Yen-Jie Lee

Exam 1 :

81 ± 16

Last time:

* Wave Equation

$$\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial x^2}$$

Normal modes: Standing waves!

(slide 2-6)

Yen-Jie Lee

Reminder: Wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = v_p^2 \frac{\partial^2 \psi}{\partial x^2}$$

We discussed about normal modes, boundary condition.

(1) $\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m)$

Today:

(2) There is a special kind of solution:

$$\psi(x,t) = f(x - v_p t)$$ for any functional form of f

Let $\tau = x - v_p t$

(i) $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{\partial f}{\partial \tau} \cdot 1 = f'(\tau)$$

$$\frac{\partial^2 f}{\partial x^2} = f''(\tau)$$

(ii) $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial t} = -v_p \frac{\partial f}{\partial t} = -v_p f'(\tau)$$

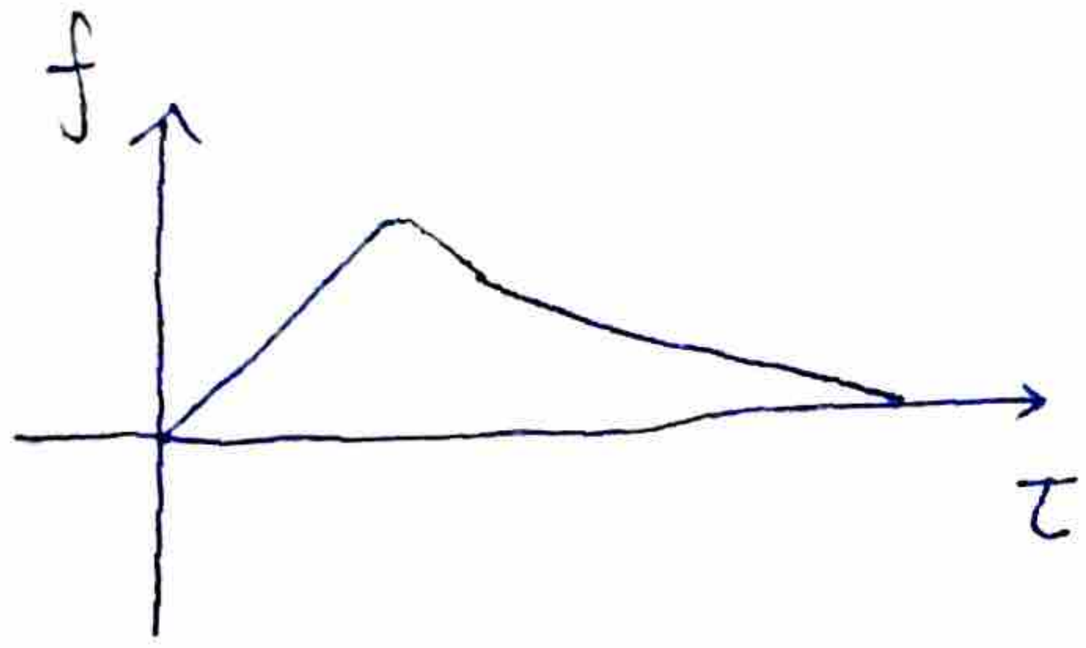
$$\frac{\partial^2 f}{\partial t^2} = v_p^2 f''(\tau)$$

From (i) and (ii) $$\frac{\partial^2 f}{\partial t^2} = v_p^2 \frac{\partial^2 f}{\partial x^2}$$

Satisfy the wave equation!

(You can also show that $f(kx \pm \omega t)$ gives the same result.
(Given $\omega = v_p \cdot k$)

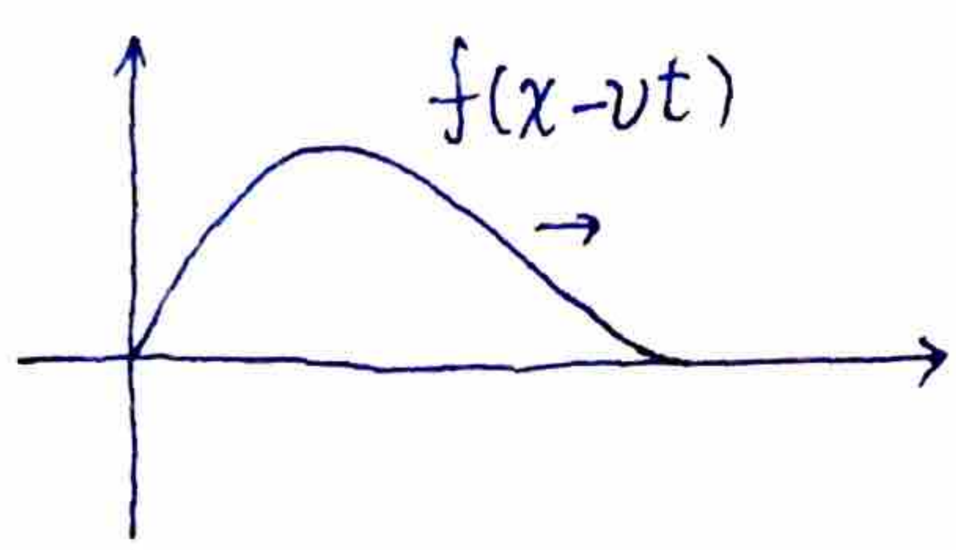
Which direction does it move in ?

