

Lecture 13: Delta Function Potential, Node Theorem, and Simple Harmonic Oscillator

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Contents

1	The Delta Function Potential	1
2	The Node Theorem	4
3	Harmonic Oscillator	7

1 The Delta Function Potential

Consider a particle of mass m moving in a one-dimensional potential. The potential $V(x)$ is rather singular: it vanishes for all x except for $x = 0$ at which point it has infinite strength. More precisely, the potential is delta function localized at $x = 0$ and is written as

$$V(x) = -\alpha \delta(x), \quad \alpha > 0, \quad (1.1)$$

Here α is a constant chosen to be positive. Because of the explicit minus sign, the potential is infinitely negative at $x = 0$; the potential is attractive. The potential is shown in Fig. 1, where we represent the delta function by an arrow pointing downwards.

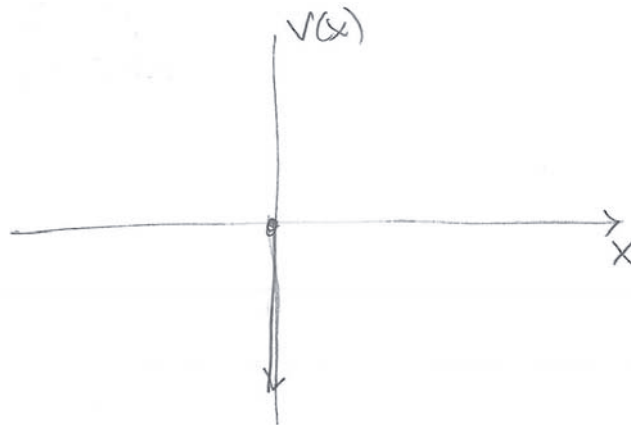


Figure 1: A delta function well.

We want to know if this potential admits bound states. For a bound state the energy E must be negative: this ensures that all of $x \neq 0$ is classically forbidden, and the wavefunction will decay rapidly allowing a normalized solution. A bit of intuition comes by thinking of the delta function as approximated by a finite square well in the limit as the width of the well goes to zero and the depth goes to infinity in such a way that the product, representing the "area" is finite (the delta function is a function with unit area, as it is clear from its integral). In Figure 2 we show two finite-well representations and sketch the wavefunction. We can see that the middle region provides the curving

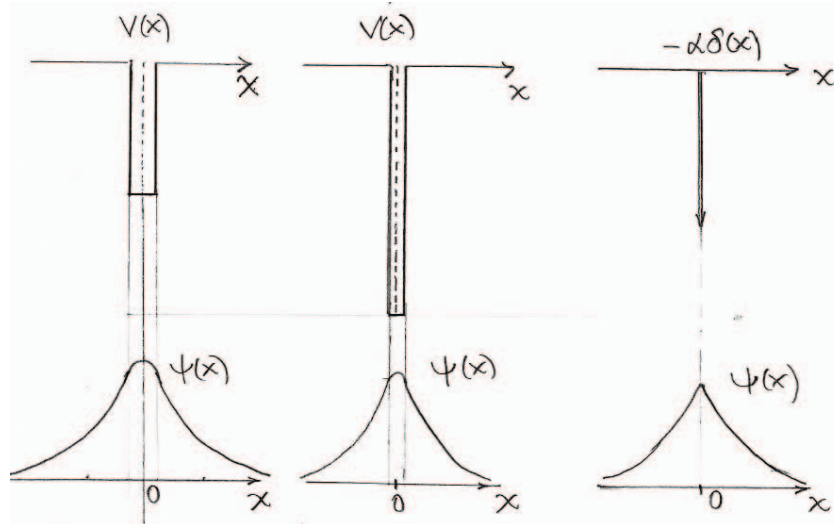


Figure 2: The delta function potential viewed as the limit as the finite square well becomes narrower and deeper simultaneously. We expect to get a wavefunction with discontinuous derivative.

of the wavefunction needed to have a smooth derivative. In the limit as the width of the region goes to zero we would expect, if there is a bound state, to have *discontinuous* derivative.

We can get further insight by considering units. The dimension-full constants in the problem are α , m , and \hbar . Since a delta function has units of one over length, the constant α must have units of energy times length for the potential to have units of energy. Thus we have, as units

$$E = \frac{\alpha}{L}, \quad (1.2)$$

but, as usual, the units of energy are

$$E = \frac{\hbar^2}{mL^2}. \quad (1.3)$$

From these two equations we find

$$L = \frac{\hbar^2}{m\alpha} \rightarrow E = \frac{m\alpha^2}{\hbar^2}. \quad (1.4)$$

The units of energy must be carried by the above combination of the constants of the problem. Therefore the energy E_b of any bound state must be a number times that combination:

$$E_b = -\# \frac{m\alpha^2}{\hbar^2}. \quad (1.5)$$

where $\#$ is a unit free positive number that we aim to determine. It is good to see α appearing in the numerator. This means that as the strength of the delta function increases, the depth of the bound state also increases, as we would naturally expect!

