Lecture 19

B. Zwiebach
April 28, 2016

Contents

1 Levinson's Theorem

Levinson's theorem relates the number N_b of bound states of a given potential to the excursion of the phase shift $\delta(E)$ as the energy goes from zero to infinity:

$$\boxed{N_b \;=\; \frac{1}{\pi}(\delta(0)-\delta(\infty))\,.} \tag{1.1}$$

To prove this result consider an arbitrary potential $V(x)$ of range R, with a wall at $x = 0$. This potential, shown to the left in Figure 1, has a number of bound states, all of which are non-degenerate and can be counted. There is also a set of positive energy eigenstates: the scattering states that, belonging to a continuum, cannot be counted. Our proof requires the possibility of counting states, so we will introduce a second infinite wall, placed at $x = L$ for large L. Of course, this will change the spectrum, but as L becomes larger and larger the changes will become smaller and smaller. We think of L as a *regulator* for the potential that discretizes the spectrum and thus enables us to enumerate the states. It does so because with two walls, the potential becomes a wide infinite well and all states become bound states. The potential with the regulator wall is shown to the right in Figure 1.

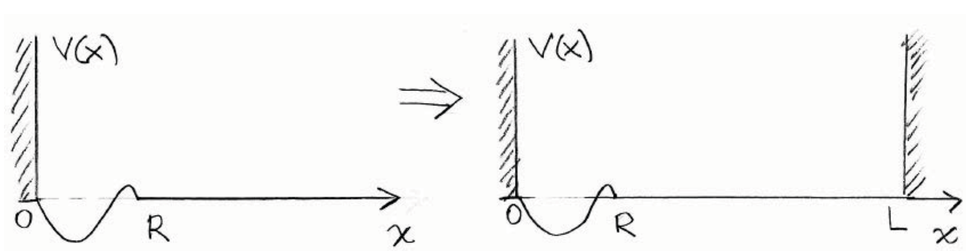

Figure 1: Left: An arbitrary one-dimensional potential $V(x)$ of range R. Right: The same potential with a regulator wall placed at $x = L$.

The key of the proof will be to compare the counting of states in the regulated $V \neq 0$ potential to the counting of states in the $V = 0$ potential, also regulated with a second wall at $x = L$. Consider therefore the regulated $V = 0$ potential and the *positive* energy eigenstates. These correspond to the wavefunction $\phi(x) = \sin kx$, with the second wall requiring $\phi(x = L) = 0$. We thus have

$$kL = n\pi, \quad \text{with} \quad n = 1, 2, \ldots \tag{1.2}$$

The values of k are now quantized. Let dk be an infinitesimal interval in wavenumber with dn the number of states in dk when $V = 0$. Thus,

$$dk\, L = dn\, \pi \quad \rightarrow \quad dn \;=\; \frac{L}{\pi}\, dk\,. \tag{1.3}$$

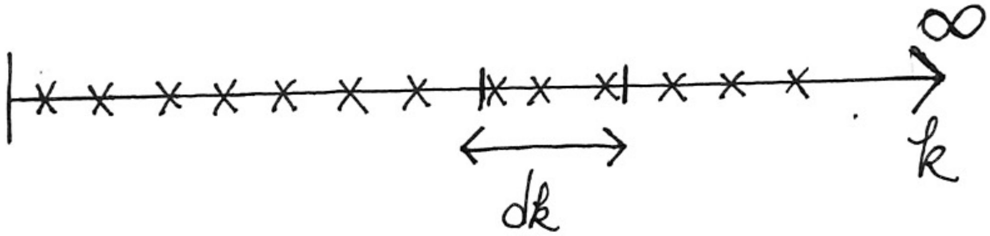

Figure 2: With the regulator wall the wavenumber k takes discrete values. dk is an infinitesimal interval in k space.

