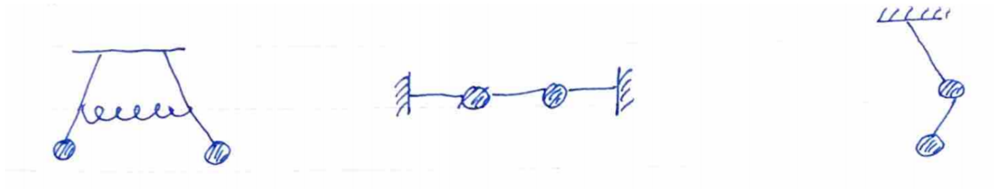
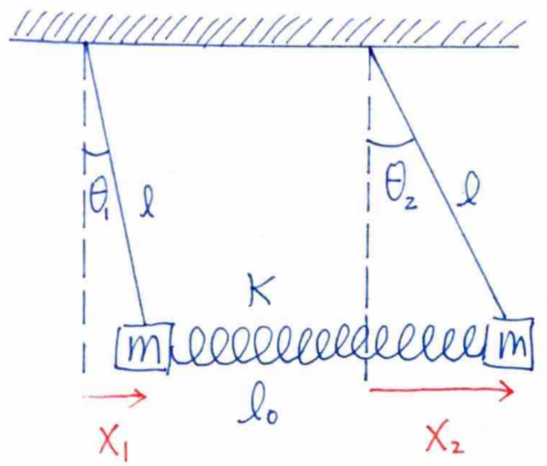
8.03 Lecture 6

Examples of compuled oscillationrs:



	Arbitrary Excitation	Normal Mode Excitation
Motion	Not Harmonic	Harmonic
Amplitude Ration	Varies	Constant
Energy	Migrates	stays

Next we will look at driven coupled oscillators.



Last time:
We solved the normal mode of this system. Now we would like to add a driving force on left mass.

$$\vec{F}_d = F_0 \cos{(\omega_d t)} \hat{x}$$

Equations of motion:

$$m\ddot{x}_1 = -\left(k + \frac{mg}{l}\right) x_1 + kx_2 + \mathbf{F_0} \cos(\omega_\mathbf{d} \mathbf{t})$$

$$m\ddot{x}_2 = kx_1 - \left(k + \frac{mg}{l}\right) x_2$$

Putting the equation of motion into matrix from we have:

$$M\ddot{X} = -KX + F\cos(\omega_d t)$$

where

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \ddot{X} = -M^{-1}KX + M^{-1}F\cos(\omega_d t)$$

$$M^{-1}K = \begin{pmatrix} \frac{k}{m}+\frac{g}{l} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m}+\frac{g}{l} \end{pmatrix} \qquad M^{-1}F = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

Last time we solved the homogeneous equation:

$$\det(M^{-1}K - \omega^2 I) = 0$$

Recall the solutions:

$$\omega_1^2 = \frac{g}{l} \qquad A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + \frac{2k}{m} \quad A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\det(M^{-1}K - \omega^2 I) = (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = 0$$

Homogeneous solution:

$$x = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \phi_1) + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \phi_2)$$

Now we have an additional driving force:

$$\ddot{X} + M^{-1}KX = M^{-1}F\cos(\omega_d t)$$

Similar to driven oscillator problem, we want to eliminate the $\cos(\omega_d t)$ term...
Go to complex notation: $X = \text{Re}[Z]$ $\quad \ddot{Z} + M^{-1}KZ = M^{-1}Fe^{i\omega_d t}$
Guess: $Z = Be^{i\omega_d t}$ where $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$
Plug our guess for Z into the equation:

$$\Rightarrow (M^{-1}K - \omega_d^2 I)Be^{i\omega_d t} = M^{-1}Fe^{i\omega_d t}$$
$$\Rightarrow (M^{-1}K - \omega_d^2 I)B = M^{-1}F$$

These are just two simultaneous equations:

$$\begin{pmatrix} \frac{k}{m}+\frac{g}{l}-\omega_d^2 & \frac{-k}{m} \\ \frac{-k}{m} & \frac{k}{m}+\frac{g}{l}-\omega_d^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

$$\left(\frac{k}{m} + \frac{g}{l} - \omega_d^2\right) B_1 - \frac{k}{m} B_2 = \frac{F_0}{m}$$
$$-\frac{k}{m} B_1 \left(\frac{k}{m} + \frac{g}{l} - \omega_d^2\right) B_2 = 0$$

We can go ahead and solve it directly to get B_1 and B_2 or we can use "Cramer's Rule" which is a useful rule when solving a large number of coupled oscillators.
First define:

$$\overleftrightarrow{E} = \begin{pmatrix} \frac{k}{m}+\frac{g}{l}-\omega_d^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m}+\frac{g}{l}-\omega_d^2 \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} \frac{F_0}{m} \\ 0 \end{pmatrix}$$

To use Cramer's rule, use one column from $\overleftrightarrow{E}$ and $\vec{D}$

$$
\begin{aligned}
B_1 &= \frac{\left|(\vec{D})()\right|}{\det \overleftrightarrow{E}} \\
&= \frac{\begin{pmatrix} \frac{F_0}{m} & \frac{-k}{m} \\ 0 & (\frac{k}{m} + \frac{g}{l} - \omega_d^2) \end{pmatrix}}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)} \\
&= \frac{\frac{F_0}{m}(\frac{k}{m} + \frac{g}{l} - \omega_d^2)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}
\end{aligned}
$$

