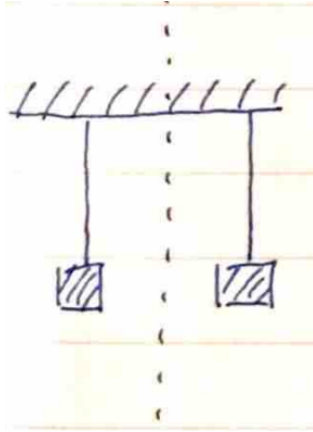


## 8.03 Lecture 7

Symmetry is a very important concept in physics, mathematics and art! In the example we just discussed, we solved the eigenvalue problem  $(M^{-1}K)A = \omega_2 A$

We get normal mode frequencies, and amplitude ratios. In fact, we can find the normal modes much easier by symmetry.



This system is **invariant** under reflection!  
(Physics is unchanged) when we:

$$x_1 \rightarrow -x_2$$

$$x_2 \rightarrow -x_1$$

This means that if  $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$  is a solution then  $\tilde{x}(t) = \begin{pmatrix} -x_2(t) \\ -x_1(t) \end{pmatrix}$  is also a solution.

To describe this mathematically, we define the “symmetry matrix”  $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$  and we have

$$\tilde{x}(t) = Sx(t)$$

Original equation of motion:

$$\ddot{x}(t) = -M^{-1}kx(t) \quad \text{where } x(t) = A \cos(\omega t + \phi) \text{ is a solution} \quad (1)$$

If  $\tilde{x}(t) = Sx(t)$  is also a solution, then

$$\begin{aligned} \ddot{\tilde{x}}(t) &= -M^{-1}k\tilde{x}(t) \\ \Rightarrow S\ddot{x}(t) &= -M^{-1}kSx(t) \end{aligned}$$

We also know (from multiplying equation 1 by S):

$$S\ddot{x}(t) = -SM^{-1}kx(t)$$

Therefore we can derive the condition: if  $SM^{-1}k = M^{-1}kS \Rightarrow \tilde{x}$  is also a solution.

To introduce a new term, we ask “Does  $S$  commute with  $M^{-1}k$ ?” Essentially we’re asking if we can pull  $S$  through  $M^{-1}k$ . Mathematically, we denote this

$$[S, M^{-1}k] = 0$$

Where the commutator is defined  $[A, B] \equiv AB - BA$ . If the commutator is zero, we say the matrices commute.

If  $x(t) = A^{(1)} \cos \omega_1 t$  and  $\omega_1 \neq \omega_2$

$$\tilde{x}(t) \propto A^{(1)} \cos \omega_1 t$$

Any solution oscillatory with  $\omega_1$  must be proportional to  $A^{(1)}$

$$\begin{aligned} Sx(t) &= SA^{(1)} \cos \omega_1 t \propto A^{(1)} \cos \omega_1 t \\ \Rightarrow SA^{(1)} &= \beta_1 A^{(1)} \quad \text{similarly} \quad SA^{(2)} = \beta_2 A^{(2)} \end{aligned}$$

$A^{(1)}$  and  $A^{(2)}$  are also eigenvectors of  $S$ !! This means that they are solutions of the eigenvalue problem  $SA = \beta A$  Now we can run the logic in the opposite direction: Given (i)  $SA = \beta A$  and (ii)  $[S, M^{-1}k] = 0$  then we can say:

$$SM^{-1}kA = M^{-1}kSA = \beta M^{-1}kA$$

We've learned that  $M^{-1}kA$  is an eigenvector of  $S$  with the same eigenvalue as  $A$

If the eigenvalues of  $S$  are different, then:  $M^{-1}kA \propto A$  and  $A$  is also an eigenvector of  $M^{-1}kA$ !! Now instead of solving the difficult equation

$$M^{-1}kA^{(n)} = \omega_{(n)}^2 A^{(n)}$$

We can solve

$$SA^{(n)} = \beta_n A^{(n)}$$

which is much easier to solve!