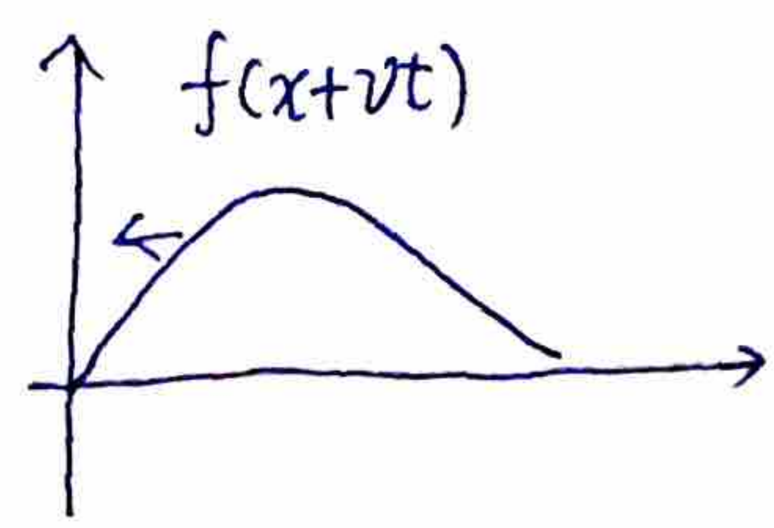


$f(\tau)$: shape of the progressing wave

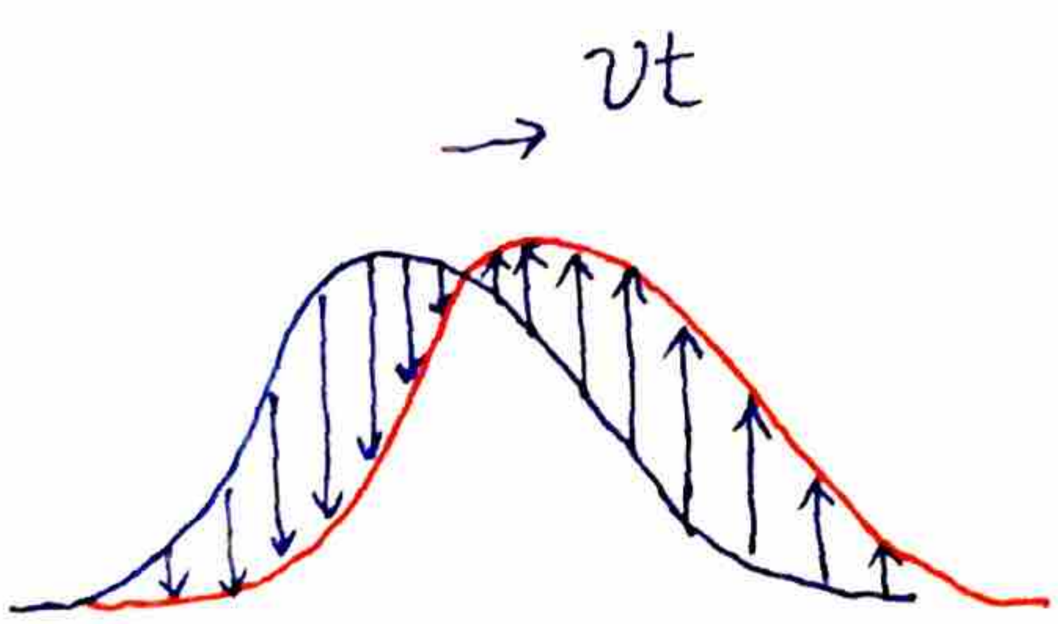
$f(x-vt)$: moving to right !

$f(x+vt)$: moving to left !

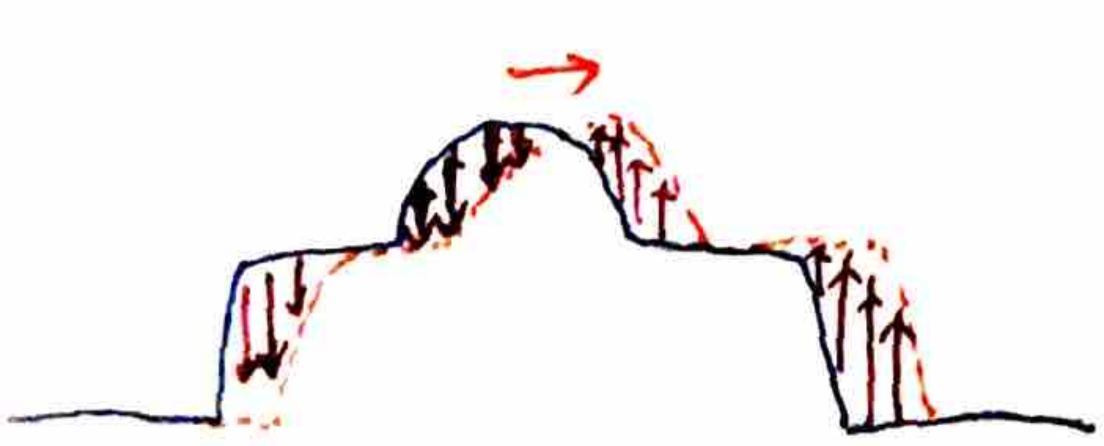




What is moving ? The particles in the spring only move up and down !



Works for any shape !! Not obvious, but it is what's happening



MIT

Cute !!

END

Wave Equations are Linear :

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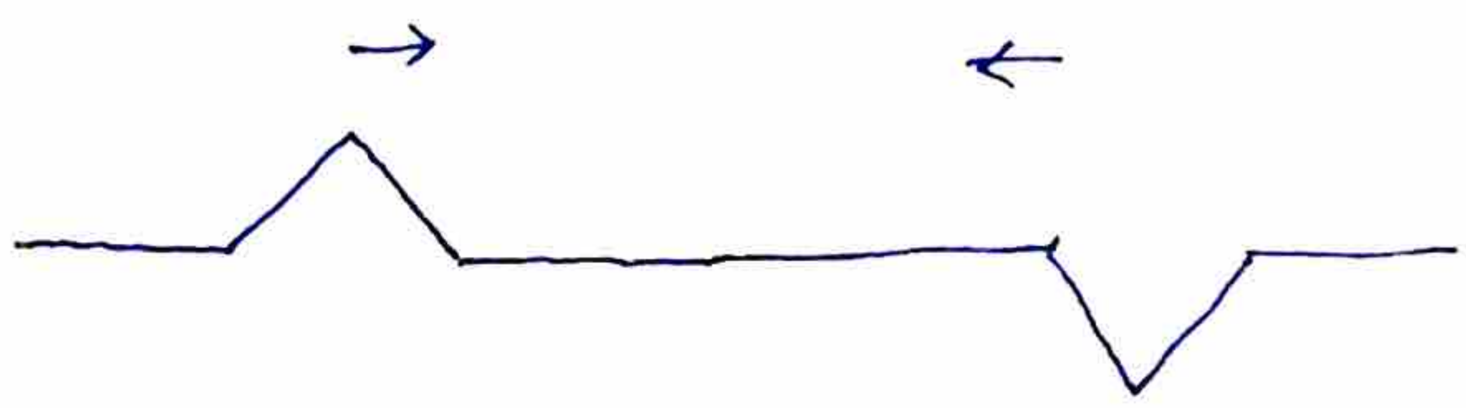
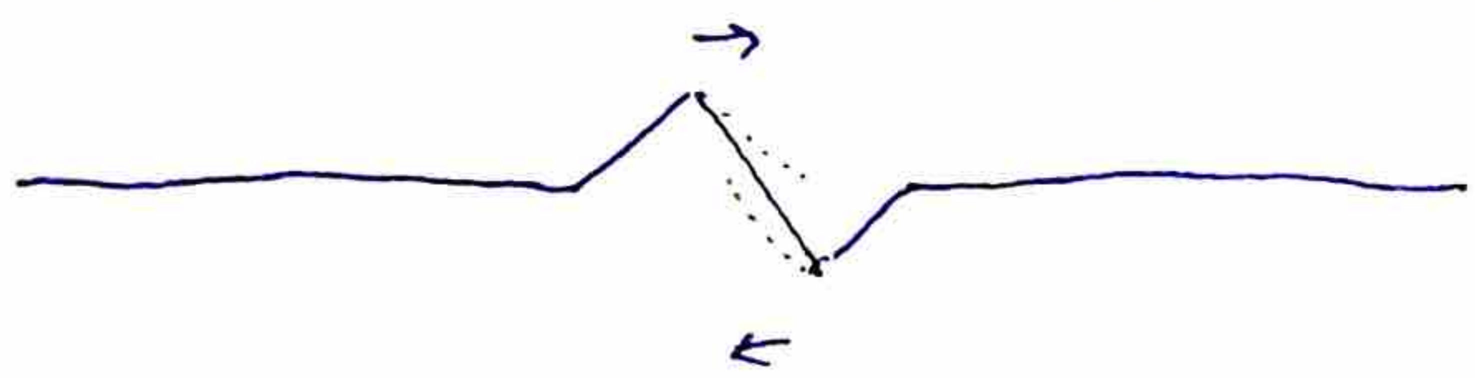
Linear combination of solutions is a solution