When $V(x) \neq 0$, the solutions for $x > R$, all of which are *positive* energy solutions, have the form

$$\psi(x) = e^{i\delta} \sin(kx + \delta)\,. \tag{1.4}$$

The boundary condition $\psi(L) = 0$ implies a quantization

$$kL + \delta(k) \;=\; n'\pi\,, \tag{1.5}$$

with n' integer. We can again differentiate to determine the number of positive energy states dn' in the interval dk, with $V \neq 0$:

$$dkL + \frac{d\delta}{dk}\, dk \;=\; dn'\pi \quad \rightarrow \quad dn' = \frac{L}{\pi} dk + \frac{1}{\pi}\left(\frac{d\delta}{dk}\right) dk\,. \tag{1.6}$$

The number of positive energy solutions *lost* in the interval dk as we turn on the potential V is given by $dn - dn'$, which can be evaluated using (1.3) and (1.6):

$$dn - dn' \;=\; -\frac{1}{\pi}\left(\frac{d\delta}{dk} dk\right). \tag{1.7}$$

The total number of positive energy solutions lost as the potential V is turned on is given by integrating the above over the full range of k:

$$\#\text{ of positive energy solutions lost as } V \text{ turns on } \;=\; -\int_0^\infty \frac{1}{\pi}\frac{d\delta}{dk} dk \;=\; -\frac{1}{\pi}(\delta(\infty) - \delta(0))\,. \tag{1.8}$$

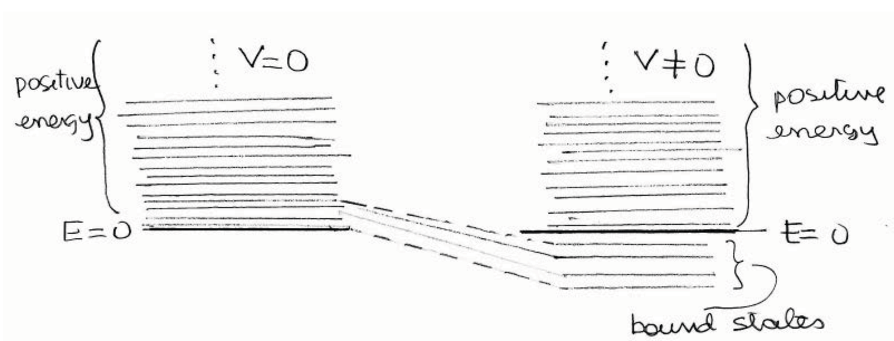

Figure 3: The positive energy states of the $V = 0$ setup shift as the potential is turned on and some can become bound states.

Although we lose a number of positive energy solutions as the potential V is turned on, states do not disappear. As one turns on the potential from zero to V continuously, we can track each energy eigenstate and no state can disappear! If we lose some positive energy states those states must now appear as negative energy states, or bound states! Letting N_b denote the number of bound states in the $V \neq 0$ potential, the result in (1.8) implies that

$$N_b \;=\; \frac{1}{\pi}(\delta(0) - \delta(\infty))\,. \tag{1.9}$$

This is what we wanted to prove!

2 Resonances

We have calculated the time delay $\Delta t = 2\hbar\delta'(E)$ associated with the **reflected** wavepacket that emerges from the range R potentials we have considered. If the time delay is negative, the **reflected** wavepacket emerges ahead of time. We can ask: Can we get an arbitrarily large *negative* time delay? The answer is no. A very large time delay would be a violation of causality. It would mean that the incoming packet is reflected even before it reaches $x = R$, which is impossible. In fact, the largest negative time delay would be realized (at least classically) if we had perfect **reflection** when the incoming packet hits $x = R$. If this happens, the time delay would be $-\frac{2R}{v_0}$, where v_0 is the velocity of the packet. Indeed, $\frac{2R}{v_0}$ is the time saved by the packet that did not have to go in and out the range. Thus we expect

$$\text{time delay} \;=\; 2\hbar\frac{d\delta}{dE} \geq -\frac{2R}{v_0}\,. \tag{2.1}$$

This can be simplified a bit by using k derivatives

$$2\hbar\frac{d\delta}{dE} \;=\; 2\hbar\frac{1}{\frac{dE}{dk}}\frac{d\delta}{dk} \;=\; \frac{2}{v_0}\frac{d\delta}{dk} \geq -\frac{2R}{v_0}\,, \tag{2.2}$$

which then gives the constraint

$$\frac{d\delta}{dk} \;\geq\; -R\,. \tag{2.3}$$

The argument was not rigorous but the result is rather accurate, receiving corrections that vanish for packets of large energy.