Let us now turn to the relevant equations. We want to find an $E < 0$ state. The wavefunction is constrained by the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = (E - V(x))\psi. \quad (1.6)$$

For $x \neq 0$, we have $V(x) = 0$, so this becomes

$$\frac{d^2\psi}{dx^2} = \left(-\frac{2mE}{\hbar^2}\right)\psi = \kappa^2\psi, \quad \kappa^2 \equiv -\frac{2mE}{\hbar^2} > 0. \quad (1.7)$$

The solutions to this differential equation are of the form

$$e^{\kappa x}, \quad e^{-\kappa x}, \quad \kappa > 0. \quad (1.8)$$

The potential is even: $\delta(-x) = \delta(x)$, so if we have a ground state it must be even and, of course, have no nodes. If there is an excited state, it must be odd and thus have a node at $x = 0$. The only odd solution we can build with the above exponentials is $\sinh \kappa x$. But a $\psi \sim \sinh \kappa x$ cannot be normalized, it blows up at $x = \pm\infty$. Therefore there cannot be an excited state in the delta function potential. If there are bound states there is just one of them!

Let us use the above solutions to build the the ground state wavefunction. First, we can see that for $x > 0$ we must discard the solution $e^{\kappa x}$, because it diverges as $x \rightarrow \infty$. Similarly we must discard $e^{-\kappa x}$ for $x < 0$. Since the wavefunction must be continuous at $x = 0$ the solution must be of the form

$$\psi(x) = \begin{cases} A e^{-\kappa x} & x > 0, \\ A e^{\kappa x} & x < 0. \end{cases} \quad (1.9)$$

Is any value of κ allowed for this solution? No, we will get another constraint by considering the derivative of the wavefunction and learning that, as anticipated, it is discontinuous. Indeed, the Schrödinger equation gives us a constraint for this discontinuity. Starting with

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \quad (1.10)$$

we integrate this equation from $-\epsilon$ to ϵ , with $0 < \epsilon \ll 1$, a range that includes the position of the delta function. This gives

$$-\frac{\hbar^2}{2m} \left(\left. \frac{d\psi}{dx} \right|_{\epsilon} - \left. \frac{d\psi}{dx} \right|_{-\epsilon} \right) + \int_{-\epsilon}^{\epsilon} dx (-\alpha\delta(x))\psi(x) = E \int_{-\epsilon}^{\epsilon} dx \psi(x). \quad (1.11)$$

The integral on the left-hand side returns a finite value due to the delta function. In the limit as $\epsilon \rightarrow 0$ the integral on the right-hand side vanishes because $\psi(x)$ is finite for all x , while the region of integration is contracting away. This yields

$$-\frac{\hbar^2}{2m} \lim_{\epsilon \rightarrow 0} \left(\left. \frac{d\psi}{dx} \right|_{\epsilon} - \left. \frac{d\psi}{dx} \right|_{-\epsilon} \right) - \alpha\psi(0) = 0. \quad (1.12)$$

We define the discontinuity Δ_0 of ψ' at $x = 0$ as

$$\Delta_0 \left(\frac{d\psi}{dx} \right) \equiv \lim_{\epsilon \rightarrow 0} \left(\left. \frac{d\psi}{dx} \right|_{\epsilon} - \left. \frac{d\psi}{dx} \right|_{-\epsilon} \right). \quad (1.13)$$

We have therefore learned that

$$\boxed{\Delta_0 \left(\frac{d\psi}{dx} \right) = -\frac{2m\alpha}{\hbar^2} \psi(0)}. \quad (1.14)$$

Note that the discontinuity in ψ' at the position of the delta function is proportional to the value of the wavefunction at that point. At a node a delta function would have no effect; ψ' would also be continuous.

