YEN-JIE LEE: OK, so welcome back, everybody. Welcome back to 8.03. Today, we are going to continue the discussion of the harmonic oscillators. And also, we will add damping force into the game and see what will happen, OK? So this is actually what we have learned last time from this slide. We have analyzed the physics of a harmonic oscillator, which we actually demonstrated last time. And you can see the device still there. And Hooke's law, actually the Hooke's law is actually far more general than what we saw before. It works for all small oscillations around about a point of equilibrium position, OK?

And that can be demonstrated by multiple different kinds of physical systems. For example here, I have a mass, which actually can only move along this track here. And if I put this mass set free, then this thing is actually exercising harmonic oscillation, OK? We can do this with large amplitude. We can also do it with small amplitude. And you see that, huh, really, it works. Hooke's law actually works. And it predicts exactly the same motion as to what you see on the slide, OK?

And we also have a little bit more complicated system. For example, this is some kind of rod. And you can actually fix one point and make it oscillate. And you see that, huh, it also does some kind of harmonic oscillation. But now, what is actually oscillating is the amplitude. The amplitude is actually the angle with respect to the downward direction. And finally this is actually the vertical version of this spring mass system, which you will be analyzing that in your P-set. And you see that, huh, it actually oscillates up and down harmonically.

So that's all very nice. And we also have learned one thing which is very, very interesting. It's that a complex exponential is actually a pretty beautiful way to present the solution. And you will see it works also when describing the damped oscillators. And we will see how it works in the lecture today. I received several questions during my office hour and through email or Piazza.

There were some confusions about doing the Taylor expansion, OK? So in lecture last time, the equilibrium position is at x equal to 0. Therefore, I do Taylor expansion around 0, OK? But in this case, if the equilibrium position or the minima of the potential is at x equal to L, then what you need to do is to do a Taylor expansion around x equal to L, just to make that really, really clear, OK? OK, I hope that will help you with the P-set question.

OK, so let's get started immediately. So let's continue the discussion of the equation of motion we arrived at last time. So we have M x double-dot and this is equal to minus kx, OK? That is actually the formula from last time. And we can actually calculate the kinetic energy of this spring and mass system. And basically, this is going to be equal to 1/2 M times x dot squared.

OK, and we can also calculate the potential energy of the spring. Potential energy, and that is equal to 1/2 kx squared. We also know what would be the total energy. The total energy would be a sum of the kinetic energy and of the potential. Basically, you get this formula, 1/2 M x dot squared plus 1/2 kx squared.

One last time, we have solved this equation of motion, right? So the solution we got is x equal to A cosine omega 0 t plus phi. Well, omega 0 is equal to a square root of k over M. Therefore, we can actually calculate what would be the total energy as a function of time, right? So if we calculate that, we'll get E will be equal to 1/2 M A squared omega 0 squared sine squared omega 0 t plus phi-- so this is actually the first term here-- plus 1/2 kA squared cosine squared omega 0 t plus phi, OK?

Then, we also know that this coefficient here is just kA squared, right? Because omega 0 is actually equal to the square root of k over M. And if you replace this omega 0 squared by k over M, then you actually arrive at kA squared, OK? So that is actually very good. So that means I can simplify the total energy. And what we are going to get is 1/2 kA squared. I can take this factor out. And that will give me, inside these brackets, I will get sine squared omega 0 t plus phi plus cosine squared omega 0 t plus phi. And this is actually equal to 1, right?

Just a reminder, sine squared of theta plus cosine squared of theta is always equal to 1. So that gives me this result. This is actually 1/2 kA squared, OK? So that is actually the result. What does that mean? That means, if I actually pull this mass harder, so that initially it has larger amplitude, then the total energy is actually proportioned to amplitude squared, OK?

So I am storing more and more energy. If I increase the amplitude even more, then I am storing the energy in this system. And it's proportional to A squared. And also, if the spring constant is larger, the same amplitude will give you more energy. So that means that you can store more energy if you have a larger string constant, OK?