Alternatively, we can ask: Can we get an arbitrarily large *positive* time delay? The answer is yes. This can happen if the wave packet gets temporarily trapped in the potential. In that case we would expect the probability amplitude to become large in the $0 < x < R$ region. If the wavepacket is trapped for a long time we have a **resonance**. The state is a bit like a bound state in that it gets localized in the potential, at least for a while. In order to get a resonance it helps to have an attractive potential and a positive energy barrier. We can achieve that with the potential

$$V(x) \;=\; \begin{cases} \infty & \text{for} \quad x \leq 0 \\ -V_0 & \text{for} \quad 0 < x < a \\ V_1 & \text{for} \quad a < x < 2a \\ 0 & \text{for} \quad x > 2a\,. \end{cases} \tag{2.4}$$

The potential, with $V_0, V_1 > 0$, is shown in figure 4. In order to have a resonance we take explore energies in the range zero to V_1. In such range of energies we can expect to find some particular values that lead to resonant behavior, namely, large time delay and large amplitude for the wavefunction in the well.

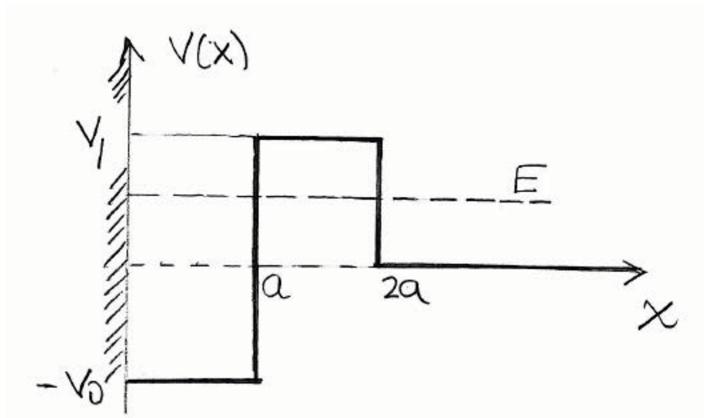

Figure 4: We search for resonances with energy E in the range $(0, V_1)$. In this range the V_1 barrier produces a classically forbidden region $x \in (a, 2a)$, that can help localize the amplitude around the well.

Given the three relevant regions in the potential we **define**

$$k'^2 = \frac{2m(E+V_0)}{\hbar^2}, \qquad \kappa^2 = \frac{2m(V_1-E)}{\hbar^2}, \qquad k^2 = \frac{2mE}{\hbar^2}. \tag{2.5}$$

In the region $0 < x < a$ we must use trigonometric functions of $k'x$. In the region $a < x < 2a$ we use hyperbolic functions of κa and in the region $x > 2a$ we use the canonical solution with phase shift and wavenumber k. In the middle region $a < x < 2a$ we could use a combination of solutions

$$\{e^{\kappa x},\ e^{-\kappa x}\}, \quad \text{or} \quad \{\cosh \kappa x\,,\ \sinh \kappa x\}\,, \quad \text{or} \quad \{\,\cosh \kappa(x-a)\,,\ \sinh \kappa(x-a)\,\}\,. \tag{2.6}$$

The last pair is most suitable to implement directly the continuity of the wavefunction at $x = a$. So we can write for the wavefunction $\psi(x)$:

$$\psi(x) = \begin{cases} A\sin(k'x) & 0 < x < a \\ A\sin(k'a)\cosh\kappa(x-a) \,+\, B\sinh\kappa(x-a) & a < x < 2a \\ e^{i\delta}\sin(kx+\delta) & x > 2a \end{cases} \tag{2.7}$$

After implementing the remaining boundary conditions we can solve for the phase shift δ. After a modest amount of work one finds:

$$\tan(2ka+\delta) = \frac{ka}{\kappa a}\cdot \frac{\sin k'a\,\cosh\kappa a \,+\, \frac{k'}{\kappa}\cos k'a\,\sinh\kappa a}{\sin k'a\,\sinh\kappa a \,+\, \frac{k'}{\kappa}\cos k'a\,\cosh\kappa a}\,. \tag{2.8}$$

