Solutions to Problem Set #3

Problem 1: Clearing Impurities

Since we are asked for an approximate answer we will resort to the central limit theorem. For this we need $< x >$ and $< x^2 >$ for a single sweep of the laser beam.

$$< x > \; = \; \int_{\infty}^{\infty} x\, p(x)\, dx \; = \; \frac{2}{3}\, a \underbrace{\int_{0}^{\infty} \xi \exp(-\xi)\, d\xi}_{1} \; = \; \frac{2}{3}\, a$$

$$< x^2 > \; = \; \int_{\infty}^{\infty} x^2\, p(x)\, dx \; = \; \frac{2}{3}\, a^2 \underbrace{\int_{0}^{\infty} \xi^2 \exp(-\xi)\, d\xi}_{2} \; = \; \frac{4}{3}\, a^2$$

$$\mathrm{Var}(x) \; = \; < x^2 > - < x >^2 = \left(\frac{4}{3} - \frac{4}{9} \right) a^2 \; = \; \frac{8}{9}\, a^2$$
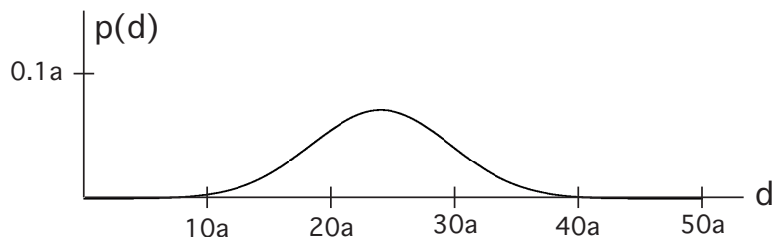
The general form of the central limit theorem is

$$p(d) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-(d - < d >)^2 / 2\sigma^2]$$

with

$$< d > \; = \; 36 \times < x > \; = \; 24\, a$$
$$\sigma^2 \; = \; 36 \times \mathrm{Var}(x) \; = \; 32\, a^2$$

Although it was not asked for, here is a sketch of the resulting probability density.

Problem 2: Probability Densities of Macroscopic verses Microscopic Variables

a) Let E_1 be the kinetic energy of a single atom in the gas. We can begin with the expression for $p(E_1)$ found in problem 4 on Problem Set 2.

$$p(E_1) \;=\; \frac{2}{\sqrt{\pi}} \frac{1}{kT} \sqrt{\frac{E_1}{kT}} \exp(-E_1/kT)$$

$$< E_1 > \;=\; \frac{2}{\sqrt{\pi}} \frac{1}{kT} \int_0^\infty E_1 \sqrt{\frac{E_1}{kT}} \exp(-E_1/kT)\, dE_1$$

$$\;=\; \frac{2}{\sqrt{\pi}} kT \underbrace{\int_0^\infty \xi\sqrt{\xi} \exp(-\xi)\, d\xi}_{\Gamma(5/2) = (3/4)\sqrt{\pi}}$$

$$\;=\; (3/2)kT$$

$$< E_1^2 > \;=\; \frac{2}{\sqrt{\pi}} \frac{1}{kT} \int_0^\infty E_1^2 \sqrt{\frac{E_1}{kT}} \exp(-E_1/kT)\, dE_1$$

$$\;=\; \frac{2}{\sqrt{\pi}} (kT)^2 \underbrace{\int_0^\infty \xi^2\sqrt{\xi} \exp(-\xi)\, d\xi}_{\Gamma(7/2) = (15/8)\sqrt{\pi}}$$

$$\;=\; (15/4)(kT)^2$$

$$\mathrm{Var}(E_1) \;=\; < E_1^2 > - < E_1 >^2 = (3/2)(kT)^2$$

$$\sigma_{E_1}/ < E_1 > \;=\; \sqrt{\frac{2}{3}} = 0.82$$

b) For the sum of N statistically independent variables, the mean is the sum of the means and the variance is the sum of the variances. Thus if E_N is the total kinetic energy of the gas

$$\sigma_{E_N}/ < E_N > = \frac{\sqrt{N}\sqrt{3/2}\, kT}{N(3/2)\, kT} = \frac{1}{\sqrt{N}} \sqrt{\frac{2}{3}} = 1.6 \times 10^{-10}$$