Applying the discontinuity equation to our solution (1.9), we have

$$\lim_{\epsilon \rightarrow 0} \left(\frac{d\psi}{dx} \Big|_{\epsilon} - \frac{d\psi}{dx} \Big|_{-\epsilon} \right) = \lim_{\epsilon \rightarrow 0} \left(-\kappa A e^{-\kappa\epsilon} - \kappa A e^{-\kappa\epsilon} \right) = -2\kappa A = -\frac{2m\alpha}{\hbar^2} A. \quad (1.15)$$

This relation fixes the value of κ

$$\kappa = \frac{m\alpha}{\hbar^2}, \quad (1.16)$$

and therefore the value E_b of the bound state energy

$$E_b = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{1}{2} \cdot \frac{m\alpha^2}{\hbar^2}. \quad (1.17)$$

As we anticipated with the unit analysis, the answer takes the required form (1.5) and the undetermined constant $\#$ takes the value $1/2$.

2 The Node Theorem

Recall the infinite well potential

$$V(x) = \begin{cases} 0 & 0 < x < a, \\ \infty & \text{elsewhere.} \end{cases} \quad (2.18)$$

The bound states take the form

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad (2.19)$$

and corresponding energies

$$E_n = \frac{\hbar^2 n^2}{2ma^2}, \quad n = 1, 2, \dots \quad (2.20)$$

Note that ψ_n has $n - 1$ nodes (zeroes). (The points $x = 0$ and $x = a$ are *not* nodes, but rather endpoints.)

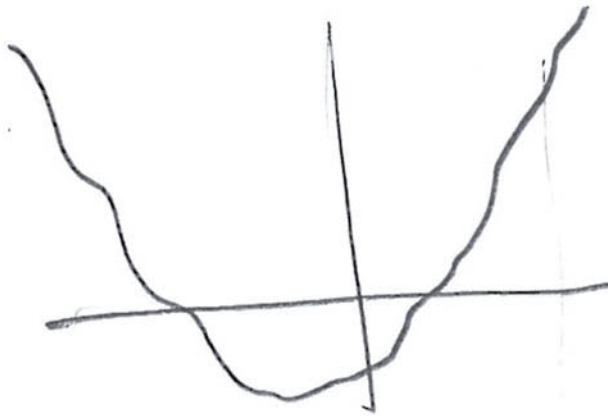


Figure 3: A smooth potential that goes to infinity as $|x| \rightarrow \infty$.

This leads us to the *node theorem*. Consider a potential $V(x)$ that is continuous and satisfies $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ (Fig. 3). This potential has a number of bound states (energy eigenstates that satisfy $\psi \rightarrow 0$ as $|x| \rightarrow \infty$), which we index $\psi_1, \psi_2, \psi_3, \dots$. Recall also that there are no degenerate bound states in one dimension. The node theorem states that ψ_n has $n - 1$ nodes. We will give an intuitive, non-rigorous explanation of this phenomenon.

For this argument we also recall that $\psi(x_0) = \psi'(x_0) = 0$ implies that $\psi(x) = 0$ for all x . One cannot have vanishing derivative at a zero of the wavefunction. That applies to nodes or finite endpoints.

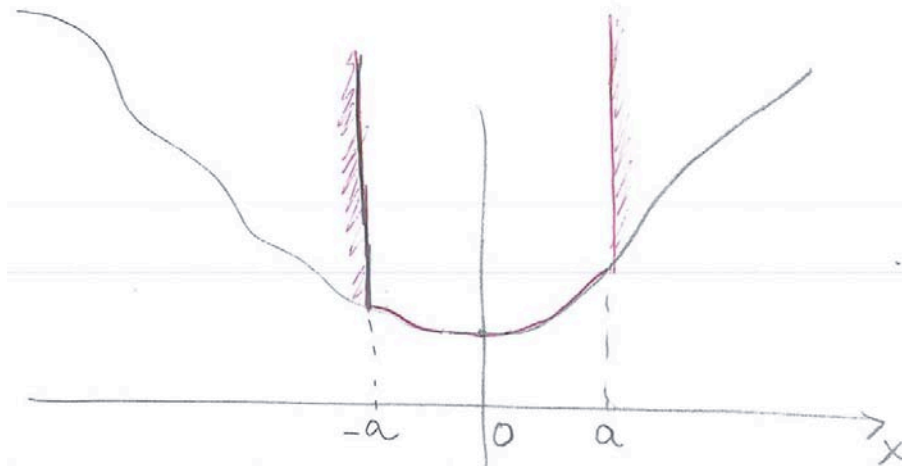


Figure 4: The screened potential $V_a(x)$.