Which explodes when $\omega_d = \omega_1, \omega_2$ which are the frequencies of the normal modes. Similiarly:

$$
\begin{aligned}
B_2 &= \frac{\left|()(\vec{D})\right|}{\det \overleftrightarrow{E}} \\
&= \frac{\begin{pmatrix} \frac{k}{m} + \frac{g}{l} - \omega_d^2 & \frac{F_0}{m} \\ -\frac{k}{m} & 0 \end{pmatrix}}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)} \\
&= \frac{\frac{F_0}{m}\left(\frac{k}{m}\right)}{(\omega_d^2 - \omega_1^2)(\omega_d^2 - \omega_2^2)}
\end{aligned}
$$

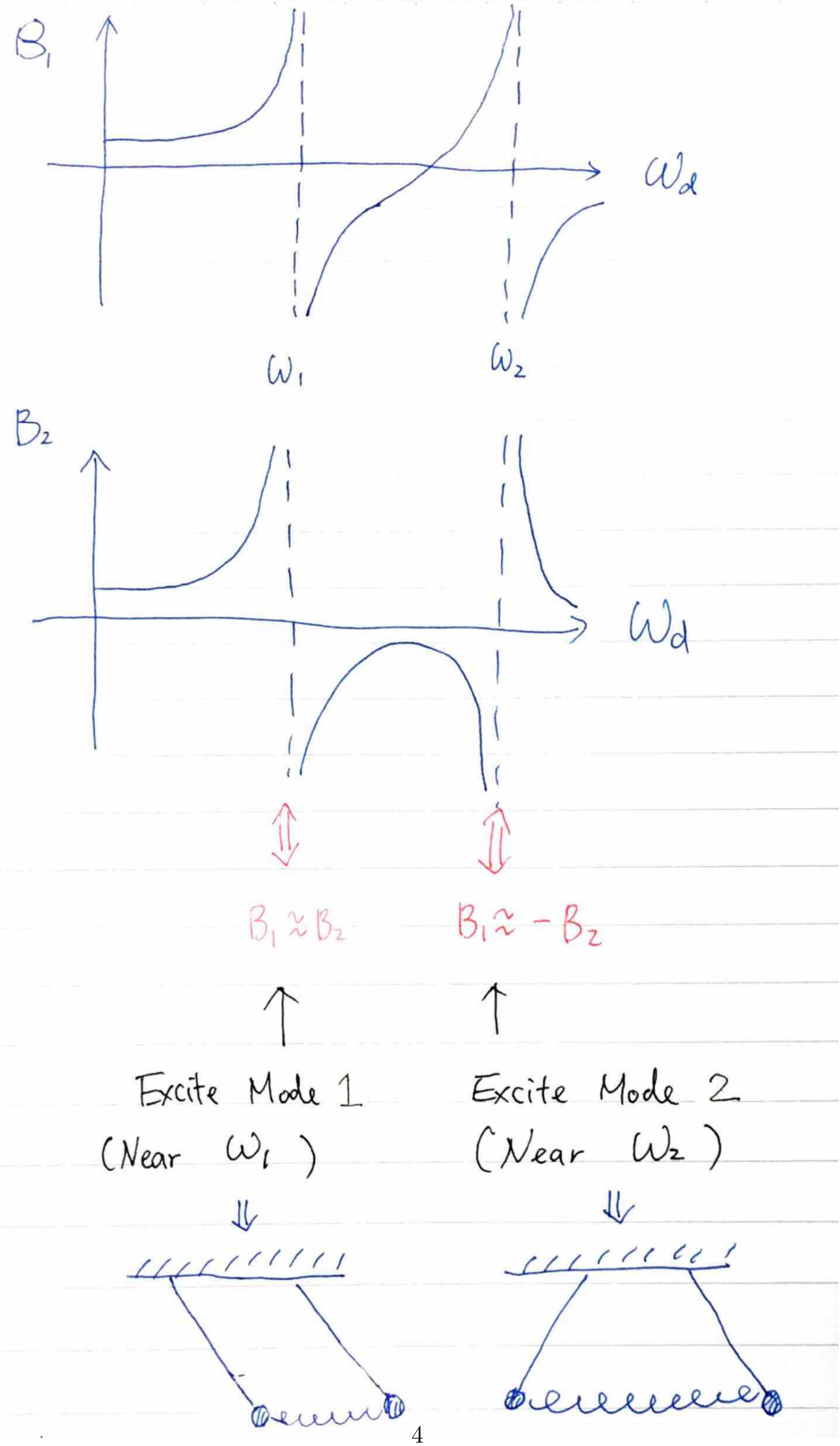
$$
\frac{B_1}{B_2} = \frac{k/m + g/l - \omega_d^2}{k/m}
$$

$$
\begin{array}{llll}
(1) & \omega_d^2 = \omega_1^2 & = \frac{g}{l} & \Rightarrow \quad \frac{B_1}{B_2} = 1 \\
(2) & \omega_d^2 = \omega_2^2 & = \frac{g}{l} + \frac{2k}{m} & \Rightarrow \quad \frac{B_1}{B_2} = -1
\end{array}
$$

Full solution:

$$
\begin{aligned}
x_1 &= \alpha \cos(\omega_1 t + \phi_1) + \beta \cos(\omega_2 t + \phi_2) + B_1 \cos(\omega_d t) \\
x_2 &= \alpha \cos(\omega_1 t + \phi_1) - \beta \cos(\omega_2 t + \phi_2) + B_2 \cos(\omega_d t)
\end{aligned}
$$

Where the term with B amplitude is the particular solution and the terms with α and β amplitude are the homogeneous solution.



B1
ωd
ω1
ω2
B2
ωd
B1 ≈ B2
B1 ≈ −B2
Excite Mode 1
(Near ω1)
Excite Mode 2
(Near ω2)