivably could have happened, but it doesn't happen. So just a major simplicity. Yes?

AUDIENCE: Does the radial equation have all the same properties as a one dimensional Schrodinger equation? Or does the divergence in the effect [INAUDIBLE] 0 change that?

PROFESSOR: Well, it changes things, but the most serious change is the fact that, in one dimensional problems, x goes from minus infinity to infinity. And here it goes from 0 to infinity, so we need to worry about what happens at 0. Basically that's the main complication. One dimensional potential, but it really just can't go below 0. r is a radial variable, and we can't forget that. Yes?

AUDIENCE: The potential v of r will depend on whatever problem you're solving, right?

PROFESSOR: That's right.

AUDIENCE: Could you find the v of r [INAUDIBLE]?

PROFESSOR: Well that doesn't quite make sense as a Hamiltonian. You see, if you have a v of r, it's something that is supposed to be v of r for any wave function. That's the definition. So it can depend on some parameter, but that parameter cannot be the l of the particular wave function.

AUDIENCE: [INAUDIBLE] or something that would interact with the--

PROFESSOR: If you have magnetic fields, things change, because then you can split levels with respect to m. Break degeneracies and things change indeed. We'll take care of those by using perturbation theory mostly. Use this solution and then perturbation theory.

OK, so let's proceed a little more on this. So the first thing that we want to talk a little about is the normalization and some boundary conditions, because otherwise we can't really understand what's going on. And happily the discussion is not that complicated.

So we want to normalize. So what do we want? Integral d cube x psi Elm of x squared equals 1. So clearly we want to go into angular variables. So again, this is r squared dr integral d solid angle, r squared Er. And this thing is now uEl squared absolute value over r squared. Look at the right most blackboard. uEl of r, I must square it because the wave function is squared. Over r squared. And then I have Ylm star of theta phi Ylm of theta phi. And if this is supposed to be normalized, this is supposed to be the number 1.

Well happily, this part, this is why we needed to talk a little about spherical harmonics. This integral is 1, because it corresponds precisely to l equal l prime m equal m prime. And look how lucky or nice this is. r squared cancels with r squared, so the final condition is the integral from 0 to infinity dr uEl of r squared is equal to 1, which shows that kind of the u really plays a role for wave function and a line. And even though it was a little complicated, there was the r here, and angular dependence, and everything, a good wave function is one that is just think of psi as being u. A one dimensional wave function psi being u, and if you can integrate it square, you've got it.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Because I had to square this, so there was u over r.

AUDIENCE: But that's [INAUDIBLE].

PROFESSOR: Oh, I'm sorry. That parenthesis is a remnant. I tried to erase it a little. It's not squared anymore. The square is on the absolute value is r squared.

So this is good news for our interpretation. So now before I discuss the peculiarities of the boundary conditions, I want to introduce really the main point that we're going to illustrate in this lecture. This is the thing that should remain in your heads. It's a picture, but it's an important one.

When you want to organize the spectrum, you'll draw the following diagram. Energy is here and l here. And it's a funny kind of diagram. It's not like a curve or a plot. It's like a histogram or kind of thing like that.

So what will happen is that you have a one dimensional problem. If these potentials are normal, there will be bound states. And let's consider the case of bound states for the purposes of this graph, just bound states.

Now you look at this, and you say OK, what am I supposed to do? I'm going to have states for all values of l, and m, and probably some energies. So m doesn't affect the radial equation. That's very important. But l does, so I have a different problem to solve for different l.

So I will make my histogram here and put here l equals 0 at this region. l equals 1, l equals 2, l equals 3, and go on. Now suppose I fix an l. l is fixed. Now it's a Schrodinger equation for a one dimensional problem. You would expect that if the potential suitably grows, which is a typical case, E will be quantized. And there will not be degeneracies, because the bound state spectrum in one dimension is not degenerate.

So I should expect that for each l there are going to be energy values that are going to appear. So for l equals 0, I expect that there will be some energy here for which I've got a state. And that line means I got a state. And there's some energy here that could be called E1, 0 is the first energy that is allowed with l equals 0. Then there will be another one here maybe. E-- I'll write it down-- 2,0. So basically I'm labeling the energies with En,l which means the first solution with l equals 0, the second solution with l equals 0, the third solution E 3,0.