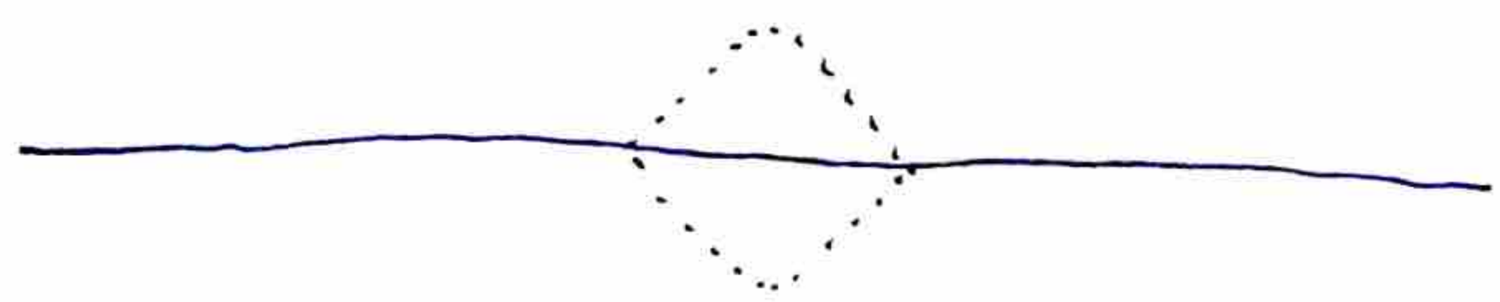
①


Energy conservation ? → poll:
① They cancell ?

String has memory ? → ② They pass each other

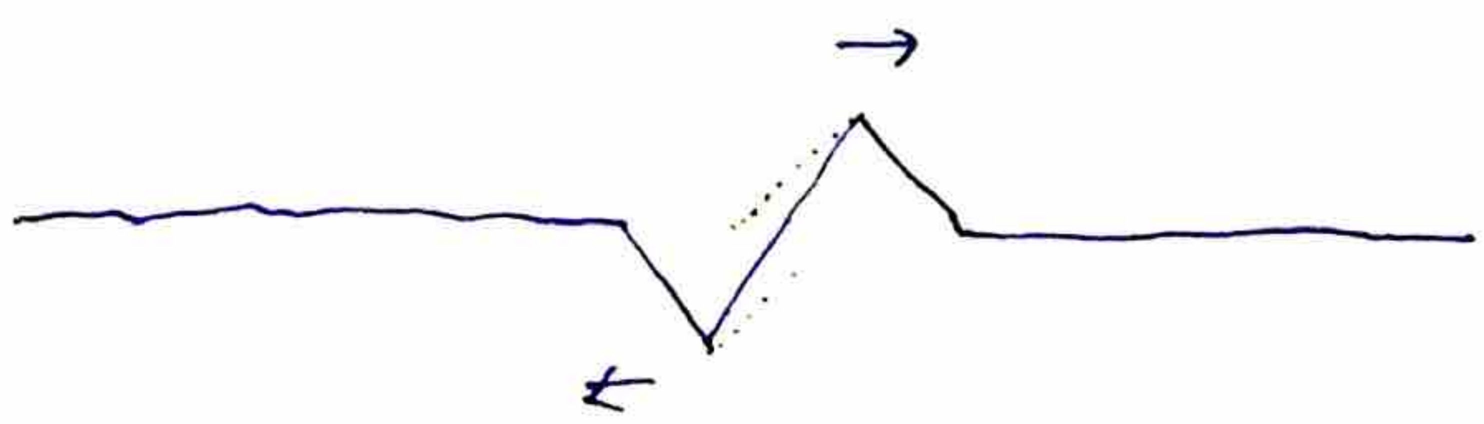
②


String gets creativity ? → ③ change shape ?