Problem 3: Temperature

a) Solve each equation for V.

$$V = \left(\frac{nR}{P}\right)\left(\frac{cH}{M}\right) \qquad V = \left(\frac{nR}{P}\right)\left(\Theta + \frac{c'H'}{M'}\right)$$

Equate these two and factor out nR/P.

$$\frac{cH}{M} = \Theta + \frac{c'H'}{M'} = \begin{array}{l} \text{some constant,} \\ \text{call it } h \end{array}$$

Substitution into the first equation gives $PV/nR = h$, so at equilibrium

$$\frac{PV}{nR} = \frac{cH}{M} = \Theta + \frac{c'H'}{M'}$$

b) $PV/nR = h$ looks like the ideal gas law with $h \to T$, so call $h \equiv T$ and thus find the following equations of state.

$$PV = nRT \qquad \text{for an ideal gas}$$

$$M = c\frac{H}{T} \qquad \text{for a Curie Law Paramagnet}$$

$$M' = c'\frac{H'}{T - \Theta} \qquad \begin{array}{l} \text{Paramagnet with ordering} \\ \text{phase transition to a} \\ \text{ferromagnet at } t = \Theta \end{array}$$

Problem 4: Work in a Simple Solid

Substitute the given model expression relating volume changes to changes in the pressure and the temperature, $dV = -V\mathcal{K}_T\,dP + V\alpha\,dT$, into the differential for work. As a simplification we are told to replace the actual volume V by its value at the starting point V_1 in the coefficients entering the differential for the work. Of course the volume itself can not really remain constant, for in that case $đW = -P\,dV = 0$.

$$đW = -P\,dV = \mathcal{K}_T P V_1\,dP - \alpha P V_1\,dT$$

Along path "a"

$$\begin{aligned} W_{1\to 2} &= \int_{\text{where dP=0}} đW + \int_{\text{where dT=0}} đW \\ &= -\alpha P_1 V_1 \int_1^2 dT + \mathcal{K}_T V_1 \int_1^2 P\,dP \\ &= \underline{-\alpha P_1 V_1(T_2 - T_1) + \frac{1}{2}\mathcal{K}_T(P_2^2 - P_1^2)V_1} \end{aligned}$$

Along path "b"

$$W_{1\to 2} = \int_{\text{where dT=0}} \dbar W + \int_{\text{where dP=0}} \dbar W$$

$$= \mathcal{K}_T V_1 \int_1^2 P\, dP - \alpha P_2 V_1 \int_1^2 dT$$

$$= \underline{\frac{1}{2}\mathcal{K}_T(P_2^2 - P_1^2)V_1 - \alpha P_2 V_1 (T_2 - T_1)}$$

Along "c" dT and dP are related at every point along the path,

$$dT = \frac{T_2 - T_1}{P_2 - P_1}\, dP,$$

so the expression for the differential of work can be written as

$$\dbar W = \mathcal{K}_T P V_1\, dP - \alpha P V_1 (\frac{T_2 - T_1}{P_2 - P_1})\, dP$$

$$= \left(\mathcal{K}_T V_1 - \alpha V_1 \frac{T_2 - T_1}{P_2 - P_1}\right) P\, dP$$

Now we can carry out the integral along the path.

$$W_{1\to 2} = \left(\mathcal{K}_T V_1 - \alpha V_1 \frac{T_2 - T_1}{P_2 - P_1}\right) \underbrace{\int_1^2 P\, dP}_{\substack{\frac{1}{2}(P_2^2 - P_1^2) = \\ \frac{1}{2}(P_1 + P_2)(P_2 - P_1)}}$$

$$= \underline{\frac{1}{2}\mathcal{K}_T(P_2^2 - P_1^2)V_1 - \frac{1}{2}\alpha(P_1 + P_2)(T_2 - T_1)V_1}$$

Note that the work done along each path is different due to the different contributions from the α (thermal expansion) term. Path "b" requires the least work; path "a" requires the most.