First, we examine the potential and fix the location of $x = 0$ at a minimum. We then define the *screened* potentials $V_a(x)$ as follows:

$$V_a(x) = \begin{cases} V(x) & |x| < a, \\ \infty & |x| > a. \end{cases} \quad (2.21)$$

As shown in Fig. 4, the screened potential $V_a(x)$ is an infinite well of width $2a$ whose bottom is the taken from $V(x)$. The argument below is based on two plausible assumptions. First: As $a \rightarrow \infty$ the bound states of $V_a(x)$ become the bound states of $V(x)$. Second: As a is increased the wavefunction and its derivative are continuously stretched and deformed.

When a is very small, $V_a(x)$ is approximately a very narrow infinite well with a flat bottom – an infinite *square* well. This is because we chose $x = 0$ to be a minimum and any minimum is locally flat. On this infinite square well the node theorem holds. The ground state, for example, will vanish at the endpoints and will have no nodes. We will now argue that as the screen is enlarged we can't generate a node. This applies to the ground state, as we explicitly discuss below, and to all other states too. If we can't generate nodes by screen enlargement the node theorem applies to $V(x)$.

Why is this the case? Consider how we might develop an additional node while stretching the screen. To start with, consider the ground state in the top part of Figure 5. There is no node at this value of the screen and we have $\psi'(-a) > 0$ (left wall) and $\psi'(a) < 0$ (right wall). Suppose that as we increase a we produce a node, shown for the larger screen a' below. For this to happen the sign of ψ' at one of the endpoints must change. In the case shown it is the right endpoint that experiences a change in the sign of ψ' . With the assumption of continuous stretching there would have to be some intermediate screen at which $\psi' = 0$ at the right endpoint. But in that case, $\psi = \psi' = 0$ at this endpoint, and then $\psi(x) = 0$ for all x , which is clearly impossible.

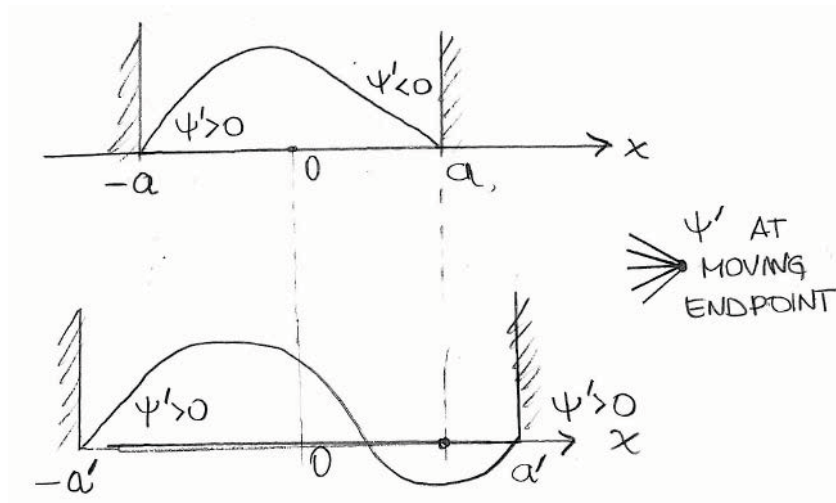


Figure 5: Introducing a single node requires changing the sign of the derivative at the right endpoint: $\psi'(a) < 0$ but $\psi'(a') > 0$. At some intermediate screen, the value of ψ' at the right endpoint must become zero. But this is impossible.

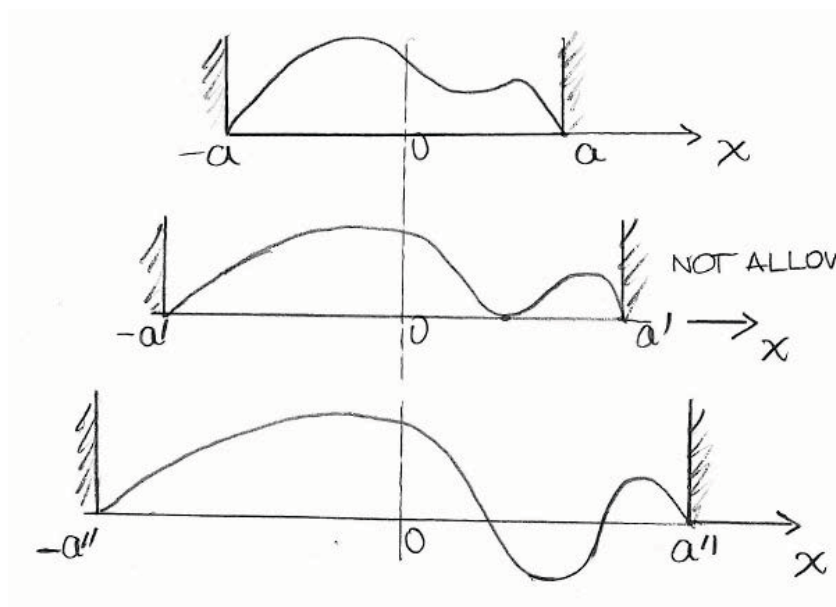


Figure 6: Introducing two nodes by having the wavefunction cross the x -axis in between the two boundaries (compare top and bottom). This is not possible as it would require in an intermediate screen (middle) in which $\psi = \psi' = 0$ at some point.