③


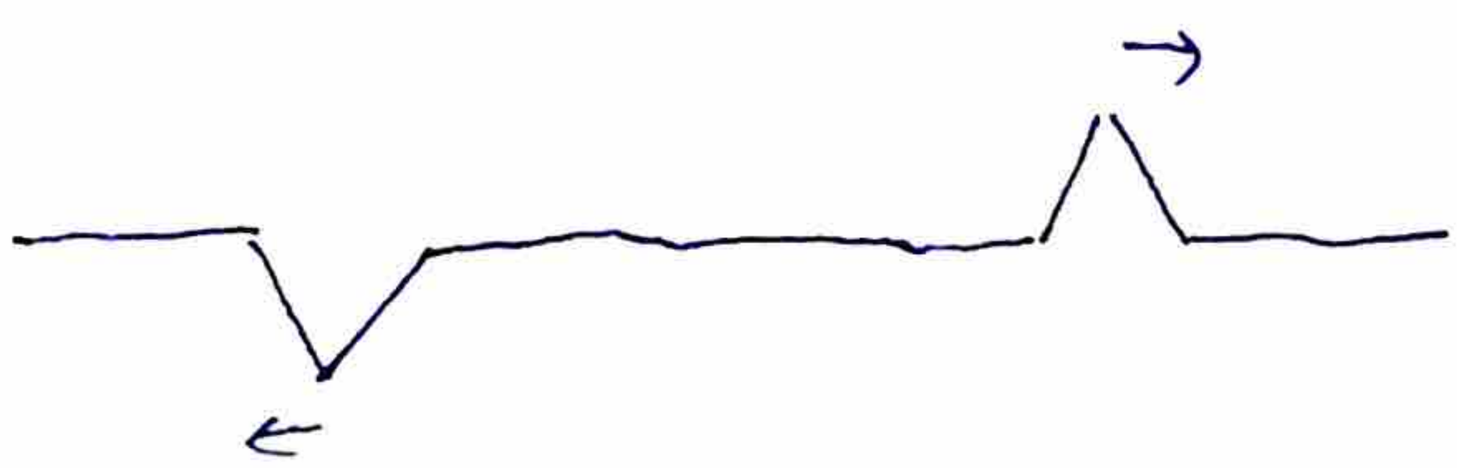
???

Exact cancellation

④


← Why?

How does this string "remember" what happened ?!

⑤


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③ is different from a stationary string :

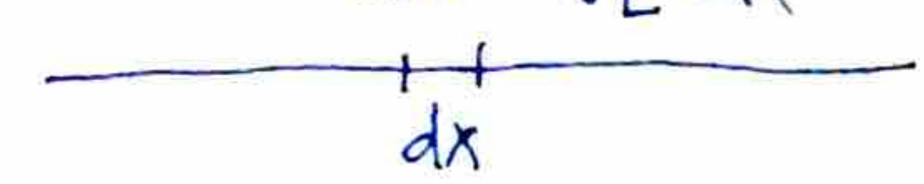
instaneous velocity in the region where cancellation happen is not zero!

i.e. the string is ready to produce the two outgoing progressing waves !

Energy stored in the string :

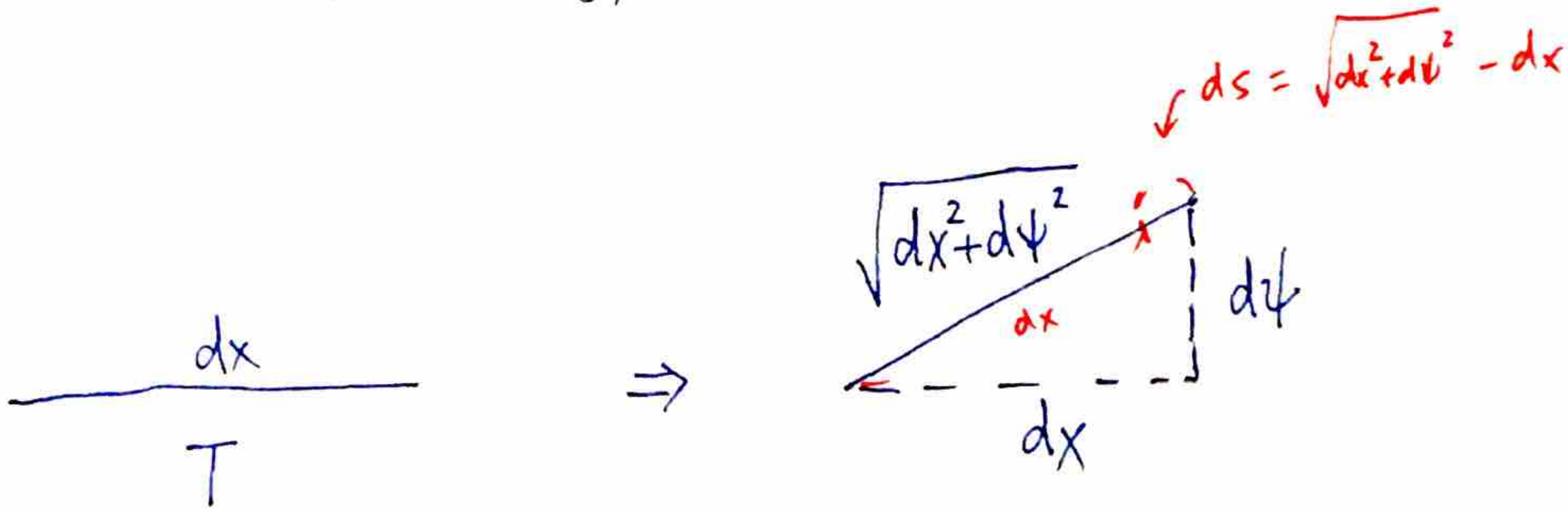
(1) Kinetic Energy $\quad \frac{1}{2} m v^2$

$dm = \rho_L \, dx$



$$\Rightarrow \int \frac{1}{2} \rho_L \, dx \left(\frac{\partial \psi}{\partial t} \right)^2$$

(2) Potential Energy : $\quad dW = F \cdot ds$



$$F \, dS \Rightarrow \quad T \cdot \left(\sqrt{dx^2 + d\psi^2} - dx \right)$$

$$= T \left(dx \sqrt{1 + \left(\frac{\partial \psi}{\partial x} \right)^2} - dx \right)$$

$\Rightarrow$ Small vibration $\left(\frac{\partial \psi}{\partial x} \right)$ small (small angle approximation)