This expression is fairly intricate so it is best to do numerical work. For this we define

$$z_0^2 \;=\; \frac{2mV_0a^2}{\hbar^2}\,, \quad z_1^2 \;=\; \frac{2mV_1a^2}{\hbar^2}\,, \quad u \;\equiv\; ka\,. \tag{2.9}$$

which allow us to express both $k'a$ and κa as functions of u

$$(k'a)^2 \;=\; z_0^2 + u^2\,, \quad (\kappa a)^2 \;=\; z_1^2 - u^2\,. \tag{2.10}$$

At this point (2.8) can be used to determine δ as a function of $u = ka$ and the constants z_0, z_1. Suppose we pick values for our parameter controlling equations. In Figure 5 we show results for $z_0^2 = 1$ and $z_1^2 = 5$.

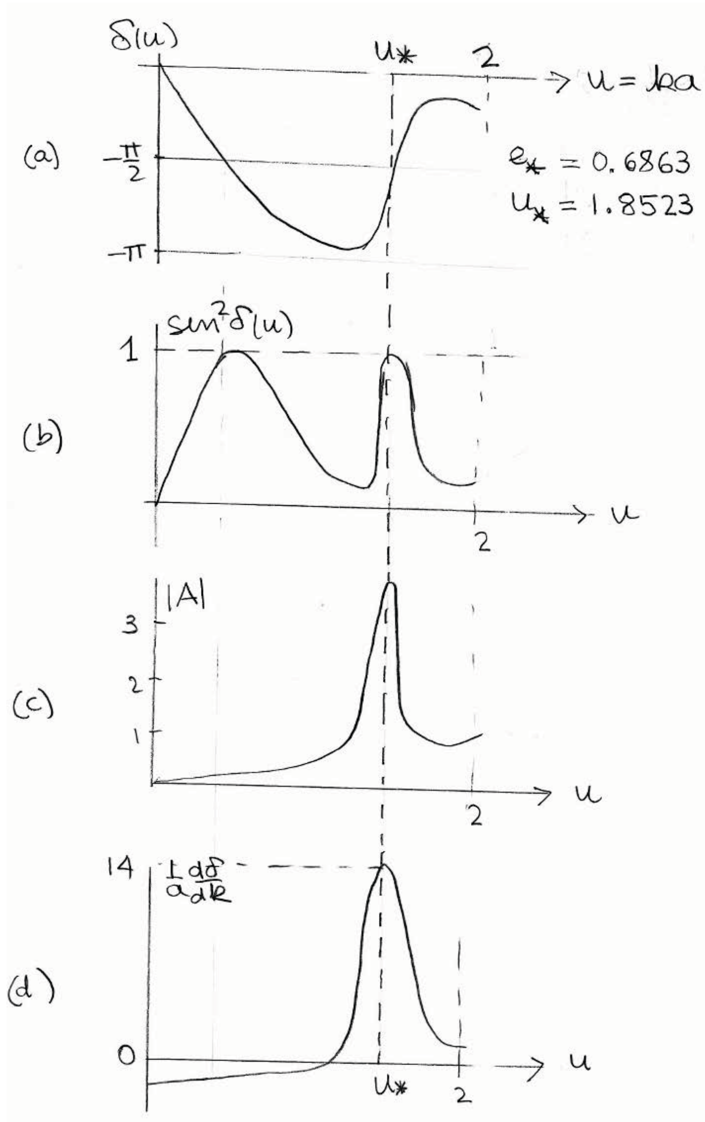

Figure 5: Plot of various quantities as a function of $u = ka$, with the potential characterized by $z_0^2 = 1$ and $z_0^2 = 5$. (a) $\delta(E)$ increases quickly around $u_* = 1.85$, or equivalently $E = 0.69V_1$, crossing $-\pi/2$ and signaling resonant behavior. (b) Plot of $|A_s|^2 = \sin^2\delta$, showing peaks each time $|\delta| = \pi/2$. (c) The coefficient $|A|$ of the wavefunction at the well peaks at the resonance, showing high probability of **finding** the particle at the well. (d) The time delay is positive and peaks at resonance.