It is possible to introduce nodes without changing the sign of ψ' at either endpoint. In this process, shown in Fig. 6, the wavefunction dips and produces two new nodes. This process can't take place, however. Indeed, for some intermediate screen the wavefunction must be tangential to the x axis and at this point we will have $\psi = \psi' = 0$, which is impossible.

We conclude that we cannot change the number of nodes of *any* wavefunction as we stretch the screen. The n -th excited state of the tiny infinite square well, with $n - 1$ nodes will turn into the n -th excited state of $V(x)$ with $n - 1$ nodes. In the tiny infinite square well the energy levels are ordered

in increasing energy by the number of nodes. The same is true at all stages of the stretching screen and therefore for true for $V(x)$. Any two consecutive energy levels cannot get permuted because, by continuity, this would require a situation where we have a degeneracy, which is not possible.

3 Harmonic Oscillator

The classical harmonic oscillator is a rich and interesting dynamical system. It allows us to understand many kinds of oscillations in complex systems. The total energy E of a particle of mass m moving in one dimension under the action of a restoring force $F = -kx$, $k > 0$, is usually written as

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \quad (3.22)$$

The first term is the kinetic energy and the second term is the potential energy

$$V(x) = \frac{1}{2}kx^2. \quad (3.23)$$

The potential is quadratic in x . In such a system the particle performs oscillatory motion with angular frequency ω given by

$$\omega = \sqrt{\frac{k}{m}} \rightarrow k = m\omega^2. \quad (3.24)$$

Trading k for ω and using the momentum to express the kinetic energy, we can rewrite E as follows

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \quad (3.25)$$

This is all for the classical harmonic oscillator.

The quadratic potential is ubiquitous in physics, as it arises to first approximation when we expand an *arbitrary* potential around a minimum. To show this consider an arbitrary potential $V(x)$ with a minimum at x_0 . For $x \approx x_0$, we can use a Taylor expansion to write

$$V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2}(x - x_0)^2V''(x_0) + \mathcal{O}((x - x_0)^3). \quad (3.26)$$

Since x_0 is a critical point $V'(x_0) = 0$. Dropping the higher order terms, we then have that the potential is approximately quadratic

$$V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2. \quad (3.27)$$

This is a good approximation for x close to x_0 . Since x_0 is a minimum $V''(x_0) > 0$ and this is a harmonic oscillator centered at x_0 and with $k = V''(x_0)$. The additive constant $V(x_0)$ has no effect on the dynamics.

Faced with the question of defining a *quantum* harmonic oscillator we are inspired by the above expression (3.25) for the energy and *declare* that \hat{x} and \hat{p} will be operators with $[\hat{x}, \hat{p}] = i\hbar$ and that the Hamiltonian \hat{H} is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \quad [\hat{x}, \hat{p}] = i\hbar. \quad (3.28)$$

The harmonic oscillator potential in here is

$$V(x) = \frac{1}{2}m\omega^2x^2. \quad (3.29)$$

Note that ω has units of frequency: $[\omega] = 1/T$. We can use this to construct a characteristic energy $\hbar\omega$. The quantum harmonic oscillator is a rather natural system directly inspired by the classical oscillator.

Our first step is finding the energy eigenstates, the solutions of the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2\varphi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \varphi(x) = E\varphi(x). \quad (3.30)$$

Here both E and $\varphi(x)$ are unknown. We expect that energy eigenstates only exist for certain quantized values of E .

As a first step we will clean the equation of dimensionful constants. This helps us appreciate better the equation at hand. Moreover, it would allow us to easily put the equation on a computer. Each term in the equation must have units of energy times units of φ , as we can see by looking at the right-hand side of the equation. Note that the units of φ are not relevant to the consistency of the equation, as φ appears on each term. We can therefore ignore the units of φ . The units of energy on the left-hand side are constructed on the first term by a combination of constants and derivatives and in the second term by a combination of constants and powers of x . If we could work with a unit free coordinate u instead of x the units of energy would have to be produced *just* by the constants in the problem, and as we have seen, the only possibility is $\hbar\omega$. A common $\hbar\omega$ factor will then simplify tremendously the structure of the equation as it will allow us to define a unit free energy!