Then you come to l equals 1, and you must solve the equation again. And then for l equal 1, there will be the lowest energy, the ground state energy of the l equal 1 potential, and then higher and higher. Since the l equal 1 potential is higher than the l equals 0 potential, it's higher up. The energies should be higher up, at least the first one should be.

And therefore the first one could be a little higher than this, or maybe by some accident it just fits here, or maybe it should fit here. Well, we don't know but know, but there's no obvious reason why it should, so I'll put it here. l equals 1. And this would be E1,1. The first state with l equals 1.

Then here it could be E2,1. The second state with l equal 1 and higher up. And then for l equal-- my diagram is a little too big. E1,1. E2,1. And then you have states here, so maybe this one, l equals 2, I don't know where it goes. It just has to be higher than this one, so I'll put it here. And this will be E1,2. Maybe there's an E2,2. And here an E1,3.

But this is the answer to your problem. That's the energy levels of a central potential. So it's a good, nice little diagram in which you put the states, you put the little line wherever you find the state. And for l equals 0, you have those states.

Now because there's no degeneracies in the bound states of a one dimensional potential, I don't have two lines here that coincide, because there's no two states with the same energy here. It's just one state. And this one here. I cannot have two things there. That's pretty important to.

So you have a list of states here. And just one state here, one state, but as you can see, you're probably are catching me in a little wrong play of words, because I say there's one state here. Yes, it's one state, because it's l equals 0. One state, one state. But this state, which is one single-- this should be called one single l equal 1 multiplet. So this is not really one state at the end of the day. It's one state of the one dimensional radial equation, but you know that l equals 1 comes accompanied with three values of m.

So there's three states that are degenerate, because they have the same energy. The energy doesn't depend on l. So this thing is an l equal 1 multiplet, which means really three states. And this is three states. And this is three states. And this is 1 l equal 2 multiplet, which has possibility of m equals 2, 1, 0 minus 1 and minus 2. So in this state is just one l equal 2 multiplet, but it really means five states of the central potential. Five degenerate states, because the m doesn't change the energy. And this is five states. And this is seven states. One l equal 3 multiplet, which contains seven states.

OK, so questions? This is the most important graph. If you have that picture in your head, then you can understand really where you're going with any potential. Any confusion here above the notation? Yes?

AUDIENCE: So normally when we think about a one dimensional problem, we say that there's no degeneracy. Not really. No multiple degeneracy, so should we think of the radial equation as having copies for each m value and each having the same eigenvalue?

PROFESSOR: I don't think it's necessary. You see, you've got your uEl. And you have here you solutions. Once the uEl is good, you're supposed to be able to put any Ylm. So put l, and now the m's that are allowed are solutions. You're solving the problem. So think of a master radial function as good for a fixed l, and therefore it works for all values of m. But don't try to think of many copies of this equation. I don't think it would help you. Any other questions? Yes?

AUDIENCE: Sorry to ask, but if you could just review how is degeneracy built one more time?

PROFESSOR: Yeah. Remember last time we were talking about, for example, what is a j equal to multiplet. Well, these were a collection of states jm with j equals 2 an m sum value. And they are all obtained by acting with angular momentum operators in each other. And there are five states. The 2,2, the 2,1, the 2,0, the 2, minus 1, and the 2, minus 2.

And all these states are obtained by acting with, say, lowering operators l minus and this. Now all these angular momentum operators, all of the Li's commute with the Hamiltonian. Therefore all of these states are obtained by acting with Li must have the same energy. That's why we say that this comes in a multiplet. So when you get j-- in this case we'll call it l-- l equals 2. You get five states. They correspond to the various values of m.

So when you did that radial equation that has a solution for l equals 2, you're getting the full multiplet. You're getting five states. 1 l equal 2 multiplet. That's why one line here. That is equivalent to five states.

OK, so that diagram, of course, is really quite important. So now we want to understand the boundary conditions. So we have here this. So this probably shouldn't erase yet. Let's do the boundary conditions.

So behavior here at r equals to 0. At r going to 0. The first claim is that surprisingly, you would think, well, normalization is king. If it's normalized, it's good. So just any number. Just don't let it diverge near 0, and that will be OK.

But it turns out that that's not true. It's not right. And you need the limit as r goes to 0 of uEl of r be equal to 0. And we'll take this and explore the simplest case. That is corresponds to saying what if the limit of r goes to 0 or uEl of r was a constant?