We know  $S^2 = I$  so

$$\begin{aligned} S^2 A^{(n)} &= \beta_n^2 A^{(n)} = A^{(n)} \\ \Rightarrow \beta_n &= \pm 1 \end{aligned}$$

Use this fact to get  $A^{(n)}$

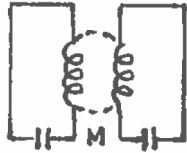
$$\begin{aligned} \beta_1 = -1 &\Rightarrow A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \beta_2 = 1 &\Rightarrow A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

$\Rightarrow$  use  $M^{-1}kA^{(n)} = \omega_{(n)}^2 A^{(n)}$  to find  $\omega_{(n)}^2$

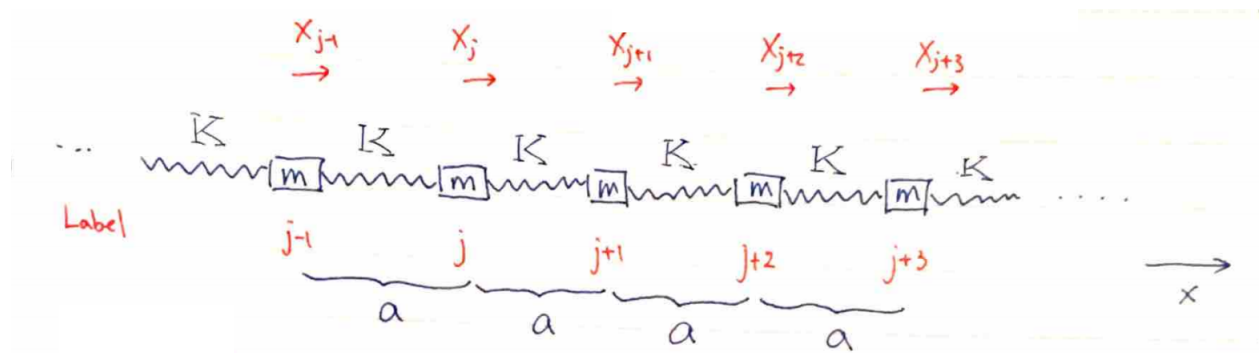
What we have learned here:

(1.) We can find the normal modes by solving the eigenvalue problem for the symmetry matrix,  $S$ , instead of  $M^{-1}k$  (given that  $SM^{-1}k = M^{-1}kS$  or  $[S, M^{-1}k] = 0$ )

(2.) This is a remarkable result: All two component system satisfy  $SM^{-1}k = M^{-1}kS$  and therefore they all have the same eigenvectors!!! Once you've solved one, you have solved them all.



Now we want to move onto an infinite number of coupled oscillators:



The masses are constrained to move only in the  $x$  direction. The springs are ideal, they have a spring constant,  $k$ , and a natural length,  $a$ . We can write down the equation of motion for the  $j$ -th mass:

$$\begin{aligned}
 m\ddot{x}_j &= -k(x_j - x_{j-1}) - k(x_j - x_{j+1}) \\
 &= kx_{j-1} - 2kx_j + kx_{j+1} \\
 \Rightarrow x_j &= A_j \cos(\omega t + \phi) = \text{Re}[A_j e^{i(\omega t + \phi)}]
 \end{aligned}$$

$$M = \begin{pmatrix} \ddots & \dots & \dots & \dots & \dots \\ \dots & m & 0 & 0 & \dots \\ \dots & 0 & m & 0 & \dots \\ \dots & 0 & 0 & m & \dots \\ \vdots & \vdots & \vdots & \vdots & m \end{pmatrix} \quad M^{-1}k = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & 0 & \dots & \dots \\ \dots & 0 & 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & 0 & \dots \\ \dots & 0 & 0 & 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix} \quad A = \begin{pmatrix} \vdots \\ A_j \\ A_{j+1} \\ \vdots \end{pmatrix}$$

Problem: we don't know how to solve an  $\infty \times \infty$  dimension matrix...Use symmetry!  
 We have already discussed reflection symmetry, now we will discuss space translation symmetry. If

we move the system of springs above by  $a$  to the left the physics of our system should be unchanged.

$$A' = SA \text{ where } S = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

$S$  here is an infinite dimensional matrix with 1s along the next to diagonal.

