

PROFESSOR: Velocity.

So we assume that we have an omega of k. That's the assumption. There are waves in which, if you give me k, the wavelength, I can tell you what is omega. And it may be as simple as omega equal to kc, but it may be more complicated.

In fact, the different waves have different relations. In mechanics, omega would be proportional to k squared. As you've seen, the energy is proportional to p squared. So omega will be proportional to k squared.

So in general, you have an omega of k. And the group velocity is the velocity of a wave packet-- packet--

constructed--

by superposition of waves.

Of waves.

All right. So let's do this here. Let's write a wave packet. Psi of x and t is going to be done by superposing waves. And superposition means integrals, summing over waves of different values of k. Each wave, I will construct it in a simple way with exponentials.

$ikx - \omega t$

And this whole thing, I will call the phase of the wave. Phi of k.

So that's one wave. It could be sines or cosines, but exponentials are nicer. And we'll do with exponentials, in this case. But you superimpose them. And each one may be superimposed with a different amplitude.

So what does it mean? It means that there is a function, phi of k, here. And for different k's,

this function may have different values. Indeed, the whole assumption of this construction is based on the statement that ϕ of k peaks. So ϕ of k --

as a function of k is 0 almost everywhere, except a little bump around some frequency that we'll call k_0 .

Narrow peak.

That is our wave. And depending on how this ϕ of k looks, then we'll get a different wave. We're going to try to identify how this packet moves in time. Now--

There is a quick way to see how it moves. And there is a way to prove how it moves. So let me do, first, the quick way to see how it moves. It's based on something called the principle of stationary phase. I doubt it was said to you in [? A03 ?] in that way.

But it's the most powerful way to see this. And in many ways, the quickest and nicest way to see. Takes a little bit of mathematical intuition, but it's simple. And intuition is something that, I think, you have.

If you're integrating--

a function--

multiplied-- a function, f of x , multiplied by maybe sine of x . Well, you have f of x , then sine of x . Sine is 1/2 the times positive, 1/2 the times negative. If you multiply these two functions, you're going to get the function that is 1/2 time positive and 1/2 the time negative.

And in fact, the integral will contribute almost nothing if this function is slowly varying. Because if it's slowly varying, the up peak and the down peak hasn't changed much the function. And they will cancel each other.

So the principle of stationary phase says that if you're integrating a function times a wave, you get almost no contribution, except in those places where the wave suddenly becomes of long wavelength and the phase is stationary. Only when the wave doesn't change much for a while,

and then it changes again.

In those regions, the function will give you some integral. So that's the principle of stationary phase. And I'll say it here, I'll write it here. Principal--

of stationary phase. We're going to use that throughout the course.

Phase. And I'll say the following. Since--

ϕ anyway only peaks around k_0 --

This is the principle of stationary phase applied to this integral. Since-- since only for k roughly equal to k_0 .

The integral--

has a chance--

To be non-zero.

So here is what I'm saying. Look. The only place where this integral contributes-- it might as well integrate from k_0 minus a little δ to k_0 plus a little δ . Because this whole thing vanishes outside.

And if we're going to integrate here, over this thing, it better be that this wave is not oscillating like crazy. Because it's going to cancel it out. So it better stop oscillating there in order to get that contribution, or send in another way.

Only when the phase stops, you get a large contribution. So on the phase stops varying fast with respect to k . So you need-- need-- that the phase becomes stationary--

with respect to k , which is the variable of integration--

at k_0 .

So around k_0 , better be that the phase doesn't change quickly. And the slower it changes, the better for your integral. You may get something. So if you want to figure out where you get the most contribution, you get it for k around k_0 , of course.

But only if this thing it's roughly stationary. So being roughly stationary will give the following result.

The result is that the main contribution comes when the phase, ϕ of k , which is kx minus ω of kt , satisfies the condition that it just doesn't vary. You have 0 derivative at k_0 .

So the relative with respect to k is x minus $d\omega$ of k dk at k_0 t must be 0. Stationary phase. Function phase has a stationary point. Look what you get. It says there that you only get a contribution if this is the case.

So the value of x , where you get a big bump in the integral, and the time, t , are related by this relation. The hump in this packet will behave obeying this relation. So x is equal to $d\omega$ dk at k_0 t .

And it identifies the packet as moving with this velocity. x equal velocity times times. This is the group velocity.

End of the answer by stationary phase. Very, extremely simple.