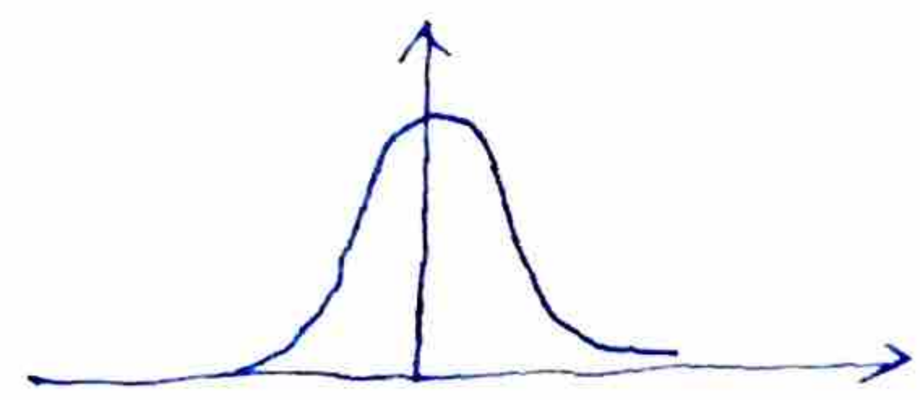
$$= T \left(\cancel{dx} + \frac{1}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 dx - \cancel{dx} \right)$$

$$\Rightarrow \quad \int \frac{1}{2} T \left(\frac{\partial \psi}{\partial x} \right)^2 dx$$

Summery :

$$\begin{cases} \text{Potential Energy} : \displaystyle\int \frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 dx \\ \text{Kinetic Energy} : \displaystyle\int \frac{\rho_L}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 dx \end{cases}$$

Example : $\psi(x,t) = \dfrac{1}{1+(x-3t)^4}$



What is the wave velocity ?

① Can be written as $f(x-3t)$

$\Rightarrow$ velocity = 3

traveling to the right

② Or, we can plug it into wave equation

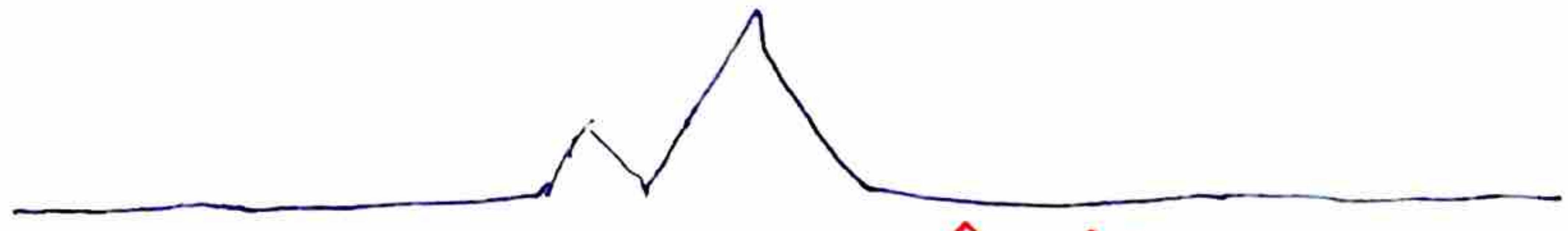
Calculate $\dfrac{\partial^2\psi}{\partial x^2}$ and $\dfrac{\partial^2\psi}{\partial t^2}$

$$\Rightarrow v^2 = \left(\frac{\partial^2\psi}{\partial t^2}\right) \Big/ \left(\frac{\partial^2\psi}{\partial x^2}\right)$$

Slides

Finally, if you start with a stationary shape

$t=0$



Tension, T

Linear density ρ_L

$\frac{\partial \psi}{\partial t} = 0$ every where

What will happen at $t = T$? Can we predict ?

$v = \sqrt{\frac{T}{\rho_L}}$

① Brute force:

decompose it into ∞ number of normal mode standing waves.

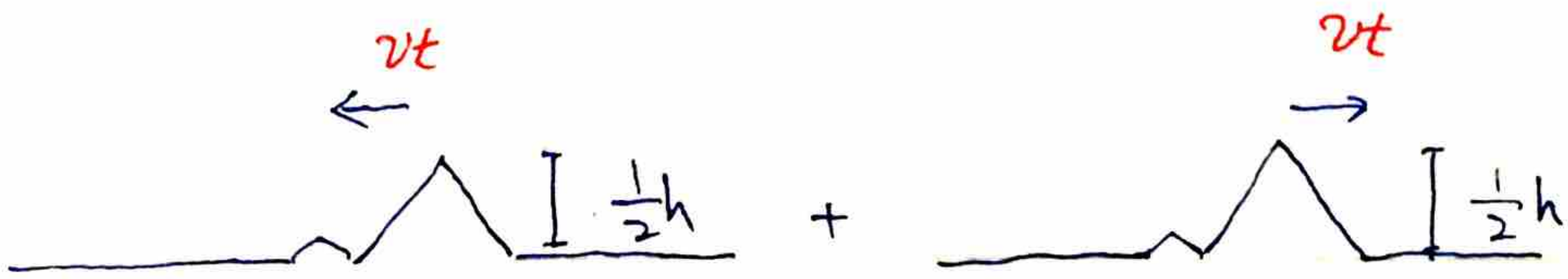
Evolve ∞ of those waves.

② $y = f(x+vt) + f(x-vt)$

velocity: $\frac{\partial y}{\partial t} = vf' - vf' = 0$

$\therefore$ Any stationary shape can be decomposed into two progressing waves !!

