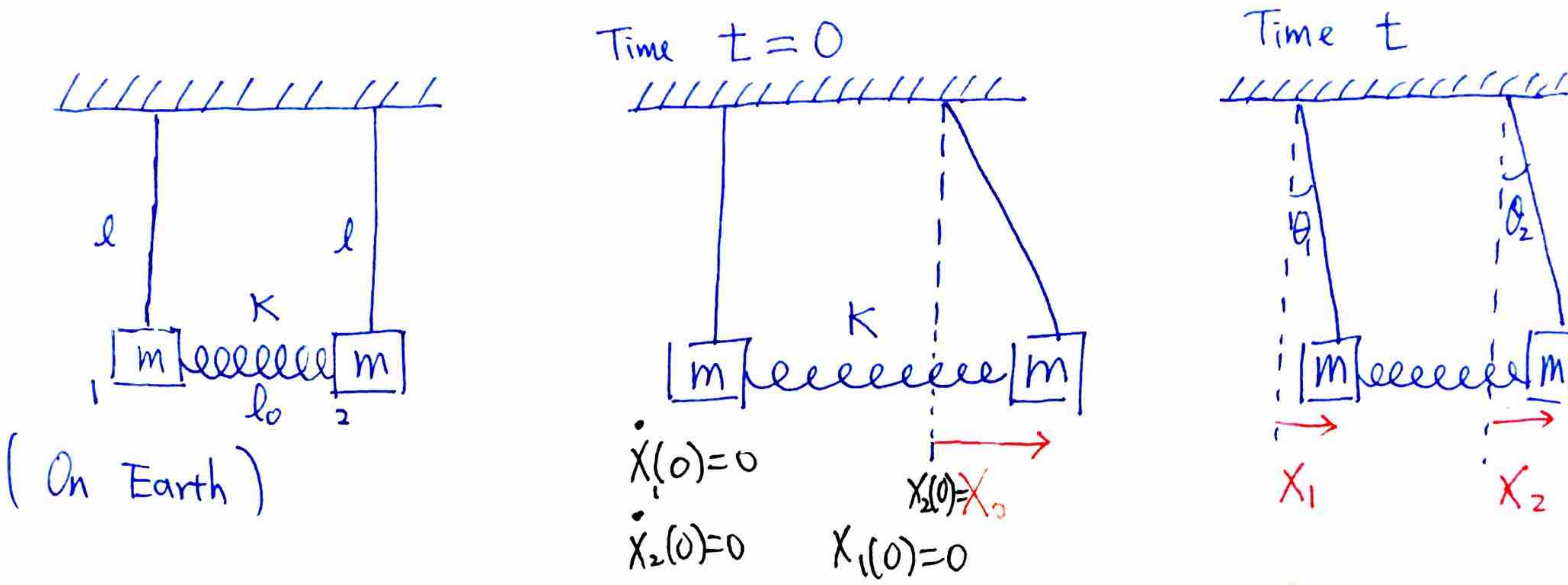
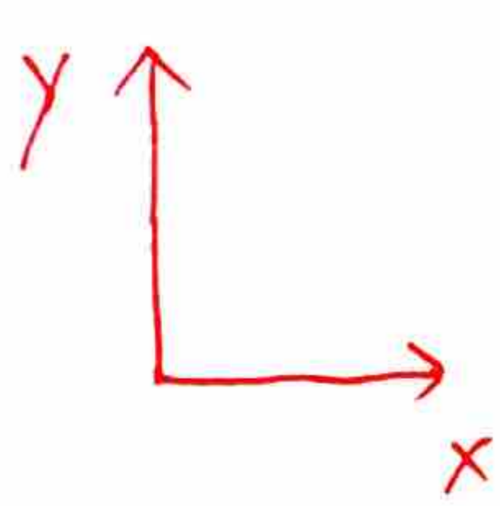