We therefore begin by introducing a unit-free coordinate u to replace the conventional coordinate x . We set

$$x = a u, \quad u \text{ unit free}, \quad [a] = L, \quad (3.31)$$

where a must be a constant with units of length. To determine a in terms of \hbar, m , and ω we equate a characteristic kinetic energy to a characteristic potential energy:

$$\frac{\hbar^2}{ma^2} = m\omega^2 a^2 \quad \rightarrow \quad a^2 = \frac{\hbar}{m\omega}. \quad (3.32)$$

Now, plugging $x = au$ into the time-independent Schrödinger equation yields

$$-\frac{\hbar^2}{2ma^2} \frac{d^2\varphi(u)}{du^2} + \frac{1}{2}m\omega^2 a^2 u^2 \varphi(u) = E\varphi(u). \quad (3.33)$$

Here, we have used

$$\frac{d}{dx} = \frac{du}{dx} \frac{d}{du} = \frac{1}{a} \frac{d}{du}. \quad (3.34)$$

Note that $\frac{\hbar^2}{ma^2} = \hbar\omega$ and $m\omega^2 a^2 = \hbar\omega$, so we have

$$-\frac{1}{2}\hbar\omega \frac{d^2\varphi(u)}{du^2} + \frac{1}{2}\hbar\omega u^2 \varphi(u) = E\varphi(u). \quad (3.35)$$

We can see that things are working. As expected the energy units on the left-hand side are carried by $\hbar\omega$! Multiplying by $\frac{2}{\hbar\omega}$, we reach

$$-\frac{d^2\varphi(u)}{du^2} + u^2 \varphi(u) = \mathcal{E}\varphi(u), \quad (3.36)$$

where we have defined a unit free energy \mathcal{E} :

$$\mathcal{E} \equiv \frac{2E}{\hbar\omega}, \quad E = \frac{1}{2}\hbar\omega \mathcal{E}. \quad (3.37)$$

If we know the pure number \mathcal{E} we then know the energy E . Rearranging, we reach the cleaned-up version of the time-independent Schrödinger equation:

$$\boxed{\frac{d^2\varphi}{du^2} = (u^2 - \mathcal{E})\varphi .} \quad (3.38)$$

This is our simplified, unit free version of the time-independent Schrödinger equation. It is clearly less cluttered than (3.30).

The above differential equation must have solutions for all values of the energy parameter \mathcal{E} , after all you could integrate it on a computer! Quantization must arise because solutions are not normalizable except for special values of \mathcal{E} . To understand this issue as it relates to the equation we examine solutions for large values of $|u|$. In this limit, \mathcal{E} can be ignored as compared to u^2 , and we have the approximate equation

$$\varphi''(u) \approx u^2\varphi(u). \quad (3.39)$$

This equation cannot be solved by any polynomial! If φ is a polynomial of degree n , the degree of the left hand side would be $n - 2$ and that of the right-hand side $n + 2$. This cannot work. Let's try a solution of the form

$$\varphi(u) = u^k e^{\alpha u^2/2}. \quad (3.40)$$

The leading term in φ'' comes when we differentiate the exponential:

$$\varphi''(u) \approx \alpha^2 u^2 \varphi(u) \quad \text{as } |u| \rightarrow \infty. \quad (3.41)$$

Comparing with (3.39) we have solutions for $\alpha = \pm 1$, in which case we have

$$\varphi(u) \approx Au^k e^{-u^2/2} + Bu^k e^{u^2/2} \quad \text{as } |u| \rightarrow \infty. \quad (3.42)$$

The solution with coefficient B would not yield an energy eigenstate because it diverges as $|u| \rightarrow \infty$ and would not be normalizable. Note that the u^k factor played no role in the analysis. This factor, however suggests that a polynomial multiplying $e^{\pm u^2/2}$ could be a solution of the differential equation.

This analysis suggests that, for our purposes, we should write

$$\varphi(u) = h(u)e^{-u^2/2}. \quad (3.43)$$

Note that there is no assumption or loss of generality in writing this expression. Indeed, any function $\varphi(u)$ can be written some other function times $e^{-u^2/2}$ as it is immediately clear $(\varphi(u)e^{u^2/2})e^{-u^2/2}$. In writing (3.43) we are only hoping that the differential equation for $h(u)$ is simpler. Clearly, if we find $h(u)$ we have found $\varphi(u)$. We actually expect that $h(u)$ may be a polynomial because the ansatz captures the large $|u|$ dependence that prevents the solution for $\varphi(u)$ from being a polynomial.