Consider part (a) of the figure, showing $\delta(ka)$. At the beginning δ decreases linearly, a sign of a negative time delay, as the low energy waves reflect at the edge $x = 2a$ of the V_1 barrier. As δ crosses $-\pi/2$ there is no resonance, even though $|A_s|^2 = \sin^2\delta$ is equal to one. Indeed we see no bump in the amplitude $|A|$. As the energy is increased and $u = u_* = 1.8523$ we get a resonance. This time δ is increasing rapidly and δ crosses $-\pi/2$ again, making $|A_s|^2 = 1$. The signal of resonance is the very high $|A|$ the peak in the time delay. This time delay reaches the value of about 14, meaning the delay is fourteen times the free transit time $4a/v_0$!

3 Modeling the Resonance

We would like to have further insight into the nature of resonances. In particular we want to appreciate the general features of the phenomenon. Additionally, so far we can identify resonances by looking at the behavior of δ but, can we find an equation that defines resonances?

As a first step, we model the behavior of a phase near resonance. Recalling that a resonance requires $|\delta|$ cross the value $\pi/2$ and that δ, physically, is the same as δ increased or decreased by multiples of π we can choose to have δ vary from nearly zero to nearly π. We can achieve this with the following simple function.

$$\delta = \tan^{-1}\left(\frac{\beta}{\alpha - k}\right), \qquad \text{with} \quad \beta > 0\,, \quad \alpha > 0\,. \tag{3.1}$$

here α and β are positive constants with the same units as k. To see what this function does, we first plot the argument of the arc tangent at the top of Figure 6. Note that the argument varies quickly in the region $(\alpha - \beta, \alpha + \beta)$. The variation of the associated phase δ is shown in the figure below. To have a sharp increase in the phase we must have small β compared to α.

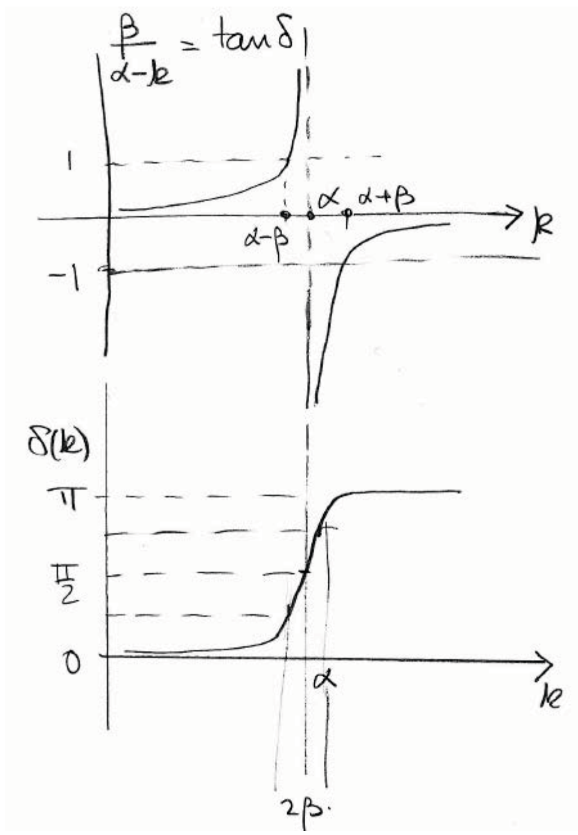

Figure 6: The constant β must be small compared to α to obtain a sharp variation. A resonance, as shown here requires δ increasing with energy.

Two relatively short calculations give us further insight:

$$\frac{d\delta}{dk}\bigg|_{k=\alpha} = \frac{1}{\beta}\,, \quad |\psi_s|^2 \;=\; \sin^2\delta \;=\; \frac{\beta^2}{\beta^2+(\alpha-k)^2}\,. \tag{3.2}$$