What goes wrong? Certainly normalization doesn't go wrong. It can be a constant. u could be like that, and it would be normalized, and that doesn't go wrong.

So let's look at the wave function. What happens with this? I actually will take for simplicity, because we'll analyze it later, the example of l equals 0. So let's put even 0. l equals 0. Well, suppose you look at the wave function now, and how does it look? Psi of E0-- if l is equal to 0, m must be equal to 0-- would be this u over r times a constant.

So a constant, because y 0, 0 is a constant. And then you uE0 of r over r. So when r approaches 0, psi goes like c prime over r, some other constant over r. So I'm doing something very simple. I'm saying if uE0 is approaching the constant at the origin, if it's uE0, well, this is a constant because it's 0,0. So this is going to constant. So at the end of the day, the wave function looks like 1 over r.

But this is impossible, because the Schrodinger equation H psi has minus h squared over 2m Laplacian on psi plus dot dot dot. And the up Laplacian of 1 over r is minus 4 pi times a delta function at x equals 0.

So this means that the Schrodinger equation, you think oh I put psi equals c over r. Well, if you calculate the Laplacian, it seems to be 0. But if you're more careful, as you know for [? emm ?] the Laplacian of 1 over r is minus 4 pi times the delta function.

So in the Schrodinger equation, the kinetic term produces a delta function. There's no reason to believe there's a delta function in the potential. We'll not try such crazy potentials. A delta function in a one dimensional potential, you've got the solution. A delta function in a three dimensional potential is absolutely crazy. It has infinite number of bound states, and they just go all the way down to energies of minus infinity. It's a very horrendous thing, a delta function in three dimensions, for quantum mechanics.

So this thing, there's no delta function in the potential. And you've got a delta function from the kinetic term. You're not going to be able to cancel it. This is not a solution.

So you really cannot approach a constant there. It's quite bad. So the wave functions will have to vanish, and we can prove that, or at least under some circumstances prove it. And as all these things are, they all depend on how crazy potentials you want to accept.

So we should say something. So I'll say something about these potentials, and we'll prove a result. So my statement will be the centrifugal barrier, which is a name for this part of the potential, dominates as r goes to 0. If this doesn't happen, all bets are off.

So let's assume that v of r, maybe it's 1 over r, but it's not worse than 1 over r squared. It's 1 over r cubed, for example, or something like that. You would have to analyze it from scratch if it would be that bad. But I will assume that the centrifugal barrier dominates.

And then look at the differential equation. Well, what differential equation do I have? Well, I have this and this. This thing is less important than that, and this is also less important, because this is u divided by r squared. And here is just u. So this is certainly less important than that, and this is less important than that, and if I want to have some variation of u, or understand how it varies, I must keep this.

So at this order, I should keep just the kinetic term h squared over 2m d second dr squared u of El. And h squared l times l plus 1 over 2 mr squared. And I will try to cancel these two to explore how the wave function looks near or equal 0. These are the two most important terms of the differential equation, so I have the right to keep those, and try to balance them out to leading order, and see what I get.

So all the h squared over 2m's go away. So this is equivalent to d second uEl dr squared is equal to l times l plus 1 uEl over r squared. And this is solved by a power uEl. You can try r to the s, some number s. And then this thing gives you s times s minus 1. Taking two derivatives is equal to l times l plus 1.

As you take two derivatives, you lose two powers of r, so it will work out. And from here, you see that the possible solutions are s equals l plus 1. And s equals 2 minus l. So this corresponds to a uEl that goes like r to the l plus 1, or a uEl that goes like 1 over r to the l.

This Is far too singular. For l equals 0, we argued that the wave function should go like a constant. I'm sorry, cannot go like a constant. Must vanish. This is not possible. It's not a solution. It must vanish. For l equals 0, uE0 goes like r and vanishes. So that's consistent, and this is good.

For l equals 0, this would be like a constant as well and would be fine. But for l equals 1 already, this is 1 over r, and this is not normalizable. So this time this is not normalizable for l greater or equal than one. So this is the answer [INAUDIBLE] this assumption, which is a very reasonable assumption. But if you don't have that you have to beware.