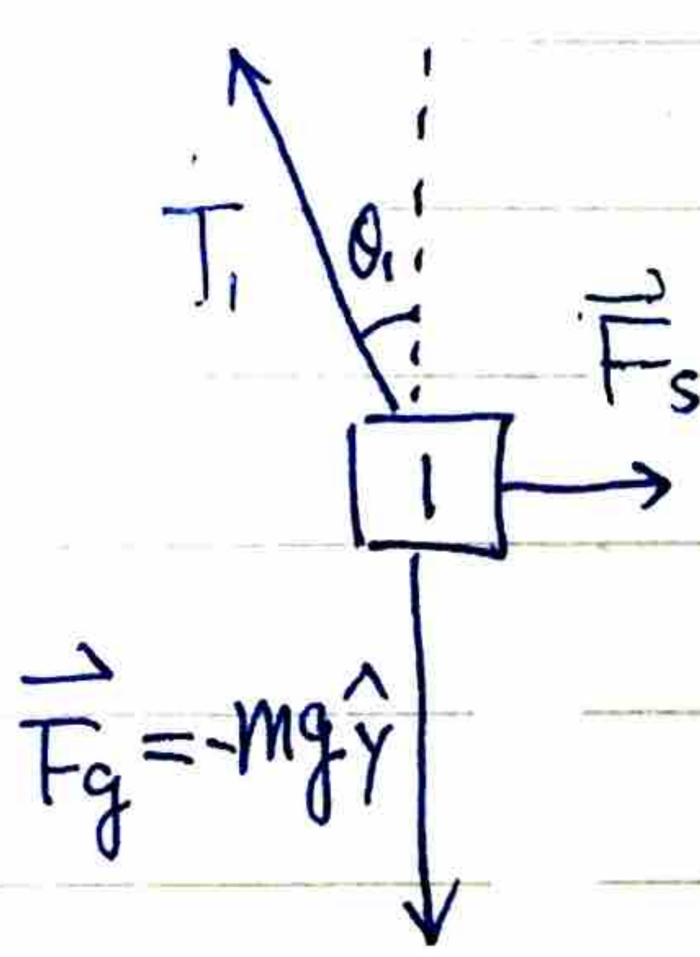
We consider this highly idealized system. Small angle.

No drag force, ideal spring. Want to predict the resulting motion at Time $= t$

Define the coordinate system: $\vec{x}_1$ and $\vec{x}_2$ measured from the equilibrium position



Force diagram



$\vec{F}_s = k(x_2 - x_1)\hat{x}$

$\hat{y}$ direction:

$$m\ddot{y}_1 = T_1 \cos\theta_1 - mg$$

$\hat{x}$ direction:

$$m\ddot{x}_1 = -T_1 \sin\theta_1 + k(x_2 - x_1)$$

Small angle approximation to $O(\theta_1)$

$\Rightarrow \cos\theta_1 \sim 1 \qquad \sin\theta_1 \sim \theta_1$

$$m\ddot{y}_1 = T_1 - mg = 0$$ (No vertical motion)

$$\Rightarrow T_1 = mg$$

$$m\ddot{x}_1 = -T_1\theta_1 + k(x_2 - x_1)$$

$$= -mg\frac{x_1}{l} + k(x_2 - x_1)$$

$$m\ddot{x}_1 = -\frac{mgx_1}{l} + k(x_2 - x_1)$$

$$\begin{cases} m\ddot{x}_1 = -\left(k + \frac{mg}{l}\right)x_1 + kx_2 \\ \text{Similarly} \quad m\ddot{x}_2 = kx_1 - \left(k + \frac{mg}{l}\right)x_2 \end{cases}$$

Write everything in matrix form! $M\ddot{X} = -KX$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad K = \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix}$$

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

$$\Rightarrow M^{-1}K = \begin{pmatrix} \frac{k}{m} + \frac{g}{l} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} + \frac{g}{l} \end{pmatrix}$$

$$\Rightarrow \ddot{X} = -M^{-1}KX$$

Solve the Eigenvalue Problem $\begin{cases} x = Re(Z) \\ Z = e^{i(\omega t+\phi)} A \end{cases}$