The most surprising thing is that actually this is actually a constant, right? What does that mean? The total energy is actually not variating as a function of time. You see? So total energy is constant, OK? So you can see from this slide the total energy is actually showing us the sum, which is the green curve. And the kinetic energy and the potential energy are shown as red and blue curves. You can see that the total energy is actually constant. But this system is very dynamical. You see?

So that energy is actually going back and forth between the spring and the mass in the form of kinetic energy and in the form of potential energy. But they are doing it so well, such that the sum is actually a constant. So the energy is actually constant, OK? So that is actually pretty beautiful. And it can be described very well by these mathematics. Any questions from here? OK, so I would like to say simple harmonic motion, actually, what you are going to get is the energy is actually conserved and independent of the time. And later, you will see an example with damping. And you will see that energy conservation is now no longer the case, OK?

So let's immediately jump to another example, which is actually involving simple harmonic motion. So let's take this rod and nail system as an example. If I actually slightly move this rod, and then I release that, then actually you will see simple harmonic motion, also for this system. So let's actually do the calculation as another example.

So this is actually my system. I have this rod, OK? Now, I am assume that the mass is actually uniformly distributed on this rod and is nailed on the wall, OK? And the length of this rod is actually l. So that means the center of mass is actually at l/2 with respect to the nail, OK? And also, this whole system is set up on Earth, right? Therefore, there will be gravitational force pointing downward, OK? So that means you have gravitational force, Fg, pointing downward, OK?

So this is actually the system, which I would like to understand. And just a reminder, what are we going to do afterwards in order to turn the whole system into a language we know describes the nature? What are we going to do? Anybody? We are going to define the coordinate system, so that I can translate everything into mathematics, right? So that's actually what we are always doing. And you will see that we are always doing this in this class, OK?

So what is actually the coordinate system which I would like to use? Since this system is going to be rotating back and forth, therefore, I would like to define theta to be that angle with respect to the axis, which essentially pointing downward, OK? So the origin of this coordinate system uses theta equal to 0. This means that the rod is actually pointing downward, OK?

And also, I need to define what is actually the positive value of the zeta, right? So I define anti-clockwise direction to be positive, OK? So it is actually important to actually first define that, then actually to translate everything into mathematics, OK? So the initial condition is the following. So I actually move this thing, rotate this thing slightly. Then, I actually release that really carefully without introducing any initial velocity, OK?

Therefore, I have two initial conditions. OK, at t equal to 0, there are two initial conditions. The first one is theta 0 is equal to theta initial. The second condition is the same as what we have been doing last time. The initial velocity or angular velocity is actually equal to 0. So that gives you theta dot equal to 0, OK?

Now, we have actually defined the coordinate system. Now, we can actually draw a force diagram, so that we can actually use our knowledge about the physics to obtain the equation of motion, right? So now, the force diagram looks like this. So this is actually the center of mass of this rod. And you have a force pointing downward, which is due to the gravitational force. Fg is equal to mg. It's pointing downward. The magnitude is actually equal to mg. And also, we know the R vector. This vector has a length, l/2. It's pointing from the center of mass of this rod to the nail, OK?

And also, we know the angle between these vectors, pointing from the center of mass to the nail, and the vertical direction, which we have already defined, which is actually called theta. Therefore, now, we can actually calculate what would be the torque. Tau will be equal to this R vector cross the force, total force acting on the center mass. In this case, it's just Fg, OK?

So now, we can actually write this down explicitly. Since the whole system is actually rotating on a single plane, so there's only one plane this is sitting on. And it's actually going back and forth only on this plane, OK? Therefore, actually, I can drop all the arrows and write down the magnitude of the tau directly. And this will be equal to minus mg l/2 sine theta t, OK? Any questions so far?