OK, this is our condition for u there. And so uEl goes like this as r goes to 0. It would be the whole answer. So f, if you care about f still, which is what appears here, goes like u divided by r. So fEl goes like cr to the l. And when l is equal to 0, f behaves like a constant. u vanishes for l equal to 0, but f goes like a constant, which means that if you take 0 orbital angular momentum, you may have some probability of finding the particle at the origin, because this f behaves like a constant for l equals 0.

On the other hand, for any higher l, f will also vanish at the origin. And that is intuitively said that the centrifugal barrier prevents the particle from reaching the origin. There's a barrier, a potential barrier. This potential is 1 over r squared. Doesn't let you go to close to the origin. But that potential disappears for l equals 0, and therefore the particle can reach the origin. But only for l equals 0 it can reach the origin.

OK, one more thing. Behavior near infinity is of interest as well. So what happens for r goes to infinity? Well, for r goes to infinity, you also have to be a little careful what you assume. I wish I could tell you it's always like this, but it's not. It's rich in all kinds of problems.

So there's two cases where there's an analysis that is simple. Suppose v of r is equal to 0 for r greater than some r0. Or r times v of f goes to 0 as r goes to infinity. Two possibilities. The potential is plane 0 after some distance. Or the potential multiplied by r goes to 0 as r goes to infinity.

And you would say, look, you've missed the most important case. The hydrogen atom, the potential is 1 over r. r times v of r doesn't go to 0. And indeed, what I'm going to write here doesn't quite apply to the wave functions of the hydrogen atom. They're a little unusual. The potential of the hydrogen atom is felt quite far away.

So never the less, if you have those conditions, we can ignore the potential as we go far away. And we'll consider the following situation. Look that the centrifugal barrier satisfies this as well. So the full effective potential satisfies. If v of r satisfies that, r times 1 over r squared of effective potential also satisfies that.

So we can ignore all the potential, and we're left ignore the effective. And therefore we're left with minus h squared over 2m d second uEl dr squared is equal to EuEl. And that's a very trivial equation. Yes, Matt?

AUDIENCE: When you say v of r goes to 0 for r greater than [INAUDIBLE] 0. Are you effectively [INAUDIBLE] the potential?

PROFESSOR: Right, there may be some potentials like this. A potential that is like that. An attractive potential, and it vanishes after some distance. Or a repulsive potential that vanishes after some distance.

AUDIENCE: But say the potential was a [INAUDIBLE] potential. Are you just approximating it to 0 after it's [INAUDIBLE]?

PROFESSOR: Well, if I'm in the [INAUDIBLE] potential, unfortunately I'm neither here nor here, so this doesn't apply. So the [INAUDIBLE] potential is an exception. The solutions are a little more--

AUDIENCE: The conditions you're saying. [INAUDIBLE].

PROFESSOR: So these are conditions that allow me to say something. If they're not satisfied, I sort of have to analyze them case by case. That's the price we have to pay. It's a little more complicated than you would think naively.

Now here, it's interesting to consider two possibilities. The case when E is less than 0, or the case when E is greater than 0. So scattering solutions or bound state solutions. For these ones, if the energy is less than 0 and there's no potential, you're in the forbidden zone far away, so you must have a decaying exponential. El goes like exponential of minus square root of 2m E over h squared r. That solves that equation.

You see, the solution of these things are either exponential decays or exponential growths and oscillatory solutions, sines and cosines, or E to the i things. So here we have a decay, because with energy less than 0, the potential is 0. So you're in a forbidden region, so you must decay like that. In this hydrogen atom what happens is that there's a power of r multiplying here. Like r to the n, or r to the k or something like that.

If E is less than 0, you have uE equal exponential of plus minus ikr, where k is square root of 2m E over h squared. And those, again, solve that equation. And they are sort of wave solutions far away.

Now with this information, the behavior of the u's near the origin, the behavior of the u's far away, you can then make qualitative plots of how solutions would look at the origin. They grow up like r to the l. Then it's a one dimensional potential, so they oscillate maybe, but then decay exponentially. And the kind of thing you used to do in 804 of plotting how things look, it's feasible at this stage.

So it's about time to do examples. I have three examples. Given time, maybe I'll get to two. That's OK. The last example is kind of the cutest, but maybe it's OK to leave it for Monday.

So are there questions about this before we begin our examples? Andrew?

AUDIENCE: What is consumption of [INAUDIBLE] [? barrier ?] dominates. But why is that a reasonable assumptions?