The E.O.Ms become:

$$\omega^2 A = M^{-1} K A$$

$$\det (M^{-1}K - \omega^2 I) A = 0$$

$$M^{-1}K - \omega^2 I = \begin{pmatrix} \frac{g}{l} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} - \omega^2 \end{pmatrix}$$

$$\det (M^{-1}K - \omega^2 I) = 0$$

$$\Rightarrow \quad = \left(\frac{g}{l} + \frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2$$

$$\Rightarrow \left(\frac{g}{l} + \frac{k}{m} - \omega^2\right) = \pm \frac{k}{m}$$

$$\Rightarrow \omega^2 = \underbrace{\frac{g}{l}}_{\omega_1^2}, \quad \underbrace{\frac{g}{l} + \frac{2k}{m}}_{\omega_2^2}$$

① $\omega^2 = \frac{g}{l}$

$$\Rightarrow (M^{-1}K - \omega^2 I) A = \begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\frac{k}{m} A_1 - \frac{k}{m} A_2 = 0 \quad \Rightarrow A_1 = A_2 \qquad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

② $\omega^2 = \frac{g}{l} + \frac{2k}{m}$

$$\Rightarrow (M^{-1}K - \omega^2 I) A = \begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\Rightarrow A_1 = -A_2 \qquad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} x = \mathrm{Re}(Z) \\ Z = e^{i(\omega t+\phi)} A \end{cases}$$

Solve the Eigenvalue Problem

The E.O.M.s become:

$$\omega^2 A = M^{-1} K A$$

$$\det (M^{-1}K - \omega^2 I) A = 0$$

$$M^{-1}K - \omega^2 I = \begin{pmatrix} \frac{g}{l} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} - \omega^2 \end{pmatrix}$$

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$$\Rightarrow \quad = \left(\frac{g}{l} + \frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2$$

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① $\omega^2 = \frac{g}{l}$

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$$\frac{k}{m} A_1 - \frac{k}{m} A_2 = 0 \quad \Rightarrow \quad A_1 = A_2 \qquad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

② $\omega^2 = \frac{g}{l} + \frac{2k}{m}$

$$\Rightarrow \quad (M^{-1}K - \omega^2 I) A = \begin{pmatrix} -\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad A_1 = -A_2 \qquad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Go back to X

$$X = Re(Z) = Re\left(e^{i(\omega t+\phi)} A\right)$$

$$X^{(1)} = \cos(\omega_1 t+\phi_1)\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{A^{(1)}} \qquad \omega_1 = \sqrt{\frac{g}{\ell}}$$

$$X^{(2)} = \cos(\omega_2 t+\phi_2)\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{A^{(2)}} \qquad \omega_2 = \sqrt{\frac{g}{\ell}+\frac{2k}{m}}$$

Full solution:

$$\boxed{\begin{aligned} X_1 &= \alpha\cos(\omega_1 t+\phi_1) + \beta\cos(\omega_2 t+\phi_2) \\ X_2 &= \alpha\cos(\omega_1 t+\phi_1) + (-\beta)\cos(\omega_2 t+\phi_2) \end{aligned}}$$

Initial conditions : can be used to determine $\alpha, \beta, \phi_1, \phi_2$

You will get $\alpha = \frac{X_0}{2}$ $\quad\beta = \frac{-X_0}{2}$

$$\phi_1 = \phi_2 = 0$$

$$\Rightarrow X_1 = \frac{X_0}{2}\left[\cos(\omega_1 t) - \cos(\omega_2 t)\right]$$

$$X_2 = \frac{X_0}{2}\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right]$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