Plugging (3.43) into (3.38) and simplifying, we find a second-order linear differential equation for $h(u)$:

$$\boxed{\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\mathcal{E} - 1)h = 0.} \quad (3.44)$$

It is actually possible at this point to see that getting a polynomial solution requires quantization of \mathcal{E} . Indeed, assume that $h(u)$ is a polynomial of degree j :

$$h(u) = u^j + \alpha_1 u^{j-1} + \alpha_2 u^{j-2} + \dots \quad (3.45)$$

In the above equation the first term is then a polynomial of degree $j - 2$. Each of the other two terms are polynomials of degree j . For the equation to make sense the coefficient of contributions to the coefficient of u^j and u^{j-1} must vanish. The coefficient of u^j is

$$\text{Coefficient of } u^j : -2j + \mathcal{E} - 1 = 0 \quad \rightarrow \quad \mathcal{E} = 2j + 1. \quad (3.46)$$

Thus, we get the quantization of energy: a polynomial solution $h(u)$ of degree j requires $\mathcal{E} = 2j + 1$. You may wonder about the subleading term of degree $j - 1$ whose coefficient must also vanish.

$$\text{Coefficient of } u^{j-1} : (-2(j-1) + \mathcal{E} - 1)\alpha_1 = 0. \quad (3.47)$$

Since the energy \mathcal{E} has already been fixed, the only way to satisfy this condition is to set $\alpha_1 = 0$. Thus the polynomial is actually of the form

$$h(u) = u^j + \alpha_2 u^{j-2} + \dots \quad (3.48)$$

If this is supposed to lead to an energy eigenstate the vanishing of α_1 could have been anticipated. Since the harmonic oscillator potential is even we know that bound states *must be* either even or odd. Since $e^{-u^2/2}$ is even, the solution $\varphi(u)$ will be either even or odd if $h(u)$ is even or odd. If α_1 had not vanished, $h(u)$ would have two consecutive powers of u and could not be either even or odd.

We can analyze the equation more systematically using a series expansion:

$$h(u) = \sum_{k=0}^{\infty} a_k u^k. \quad (3.49)$$

A simple way to plug into the differential equation (3.44) is to select from each term the contribution to the coefficient of u^j . For this one can imagine the terms $a_j u^j + a_{j+1} u^{j+1} + a_{j+2} u^{j+2}$ in $h(u)$ and select the piece that contributes to the coefficient of u^j :

$$\begin{aligned} \text{Contribution from: } \frac{d^2 h}{du^2} & : (j+2)(j+1)a_{j+2} \\ \text{Contribution from: } -2u \frac{dh}{du} & : -2ja_j \\ \text{Contribution from: } (\mathcal{E} - 1)h & : (\mathcal{E} - 1)a_j \end{aligned} \quad (3.50)$$

The total coefficient of u^j in the left hand side of the differential equation must be set to zero, for all values of j , for the differential equation to be satisfied. Therefore

$$(j+2)(j+1)a_{j+2} - 2ja_j + (\mathcal{E} - 1)a_j = 0, \quad j = 0, 1, 2, \dots \quad (3.51)$$

This can be written as a recursion relation:

$$a_{j+2} = \frac{2j+1-\mathcal{E}}{(j+2)(j+1)} a_j, \quad (3.52)$$

This is a two-step recurrence relation. If you choose some a_0 you can construct a solution that contains only odd coefficients, a_2, a_4, \dots as determined recursively by the above relation. That solution, of the form

$$a_0 + a_2 u^2 + a_4 u^4 + \dots \quad (3.53)$$

would be even. Another solution is constructed by choosing some a_1 and then using the above recursion to find a_3, a_5, \dots . That solution, of the form

$$a_1 u + a_3 u^3 + \dots \quad (3.54)$$

would be odd. For any arbitrary value of \mathcal{E} both solutions of the differential equation (3.44) exist, but neither one would be polynomial and neither one would be expected to be a good energy eigenstate. The general solution to (3.44) with arbitrary \mathcal{E} is thus determined by the two constants (a_0, a_1) as they together determine all coefficients. This makes sense, because $a_0 = h(0)$ and $a_1 = h'(0)$ and the solution of a second order differential equation is determined by knowing the function and its derivative at any point.