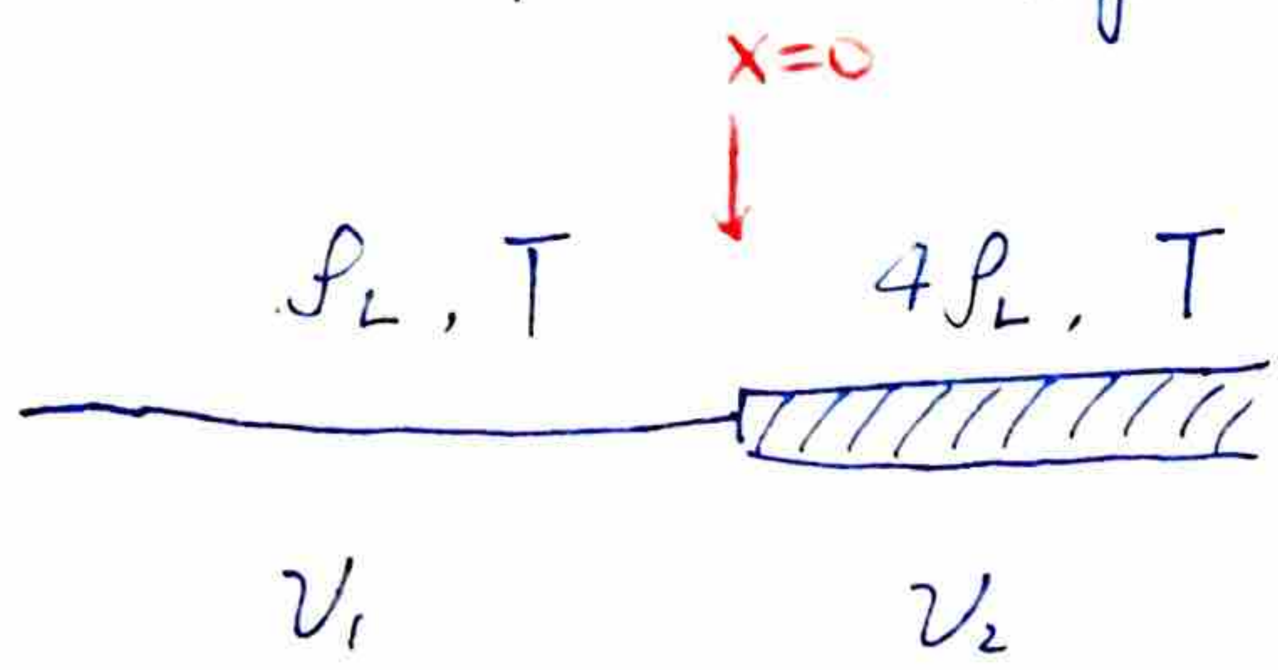
$=$



$\Rightarrow$ at $t=T$

Similarly: Normal modes: can be decomposed into two traveling sine waves.

A few examples with string:

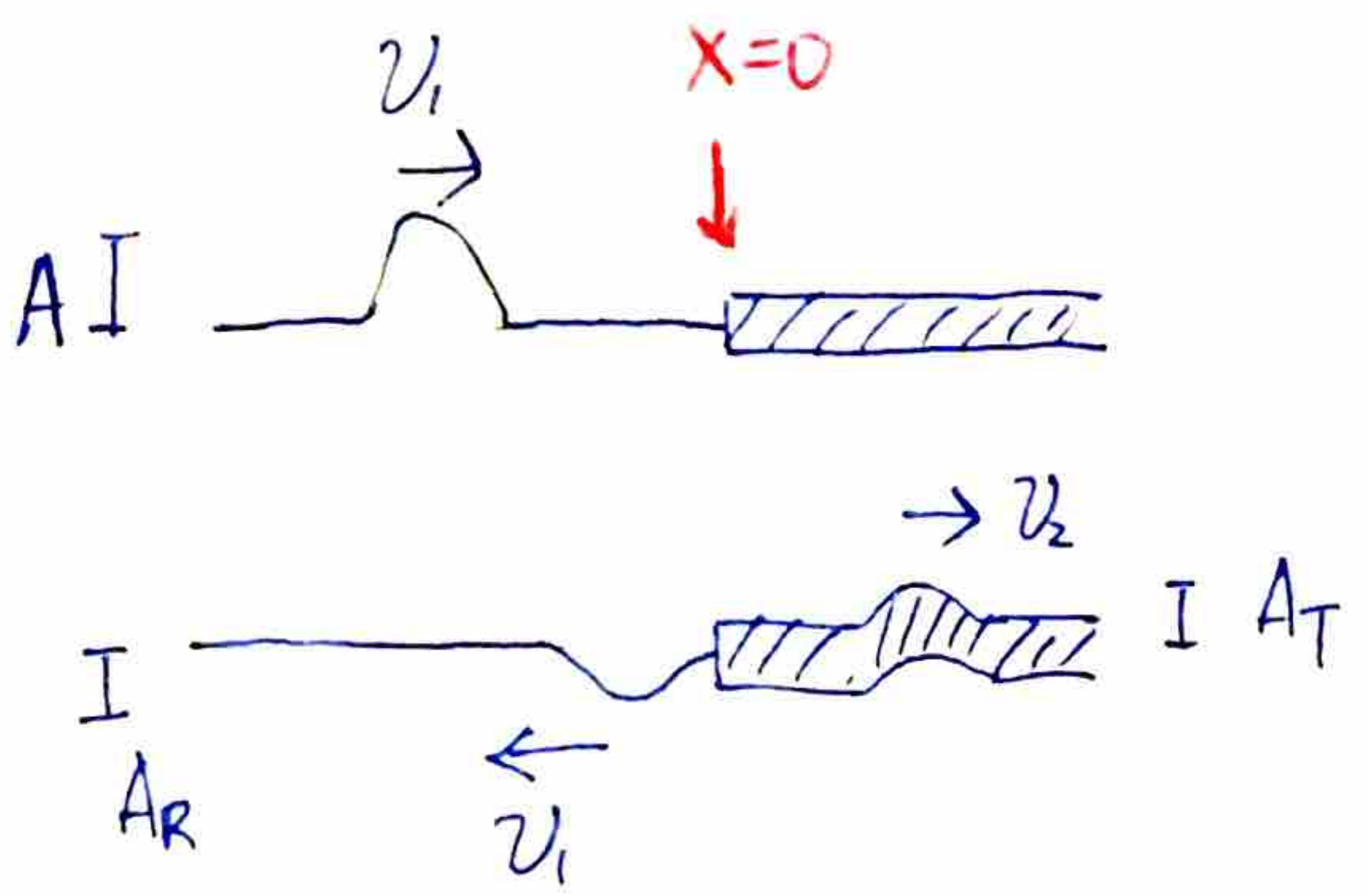


Now we are connecting two systems.
With different density. ρ_L and $4\rho_L$.

Assuming that the Tension T is <u>uniform</u>.

$$v_1 = \sqrt{\frac{T}{\rho_L}} \qquad v_2 = \sqrt{\frac{T}{4\rho_L}} = \frac{1}{2} v_1$$

Suppose we have an incident wave with amplitude A.



There will be reflected wave and transmission wave.

Boundary conditions:

(1) The string is continuous $Y_L(0^-) = Y_R(0^+)$

$\Rightarrow$ ω has to be the same.