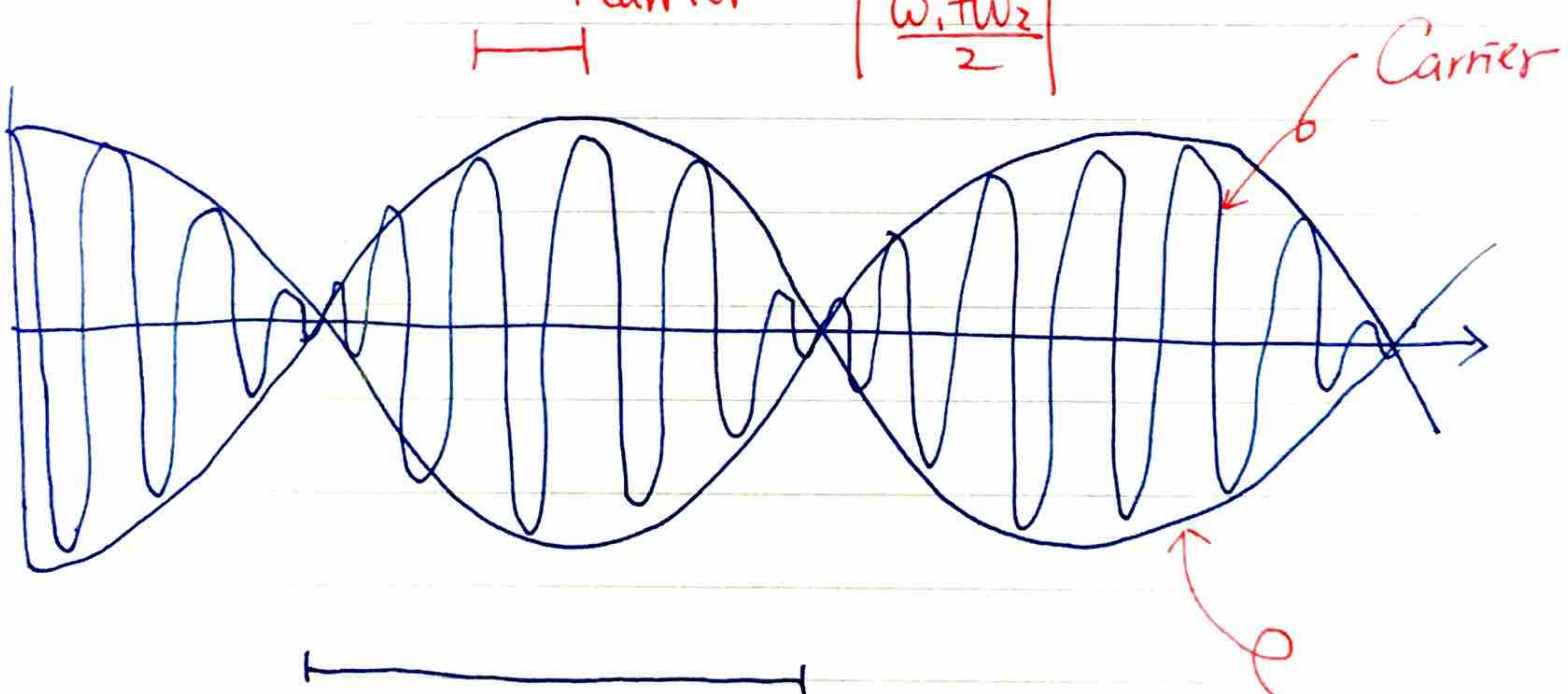
$$x_1 = -X_0 \sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$x_2 = X_0 \cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

if $\omega_1 \approx \omega_2$ (ex: $\omega_1 = 0.9\,\omega_2$)

$$\Rightarrow \frac{\omega_1+\omega_2}{2} = 0.95\,\omega_2 \qquad \frac{\omega_1-\omega_2}{2} = -0.05\,\omega_2$$

$$T_{carrier} = \frac{2\pi}{\left|\frac{\omega_1+\omega_2}{2}\right|}$$



$$T_{beat} = \frac{2\pi}{|\omega_1-\omega_2|}$$

Normal Coordinate: $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$

If I define $U_1 = X_1 + X_2$

$U_2 = X_1 - X_2$

$\Rightarrow U_1 = 2A\cos(\omega_A t + \phi_1)$

$U_2 = 2B\cos(\omega_B t + \phi_2)$

DEMO
ROOT

$$m(\ddot{x}_1 + \ddot{x}_2) = -\left(\frac{mg}{\ell}\right)(x_1 + x_2)$$

$$m(\ddot{x}_1 - \ddot{x}_2) = -\left(\frac{mg}{\ell} + 2k\right)(x_1 - x_2)$$

$$\Rightarrow m\ddot{U}_1 = -\frac{mg}{\ell} U_1$$

$$m\ddot{U}_2 = -\left(\frac{mg}{\ell} + 2k\right) U_2$$

Decoupled!

U_1 (and U_2) are oscillating harmonically at ω_1 (and ω_2) !!!!