OK, so now, we have the torque. And we can make use of the rotational version of Newton's Law to obtain the equation of motion, right? So that should be pretty straightforward. Tau will be equal to I, which is the moment of inertia of the system, times alpha t, OK? And just for your information, I already calculated the I for you. I is equal to 1/3 ml squared, OK? So you can actually go back home and actually do a check to see if I'm telling the truth. And if you trust me, then that's the answer, which is actually 1/3 ml squared, if the mass is actually uniformly distributed on this rod, OK?

So that would give me minus mgl divided by 2 sine theta t, OK? So that is actually coming from this side, OK? So now, I can actually simplify this expression. I can now plug in the I value into this equation. And I will get 1/3 ml squared theta double-dot t, which is actually alpha, OK? Now, I write it as theta double-dot.

And that will be equal to minus mgl over 2 sine theta, OK? I can move all the constants to the right-hand side. Therefore, I get theta double-dot t. This is equal to minus mgl. OK, actually, I can already simplify this, right? These actually cancel. And the 1/l actually cancels. So therefore, I get minus 3g over 2 sine theta t.

OK, as you know, we actually defined omega to replace this constant to make our life easier. So I can now define omega 0 equal to square root of 3g over 2l, OK? And that will give you theta double-dot of t equal to minus omega 0 squared sin theta t. Any questions so far? A lot of calculations. But they should all be pretty straightforward. And actually, we are done now. We are done. Because we have the equation of motion. And the rest of the job is to solve just it. So it is actually now the problem of the math department. So can anybody actually tell me the solution of the theta of t? Anybody?

AUDIENCE: Unfortunately, we'd have to approximate it.

YEN-JIE LEE: That's very unfortunate. So now, we are facing a very difficult situation. We don't know how to solve this equation in front of you. I don't know, OK? Of course, you can actually solve it with a computer, or, if you want to go fancy, solve it with your cellphone, if it doesn't explode. But it's not really nice to do this in front of you. We don't learn too much. OK, so what are we going to do?

So what we can do is actually to consider a special case. So we know that this equation of motion is exact, OK? So if you solve it, it would describe the motion of this rod. Even with a large angle, it works, OK? And now, in order to actually show you the math in the class, therefore actually I will do a small approximation. So actually, I would only work on the case that when the amplitude is very small and see what is going to happen.

So now, I'm considering a special case. Up to now, everything is exact. And now, I am now going to a special case. Theta t goes to 0, OK? Then, we can actually get this. Sine theta t is roughly theta t, OK? Based on the Taylor expansion, you can actually verify this, OK? So in this case, if we take theta equal to 1 degree, then the ratio of the sine theta and the theta is actually equal to 99.99%, which is very good. If I take it as 5 degrees, then it's actually 99%. Even at 10 degrees, it's actually 99.5%.

Now, that shows you that sine theta is so close to theta, OK? We are pretty safe. Because the difference is smaller than 1%. OK, so that's very nice. After this approximation, I get my final equation of motion. Theta double-dot t equal to minus omega 0 squared theta t. Just a reminder, omega 0 is equal to square root of 3g over 2l, OK?

We have solved this equation last time, last lecture, right? It's exactly the same. OK, it happened to be exactly the same. Therefore, I know the solution will be theta of t equal to A cosine omega 0 t plus phi. From the initial conditions, which I have one and two, I am not going to go over these calculation again. But again, we can actually plug in 1 and 2 to solve the unknown A and the phi. If you do this exercise, you will conclude that A is equal to theta initial. And phi is equal to 0, OK? So the solution would be theta of t equal to theta initial cosine omega 0 t.

You can see that this actually works for this system. Simple harmonic oscillation actually described the motion of this system as a function of time. You can also see a few more examples shown here. Two of them you are going to really work on in your P-set and also another one involving circuits. If you have a capacitor and you have an inductor, actually the size of the current is also doing a simple harmonic motion, OK? And as we actually discussed before, the energy is always conserved. And that is actually stored in different components of the system, OK?