(2) the slope is continuous

$$\left.\frac{\partial Y_L}{\partial x}\right|_{x=0} = \left.\frac{\partial Y_R}{\partial x}\right|_{x=0}$$

if the slope is not continuous $\Rightarrow$ huge acceleration at the junction!!

$$y_L(x,t) = f_i(-k_1x+\omega t) + f_r(k_1x+\omega t)$$

$$y_R(x,t) = f_t(-k_2x+\omega t)$$

$$k_1 = \frac{\omega}{v_1} \qquad k_2 = \frac{\omega}{v_2}$$

B.C.:

(1) $f_i(\omega t) + f_r(\omega t) = f_t(\omega t)$

(2) $-k_1 f_i'(\omega t) + k_1 f_r'(\omega t) = -k_2 f_t'(\omega t)$

integration on both side, replace k by v

$\Rightarrow -v_2 f_i(\omega t) + v_2 f_r(\omega t) = -v_1 f_t(\omega t)$

From (1) and (2)

$$f_r(\omega t) = \left(\frac{v_2-v_1}{v_1+v_2}\right) f_i(\omega t) \qquad R = \left(\frac{v_2-v_1}{v_1+v_2}\right)$$

$$f_t(\omega t) = \left(\frac{2v_2}{v_1+v_2}\right) f_i(\omega t) \qquad T = \left(\frac{2v_2}{v_1+v_2}\right)$$

In this example: $v_2 = \frac{v_1}{2} \Rightarrow R = -\frac{1}{3} \quad T = \frac{2}{3}$

Wave length changed, but frequency did not.

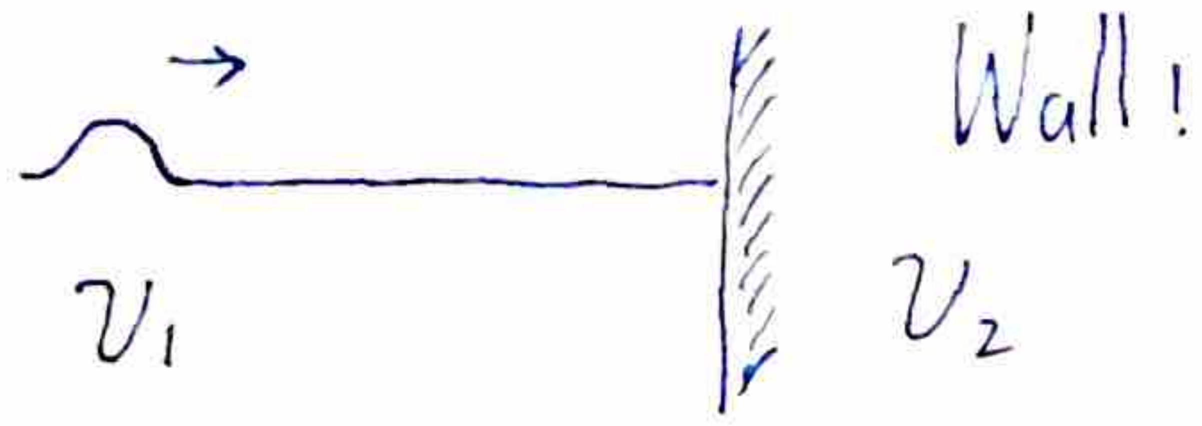
Two things we have learned:

(1) The amplitude of the transmission and reflected wave is determined by the properties of the two systems "Impedance" in this case it is $Z = T/v$

(2) Wave length changes: $k_1 \propto \frac{1}{v_1}$

Consider two extreme cases:

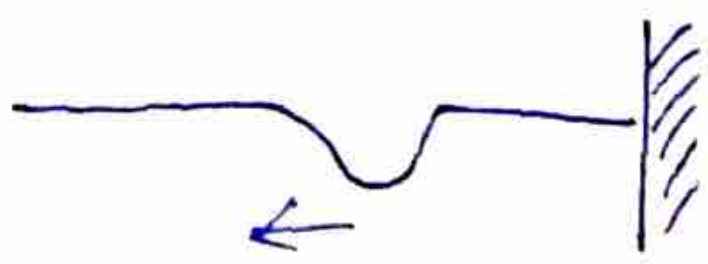
(1)



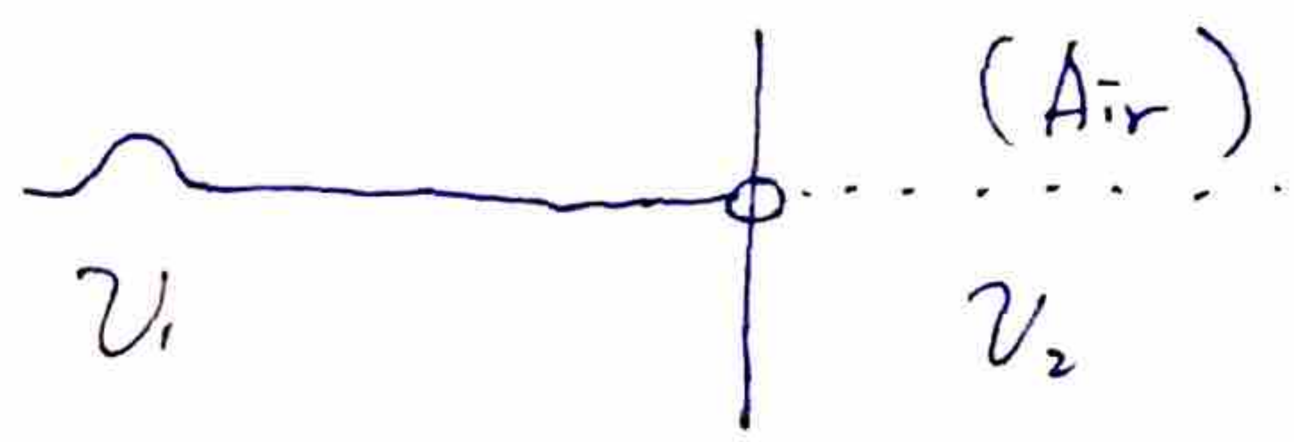
$\rho_{L,2}$ of Wall is BIG!