Problem 5: Work and the Radiation Field

The differential of work is $\bar{d}W = -P\,dV$ and one immediately thinks about trying to express P in terms of V in order to simplify the integral. However, along path "a" this is not necessary: along one part $dV = 0$ and along the other the temperature, and hence the pressure, is a constant.

$$W_{1\to 2} \;=\; \int_{1\ \text{T constant}}^{2} P\,dV - \underbrace{\int_{1\ \text{V constant}}^{2} P\,dV}_{0}$$

$$=\; -\frac{1}{3}\sigma T_1^4 \int_1^2 dV = -\frac{1}{3}\sigma T_1^4\,(V_2 - V_1)$$

$$=\; \underline{\frac{1}{3}\sigma T_1^4 V_2(\frac{V_1}{V_2} - 1)}$$

Since the figure in the problem indicates that $V_1 > V_2$, the underlined result is positive.

Along path "b" there are no shortcuts and we must prepare to carry out the integral. Since $\bar{d}W$ is expressed in terms of dV, we convert the T dependence of P into a function of V.

$$VT^3 \;=\; \text{a constant} \equiv V_1 T_1^3$$

$$T \;=\; (\frac{V_1}{V})^{1/3} T_1$$

$$T^4 \;=\; T_1^4\,(\frac{V_1}{V})^{4/3} = T_1^4\,(\frac{V}{V_1})^{-4/3}$$

Now we can carry out the integral over the path to obtain the total work done.

$$W_{1\to 2} \;=\; -\int_1^2 P\,dV = -\frac{1}{3}\sigma \int_1^2 T^4(V)\,dV$$

$$=\; -\frac{1}{3}\sigma T_1^4 \int_1^2 (\frac{V}{V_1})^{-4/3}\,dV$$

$$=\; -\frac{1}{3}\sigma T_1^4 V_1 \left[-\frac{1}{1/3}(\frac{V}{V_1})^{-1/3} \right]_1^2$$

$$=\; \sigma T_1^4 V_1 \left((\frac{V_2}{V_1})^{-1/3} - 1 \right)$$

$$=\; \underline{\sigma T_1^4 V_1 \left((\frac{V_1}{V_2})^{1/3} - 1 \right)}$$

Since $V_1 > V_2$, this quantity is also positive.

Problem 6: Equation of State for a Ferromagnet

a) We are looking for the magnetization as a function of the field and the temperature, $M(H, T)$, so we form the differential of M as follows.

$$dM = \left(\frac{\partial M}{\partial H}\right)_T dH + \left(\frac{\partial M}{\partial T}\right)_H dT$$

We are given the two coefficients in the expansion, but must make sure that their cross derivatives are equal as is required for an exact differential.

$$\frac{\partial}{\partial T}\left(\frac{\partial M}{\partial H}\right)_T = \frac{1}{T_c}\frac{a}{(1 - T/T_c)^2}$$

$$\frac{\partial}{\partial H}\left(\frac{\partial M}{\partial T}\right)_H = \frac{1}{T_c}\frac{f'(H)}{(1 - T/T_c)^2}$$

The equality of these two expressions requires that $f'(H) = a$. Integration gives $f(H) = aH + c$ but we are told that $f(0) = 0$ so we know that $c = 0$. Thus

$$\underline{f(H) = aH}$$

b) Now we must integrate the exact differential to find the state function $M(H, T)$.

$$M(H, T) = \int \left(\frac{\partial M}{\partial T}\right)_H dT + g(H)$$

$$= \frac{f(H)}{(1 - T/T_c)} + M_0(1 - T/T_c)^{1/2} + g(H)$$

$$\left(\frac{\partial M}{\partial H}\right)_T = \frac{f'(H)}{(1 - T/T_c)} + g'(H) \quad \text{by calculation from above}$$

$$= \frac{a}{1 - T/T_c} + 3bH^2 \qquad \text{as given}$$

Now we set about finding an expression for $g(H)$.

$$g'(H) = 3bH^2$$
$$g(H) = bH^3 + K$$
$$M(H = 0, T = T_c) = 0 \quad \Rightarrow \quad K = 0$$

Now putting all the pieces together gives

$$\underline{M(H, T) = M_0(1 - T/T_c)^{1/2} + \frac{aH}{(1 - T/T_c)} + bH^3}$$

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