We want to find the eigenvalues of  $S$ . We need to solve  $A' = SA = \beta A$

$$A = \begin{pmatrix} \vdots \\ A_j \\ A_{j+1} \\ A_{j+2} \\ \vdots \end{pmatrix} \quad SA = \begin{pmatrix} \vdots \\ A_{j+1} \\ A_{j+2} \\ A_{j+3} \\ \vdots \end{pmatrix}$$

In this case  $A'_j = \beta A_j = A_{j+1}$

We don't yet know what  $\beta$  is but we know if  $A_0 = 1$ ,  $A_1 = \beta$ ,  $A_2 = \beta^2$  and so on until  $A_j = \beta^j$ . This works for all non-zero values of  $\beta$ . We get an  $\infty$  number of normal modes (and eigenvalues). Does this make sense? Yes, because we have an infinite number of degrees of freedom! Now we have normal modes and eigenvectors.

From previous discussion:

If  $[S, M^{-1}k] = 0$  and eigenvalues are all different we know that  $S$  and  $M^{-1}k$  share the same eigenvectors. To get the corresponding angular frequency:

$$\begin{aligned} M^{-1}kA &= \omega^2 A \\ \omega^2 A_j &= \frac{-k}{m} A_{j-1} + \frac{2k}{m} A_j + \frac{-k}{m} A_{j+1} \\ &= \omega_0^2 (-A_{j-1} + 2A_j - A_{j+1}) \end{aligned}$$

$$\text{Where } \omega_0^2 \equiv \frac{k}{m}$$

Since  $A_j \propto \beta^j$

$$\begin{aligned} \omega^2 \beta^j &= \omega_0^2 (-\beta^{j-1} + 2\beta^j - \beta^{j+1}) \\ \omega^2 &= \omega_0^2 \left( \frac{-1}{\beta} + 2 - \beta \right) \\ \omega^2 &= \omega_0^2 (2 - (\beta + \beta^{-1})) \end{aligned}$$

$\beta$  can have any value, however, if  $|\beta| \neq 1$  the amplitude goes to infinity because  $A_j \propto \beta^j$ . Also,  $\beta = b$  and  $\beta = 1/b$  gives the same angular frequency.

Consider the  $|\beta| = 1$  case:

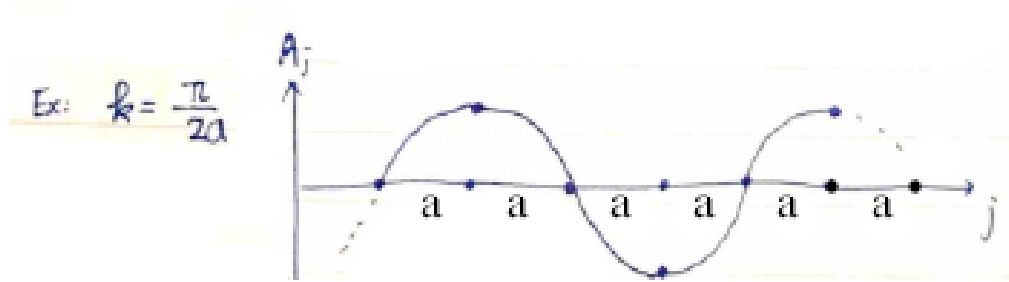
$$\beta = e^{ika} \quad A_j \propto e^{ijka}$$

If we plug this expression for  $\beta$  into our expression for  $\omega^2$  we get:

$$\begin{aligned} \omega^2 &= \omega_0^2 (2 - (e^{ika} + e^{-ika})) \\ \omega^2 &= 2\omega_0^2 (1 - \cos ka) \end{aligned}$$

This equation is known as a “dispersion relation” because it relates the wave number  $k$  and the frequency  $\omega$ . Let’s study this equation further:

- (1) There are an infinite number of normal modes: Each  $k$  gives a normal mode
- (2) Amplitude:  $A_j = \frac{1}{2i}(e^{ijka} - e^{-ijka}) = \sin jka$  (to make  $A_j$  real)



Since both  $A_j = \beta^j = e^{ijka}$  and  $A_{-j}$  are eigenvectors of  $M^{-1}k$  with eigenvalue  $\omega^2 = 2\omega_0^2(1 - \cos ka) \Rightarrow$  linear combinations of them are also eigenvectors of  $M^{-1}k!$  (see Georgi pg. 113)

- (3) Maximum frequency:  $\omega^2 = 4\omega_0^2 \rightarrow$  which occurs when  $\cos ka = -1$
- (4)  $\cos ka = 1$  or  $\beta = 1$  gives the minimum frequency:  $\omega^2 = 0$ . Physically this means all masses are moving in the same direction.

All possible motions: linear combinations of all normal modes. Each normal mode: standing waves.

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