PROFESSOR: Well, potentials that are just too singular at the origin are not common. Just doesn't happen. So mathematically you could try them, but I actually don't know of useful examples if a potential is very singular at the origin.

AUDIENCE: [INAUDIBLE] in the potential [INAUDIBLE] the centrifugal barrier. That [INAUDIBLE].

PROFESSOR: Right. An effective potential, the potential doesn't blow up-- your potential doesn't blow up more than 1 over r squared or something like that. So we'll just take it like that.

OK, our first example is the free particle. You would say come on. That's ridiculous. Too simple. But it's fairly non-trivial in spherical coordinates. And you say, well, so what. Free particles, you say what the momentum is. You know the energy. How do you label the states? You label them by three momenta. Or energy and direction. So momentum eigenstates, for example

But in spherical coordinates, these will not be momentum eigenstates, and these are interesting because they allow us to solve for more complicated problems, in fact. And they allow you to understand scattering out of central potential. So these are actually pretty important.

You can label these things by three numbers. p1, p2, p3. Or energy and theta and phi, the directions of the momenta. What we're going to label them are by energy l and m. So you might say how do we compare all these infinities, but it somehow works out. There's the same number of states really in either way.

So what do we have? It's a potential that v is equal to 0. So let's write the differential equation. v is equal to 0. But not v effective. So you have minus h squared over 2m d second uEl dr squared plus h squared over 2m l times l plus 1 over r squared uEl equal EuEl.

This is actually quite interesting. As you will see, it's a bit puzzling the first time. Well, let's cancel this h squared over 2m, because they're kind of annoying. So we'll put d second uEl over dr squared with a minus-- I'll keep that minus-- plus l times l plus 1 over r squared uEl. And here I'll put k squared times uEl. And k squared is the same k as before. And E is positive because you have a free particle. E is positive. And k squared is given by this, 2m E over h squared.

So this is the equation we have to solve. And it's kind of interesting, because on the one hand, there is an energy on the right hand side. And then you would say, look, it looks like this just typical one dimensional Schrodinger equation. Therefore that energy probably is quantized because it shows in the right hand side. Why wouldn't it be quantized if it just shows this way? On the other hand, it shouldn't be quantized.

So what is it about this differential equation that shows that the energy never gets quantized? Well, the fact is that the energy in some sense doesn't show up in this differential equation. You think it's here, but it's not really there.

What does that mean? It actually means that you can define a new variable rho equal kr, scale r. And basically chain rule or your intuition, this k goes down here. k squared r squared k squared r squared, it's all rho. So chain rule or changing variables will turn this equation into a minus d second uEl d rho squared plus l times l plus 1 rho squared is equal to-- times uEl-- is equal to uEl here.

And the energy has disappeared from the equation by rescaling, a trivial rescaling of coordinates. That doesn't mean that the energy is not there. It is there, because you will find solutions that depend on rho, and then you will put rho equal kr and the energies there. But there's no quantization of energy, because the energy doesn't show in this equation anymore. It's kind of a neat thing, or rather conceptually interesting thing that energy is not there anymore.

And then you look at this differential equation, and you realize that it's a nasty one. So this equation is quite easy without this. It's a power solution. It's quite easy without this, it's exponentials are this. But whenever you have a differential equation that has two derivatives, a term with 1 over x squared times the function, and a term with 1 times the function, you're in Bessel territory. All these functions have Bessel things.

And then you have another term like 1 over x d dx. That is not a problem, but the presence of these two things, one with 1 over x squared and one with this, complicates this equation. So Bessel, without this, would be exponential solution without this would be powers. In the end, the fact is that this is spherical Bessel, and it's a little complicated. Not terribly complicated. The solutions are spherical Bessel functions, which are not all that bad. And let me say what they are.

So what are the solutions to this thing? In fact, the solutions that are easier to find is that the uEl's are r times the Bessel function jl is called spherical Bessel functions. So it's not capital j that people use for the normal Bessel, but lower case l. Of kr. As you know, you solve this, and the solutions for this would be of the form rho jl for rho. But rho is kr, so we don't care about the constant, because this is a homogeneous linear equation. So some number here. You could put a constant if you wish.

But that's the solution. Therefore your complete solutions is like the psi's of Elm would be u divided by r, which is jl of kr times Ylm's of theta phi. These are the complete solutions.