The first one informs us that, all things being equal, the delay is large if β is small. The second gives the norm-squared of the scattering amplitude as a function of k, with a peak at $k=\alpha$. This equation is most famously expressed in terms of the energy. For this we note that

$$E - E_\alpha = \frac{\hbar^2}{2m}(k^2-\alpha^2) = \frac{\hbar^2}{2m}(k+\alpha)(k-\alpha) \;\simeq\; \frac{\hbar^2}{2m}(2\alpha)(k-\alpha)\,, \tag{3.3}$$

when working with $k \approx \alpha$. It thus follows that

$$(k-\alpha)^2 \;\simeq\; \frac{m^2}{\hbar^4\alpha^2}(E-E_\alpha)^2\,, \tag{3.4}$$

and therefore

$$|\psi_s|^2 \;\simeq\; \frac{\beta^2}{\beta^2+\frac{m^2}{\hbar^4\alpha^2}(E-E_\alpha)^2} \;=\; \frac{\frac{1}{4}\Gamma^2}{(E-E_\alpha)^2+\frac{1}{4}\Gamma^2}\,, \tag{3.5}$$

Where we have defined the constant Γ with units of energy:

$$\tfrac{1}{4}\Gamma^2 = \frac{\hbar^4\beta^2\alpha^2}{m^2} \quad\to\quad \Gamma = \frac{2\alpha\beta\hbar^2}{m}\,. \tag{3.6}$$

The energy dependence of $|\psi_s|^2$ follows the so-called *Breit-Wigner distribution*,

$$\boxed{|\psi_s|^2 \;\simeq\; \frac{\frac{1}{4}\Gamma^2}{(E-E_\alpha)^2+\frac{1}{4}\Gamma^2}\,.} \tag{3.7}$$

The distribution is shown in Figure 8. The peak value for $|\psi_s|^2$ is attained for $E=E_\alpha$ and is one. We call Γ the width at half-maximum because the value of $|\psi_s|^2$ at $E = E_\alpha \pm \frac{1}{2}\Gamma$ is one-half. Small Γ corresponds to a narrow width, or a narrow resonance.

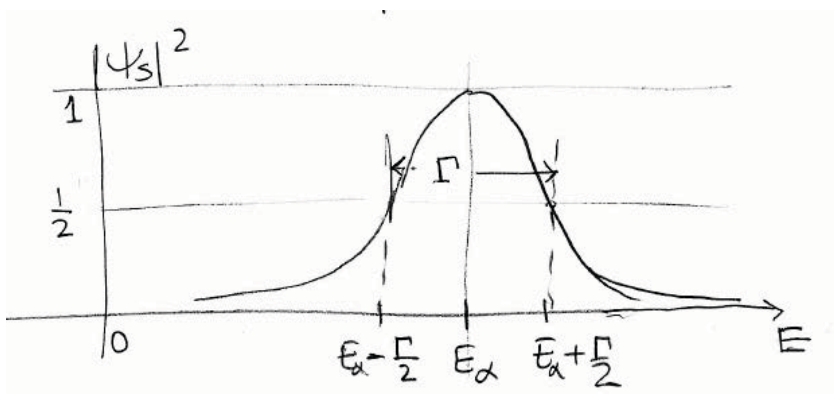

Figure 7: The Breit-Wigner distribution. Γ is the width of the distribution at half maximum.

To understand better the significance of Γ we define the associated time τ, called the **lifetime** of the resonance:

$$\tau \;\equiv\; \frac{\hbar}{\Gamma} \;=\; \frac{m}{2\alpha\beta\hbar}\,. \tag{3.8}$$

As you probably would expect, the lifetime is closely related to the time delay associated with a wavepacket of mean energy equal to the resonant energy. Indeed, we can evaluate the time delay $\boldsymbol{\Delta t}$ for $k = \alpha$ to get

$$\Delta t \;=\; 2\hbar\frac{d\delta}{dE} \;=\; 2\hbar\frac{dk}{dE}\,\frac{d\delta}{dk}\bigg|_{k=\alpha} \;=\; \frac{2\hbar}{\left(\frac{\hbar^2 k}{m}\right)}\left(\frac{1}{\beta}\right) \;=\; \frac{2\hbar}{\left(\frac{\hbar^2\alpha\beta}{m}\right)} \;=\; \frac{2m}{\alpha\beta\hbar} = 4\tau\,. \tag{3.9}$$

We therefore conclude that the lifetime and the time delay are the same quantity, up to a factor of four.

$$\boxed{\tau \;=\; \frac{\hbar}{\Gamma} \;=\; \tfrac{1}{4}\,\Delta t\,.} \tag{3.10}$$

Unstable particles are sometimes called resonances. The Higgs boson, discovered in 2012, is an unstable particle with mass 125 GeV. It can decay into two photons, or into two **tau's,** or into a $b\bar{b}$ pair, among few possibilities. The width Γ associated to the particle is 4.07 Mev ($\pm 4\%$). Its lifetime τ is about 1.62×10^{-22} seconds!