Let us now demonstrate that if the series for $h(u)$ never stops the corresponding $\varphi(u)$ is not an acceptable energy eigenstate. Let us see what would be the large u behavior of $h(u)$ if it does not terminate. For large j the recursion relation (3.52) gives

$$\frac{a_{j+2}}{a_j} \approx \frac{2}{j}. \quad (3.55)$$

What kind of function grows this way? Note that

$$e^{u^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (u^2)^n = \sum_{j \in \text{even}} \frac{1}{(j/2)!} u^j. \quad (3.56)$$

This series has coefficients $c_j = \frac{1}{(j/2)!}$ for even j , and so we see that

$$\frac{c_{j+2}}{c_j} = \frac{(j/2)!}{((j+2)/2)!} = \frac{2}{j+2} \approx \frac{2}{j}, \quad (3.57)$$

for large j . This is just the behavior noted in (3.55) for $h(u)$. So, if the series for $h(u)$ does not terminate the wavefunction behaves like

$$\varphi(u) = h(u)e^{-u^2/2} \sim e^{u^2} e^{-u^2/2} \sim e^{u^2/2}, \quad (3.58)$$

which is the bad solution we identified in (3.42). This proves that $h(u)$ must be a polynomial and the recursion relation must terminate for us to get an energy eigenstate!

Now we discuss how to get a polynomial $h(u)$, although the main conclusion was anticipated earlier in (3.46). If $h(u)$ is to be of degree j it must have non vanishing a_j and *vanishing* a_{j+2} , as determined from the recursion relation (3.52). The numerator in this recursion relation must vanish and we must choose \mathcal{E} such that

$$2j + 1 - \mathcal{E} = 0. \quad (3.59)$$

The solution will then take the form:

$$h(u) = a_j u^j + a_{j-2} u^{j-2} + \dots, \quad (3.60)$$

with powers decreasing in steps of two because this is what the recursion relation demands for having a solution. The solution will therefore be automatically even (if j is even) or odd (if j is odd). Say j is even and the solution is even with energy $2j + 1$ as required. The second solution of the differential equation for that value of the energy would be odd, but the energy $2j + 1$ that made the even solution

terminate will not make the odd solution terminate. This means that the second solution of the differential equation is not an energy eigenstate.

We usually call the degree j of the solution using the letter n . Then,

$$\mathcal{E} = 2n + 1, \quad n = 0, 1, 2, \dots \quad (3.61)$$

corresponds to the polynomial solution

$$h_n(u) = a_n u^n + a_{n-2} u^{n-2} + \dots, \quad n = 0, 1, 2, \dots \quad (3.62)$$

The energy of the solution $\varphi_n(u) = h_n(u)e^{-u^2/2}$ is

$$E = \frac{\hbar\omega}{2}\mathcal{E} = \frac{\hbar\omega}{2}(2n + 1). \quad (3.63)$$

We have

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots \quad (3.64)$$

We see that the energies are quantized and the energy levels are evenly spaced. The ground state energy is $E_0 = \hbar\omega/2$. The corresponding power series solutions $h_n(u)$ are the Hermite polynomials, usually denoted as $H_n(u)$

$$H_n(u) = 2^n u^n \pm \dots \quad (3.65)$$

The factor of 2^n here is a convention choice. The Hermite polynomials are solutions of (3.44) with $\mathcal{E} = 2n + 1$ therefore they satisfy the differential equation

$$\frac{d^2 H_n}{du^2} - 2u \frac{dH_n}{du} + 2n H_n = 0. \quad (3.66)$$

The first several Hermite polynomials are

$$\begin{aligned} H_0(u) &= 1 \\ H_1(u) &= 2u \\ H_2(u) &= 4u^2 - 2 \\ H_3(u) &= 8u^3 - 12u. \end{aligned} \quad (3.67)$$

The generating function for the Hermite polynomials is an exponential, with formal parameter z :

$$e^{-z^2+2zu} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(u). \quad (3.68)$$

It is not too hard to show that the polynomials defined by this expansion satisfy the requisite differential equation (3.66) and are normalized as claimed in (3.65).

Let us write the energy eigenstates in terms of x . Recalling that $u^2 = x^2/a^2$, where $a^2 = \frac{\hbar}{m\omega}$ the relation

$$\varphi_n(u) \sim H_n(u)e^{-u^2/2}, \quad (3.69)$$

then gives us

$$\varphi_n(x) = N_n H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right) e^{-\frac{m\omega}{2\hbar}x^2} \quad n = 0, 1, 2, \dots, \quad (3.70)$$

where N_n is a normalization constant.

Andrew Turner transcribed Zwiebach's handwritten notes to create the first LaTeX version of this document.

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8.04 Quantum Physics I
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