This is a second order differential equation. Therefore it has to have two solutions. And this is what is called a regular solution at the origin. The Bessel functions come in j and n type. And the n type is singular at the origins, so we won't care about it.

So what do we get from here? Well, some behavior that is well known. Rho jl of rho behaves like rho to the l plus 2 over 2l plus 1 double factorial as rho goes to 0. So that's a fact about these Bessel functions. They behave that way, which is good, because rho jl behaves like that, so u behaves like r to the l plus 1, which is what we derived a little time ago. So this behavior of the Bessel function is indeed consistent with our solution.

Moreover, there's another behavior that is interesting. This Bessel function, by the time it's written like that, when you go far off to infinity jl of rho, it behaves like sine of rho minus l pi over 2. This is as rho goes to infinity. So as rho goes to infinity, this is behaving like a trigonometric function. It's consistent with this, because rho-- this is rho jl is what we call u essentially. So u behaves like this with rho equal kr. And that's consistent. This superposition of a sine and a cosine.

But it's kind of interesting though that this l pi over 2 shows up here. You see the fact that this function has to vanish at the origin. It vanishes at the origin and begins to vary. And by the time you go far away, you contract. And the way it behaves is this way. The face is determined. So that actually gives a lot of opportunity to physicists because the free particle-- so for the free particle, uEl behaves like sine of kr minus l pi over 2 as r goes to infinity.

So from that people have asked the following question. What if you have a potential that, for example for simplicity, a potential that is localized. Well, if this potential is localized, the solution far away is supposed to be a superposition of sines and cosines. So if there is no potential, the solution is supposed to be this.

Now another superposition of sines and cosines, at the end of the day, can always be written as some sine of this thing plus a change in this phase So in general, uEl will go like sine of kr minus l pi over 2 plus a shift, a phase shift, delta l that can depend on the energy.

So if you haven't tried to find the radial solutions of a problem with some potential, if the potential is 0, there's no such term. But if the potential is here, it will have an effect and will give you a phase shift. So if you're doing particle scattering experiments, you're sending waves from far away and you just see how the wave behaves far away, you do have measurement information on this phase shift. And from this phase shift, you can learn something about the potential.

So this is how this problem of free particle suddenly becomes very important and very interesting. For example, as a way through the behavior at infinity learning something about the potential. For example, if the potential is attractive, it pulls the wave function in and produces some sign of delta that the corresponds to a positive delta. If the potential is repulsive, it pushes the wave function out, repels it and produces a delta that is negative. You can track those signs thinking carefully. But the potentials will teach you something about delta.

The other case that this is interesting-- I will just introduce it and stop, because we might as well stop-- is a very important case. The square well. Well, we've studied in one dimension the infinite square well. That's one potential that you now how to solve, and sines and cosines is very easy.

Now imagine a spherical square well, which is some sort of cavity in which a particle is free to move here, but the potential becomes infinite at the boundary. It's a hollow sphere, so the potential v of r is equal to 0 for r less than a. And it's infinity for r greater than a. So it's like a bag, a balloon with solid walls impossible to penetrate. So this is the most symmetric simple potential you could imagine in the world.

And we're going to solve it. How can we solve this? Well, we did 2/3 of the work already in solving it. Why? Because inside here the potential is 0, so the particle is free. So inside here the solutions are of the form uEl go like rjl of kr. And the only thing you will need is that they vanish at the end.

So you will fix this by demanding that ka is a number z such-- well, the jl of ka will be 0. So that the wave function vanishes at this point where the potential becomes infinite. So you've solved most of the problem. And we'll discuss it in detail, because it's an important one. But this is the most symmetric potential, you may think. This potential is very symmetric, very pretty, but nothing to write home about.

If you tried to look-- and we're going to calculate this diagram. You would say well it's so symmetric that something pretty is going to happen here. Nothing happens. These states will show up. And these ones will show up, and no state ever will match another one. There's no pattern, or rhyme, or reason for it.

On the other hand, if you would have taken a potential v of r of the form beta r squared, that potential will exhibit enormous amounts of degeneracies all over. And we will have to understand why that happens.

So we'll see you next Monday. Enjoy your break. Homework will only happen late after Thanksgiving. And just have a great time. Thank you for coming today, and will see you soon.

[APPLAUSE]