We now try to understand resonances more mathematically. We saw that, at resonance, the norm of A_s reaches a maximum value of one. Let us explore when A_s is large. We have

$$A_s \;=\; \sin\delta\, e^{i\delta} \;=\; \frac{\sin\delta}{e^{i\delta}} \;=\; \frac{\sin\delta}{\cos\delta - i\sin\delta} \;=\; \frac{\tan\delta}{1 - i\,\tan\delta}\,. \tag{3.11}$$

At resonance $\delta = \pi/2$ and $A_s = i$, using the first equality. On the other hand, while we usually think of δ as a real number, the final expression above indicates that A_s becomes infinite for

$$\tan\delta \;=\; -i\,, \tag{3.12}$$

whatever that means! If we recall that $\tan iz = i\tanh z$ we deduce that the above condition requires $\delta \to -i\infty$, a rather strange result. At any rate, A_s becomes infinite, or has a pole, at $\tan\delta = -i$. We will see that the large value $|A_s| = 1$ at resonance can be viewed as the "shadow" of the infinite value A_s reaches nearby in the complex plane.

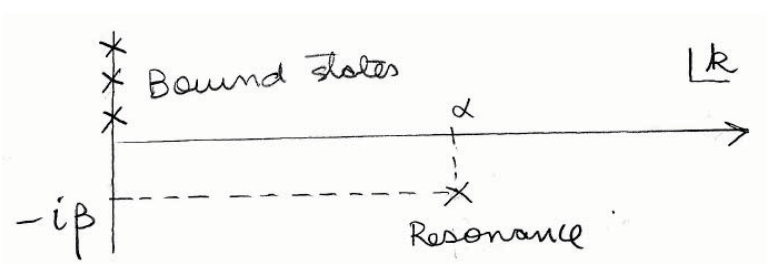

Figure 8: In the complex k plane, resonances are identified as poles of the scattering amplitude A_s located slightly below the real axis. Bound states appear as poles on positive imaginary axis.

Indeed, we can see how A_s behaves near resonance by inserting the near-resonance behavior (3.1) of δ into (3.11):

$$A_s \;=\; \frac{\frac{\beta}{\alpha-k}}{1 - i\frac{\beta}{\alpha-k}} \;=\; \frac{\beta}{(\alpha - i\beta) - k}\,. \tag{3.13}$$

When $k = \alpha$, meaning at the resonant energy, we get $A_s = i$, as expected. If we now think of the wavenumber k as a complex variable, we see that the pole of A_s is a pole at $k = k_* = \alpha - i\beta$. The real part of k_* is the resonant energy, and the imaginary part β encodes the lifetime. For small β the resonance is a pole near the real axis, as illustrated in Figure 8. The smaller β the sharper the resonance. As we can see, the value of $|A_s|$ on the real line becomes large for $k = \alpha$ because it is actually infinite a little below the axis.

The lesson in all of this is that we can indeed take (3.12) seriously and look for resonances by solving for the *complex* k values for which

$$\boxed{\text{Resonance condition:} \quad \tan\delta(k) \;=\; -i\,.} \tag{3.14}$$

The real part of those k's are the resonant energies. The imaginary parts give us the lifetime.

The idea of a complex k plane is very powerful. Suppose we consider purely imaginary k values of the form $k = i\kappa$, with $\kappa > 0$. Then the energy takes the form

$$E \;=\; -\frac{\hbar^2\kappa^2}{2m} < 0\,, \tag{3.15}$$

which is suitable for bound states. Indeed one can show that bound states appear as poles of A_s along the positive imaginary axis, as shown in Figure 8. The complex k-plane has room to fit scattering states, resonances, and bound states!

Sarah Geller transcribed Zwiebach's handwritten notes to create the first LaTeX version of this document.