$\Rightarrow \quad v_2 \to 0$

$\Rightarrow \quad R = -1 \quad , \quad T = 0$



(2)



$\rho_{L,2} \longrightarrow 0 \quad \Rightarrow \quad v_2 \to \infty$

$\Rightarrow \quad R = 1 \quad , \quad T = 2$

$\Rightarrow$

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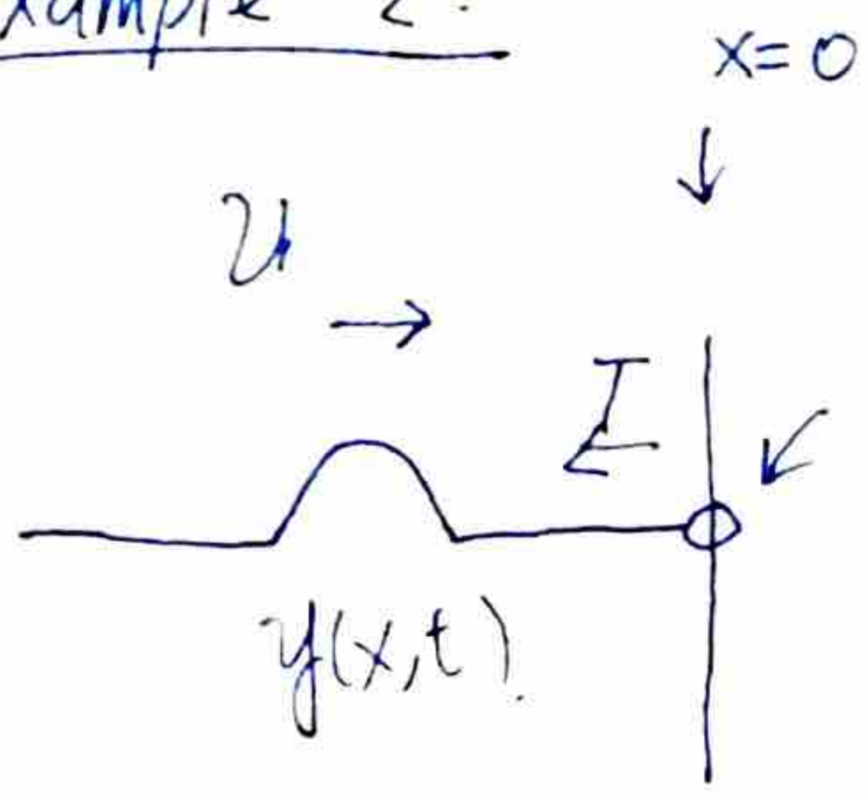
(More examples)

Example 2:



$p(t)$

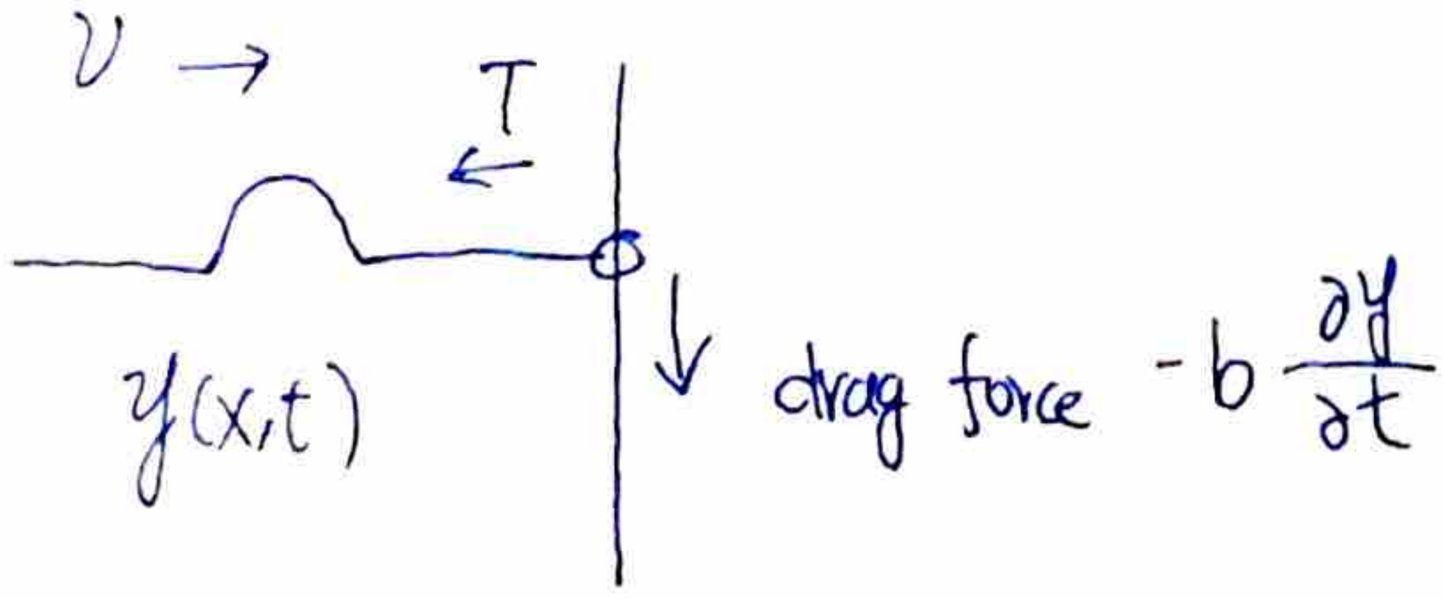
Boundary conditions:

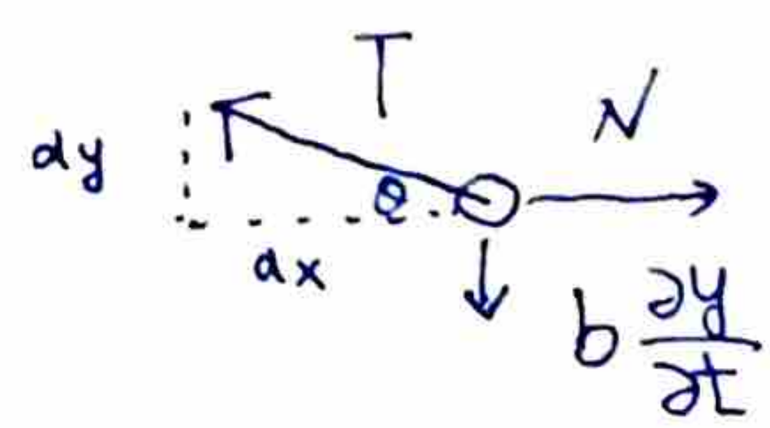
(1) $y\Big|_{x=0} = p(t)$

(2) Tension force cancel normal force

$\Rightarrow \quad -T\frac{\partial y}{\partial x} = 0$

Example 3:





(1) $y\Big|_{x=0} = p(t)$

(2) $\Rightarrow \; -T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t} = 0$

$\theta \to 0 \Rightarrow \sin\theta \approx \frac{dy}{dx}$