So we have done this. What is actually new today? What we are going to do today is let's actually observe this phenomenon here. So this thing is actually going to go back and forth. But it's actually not going to do that forever, right? Something is happening, which actually slows the motion down. I can also make use of this system, OK? I start from here. And I'm not worried that this actually goes out of this track. Because I know for sure it will stop there. Why? Because the initial amplitude is not going to-- the amplitude is not going to be larger than the initial amplitude, right? So I'm not worried at all, OK?

But you can see that the amplitude is changing as a function of time. Apparently, something is missing. And that is actually a direct force, or friction, which is actually not included in our calculation. So let's actually try to make the calculation more realistic and see what is going to happen. So now, I will introduce a drag force, which actually introduces a torque tau drag, t, which is equal to minus b-- b is actually some kind of constant, which is given to you-- theta dot t, which is actually proportional to angular velocity of that rod, OK?

And also of course, I keep the original approximation. The theta is very small, such that I don't have to deal with the integration of sine theta, OK? So solving this, theta double-dot equal to minus omega 0 squared sine theta is a complicated function. You may ask, why do I actually introduce a drag force proportional to the velocity? And why do I put a minus sign there?

That is actually because, if you have a minus sign, that means, when this mass or that rod is actually going downward, then the drag force is really dragging it. Because it's actually in the opposite direction of the velocity of the mass or the angular velocity of the rod, OK? So I need a minus sign there, OK? Otherwise, it's not a drag force anymore. It's actually accelerating the whole thing.

Secondly, why do I choose that to be proportional to theta dot or velocity? There's really no much deeper reason. I choose this form because I can actually solve it in front of you, OK? The reality is actually between proportional to theta dot and theta dot squared, for example, OK? This is actually a model which I introduced here, which I can actually solve it in front of you.

On the other hand, you'll see that it's actually not bad at all. It actually works and describes the system, which will actually work to perform the demo here, OK? And once we have introduced this, the equation of motion will be modified. So let's come back to the equation of motion. So you have to theta double-dot t originally would be equal to tau total t divided by I, OK? And now, this will become tau t plus tau drag t divided by I. So there's an additional time here.

OK, if I simplify this whole equation, then I get minus mgl over 2 sine theta. And this is actually roughly theta minus b theta dot divided by 1/3 of ml squared, OK? So you can see that I still make this approximation sine theta roughly equal to theta. Then, I can actually write this equation in the small angle case. OK, I get minus 3g over 2l theta t minus 3b over ml squared theta dot t. OK, and now, as usual, I define omega 0 squared equal to 3g over 2l. And I can also define gamma is equal to 3b over ml squared, just to make my life easier, right?

Finally, we will arrive at this expression, theta double-dot plus gamma theta dot plus omega 0 squared theta. And that is equal to 0. So what you can see from here is that we have actually derived the equation of motion, OK? We have derived the equation of motion. And actually, part of the work is actually really just solving this equation of motion. And you don't really have to solve it. Because you already get the result from 18.03 actually, if you remember. And we are going to discuss the result.

But before that, before I really try to solve this equation, I would like to take a vote, OK? So here, I have two different systems. They have equal amounts of mass. They are attached to a spring. If you do the same equation of motion derivation, you will actually get exactly the same equation of motion in that format, OK? So the form of the equation of motion will be the same between this system and that system, OK?

I would like to ask you a question about the oscillation frequency. So you can see that one of them is actually a better mass. It's like a point-like particle. And the other one is wearing a hat, OK? What is going to happen is that this Mexican hat is going to be trying to push the air away, right? Then, you may think, OK, this Mexican thing is not really very important. Therefore, the oscillation frequency may be the same, right?

How many of you think the oscillation frequency, if I actually tried to perturb these two systems, would be the same? Raise your hands. 1, 2, 3, 4, 5, 6, 7, 8-- OK, we have 11. So the omega, the predicted omega, will be equal to omega 0-- 11 of you. How many of you will think that, because of this hat, this pushing this air away, it's a lot of work to be done. Therefore, this is going to slow down the oscillation. How many of you think that is going to happen? 1, 2, 3-- OK, 17.

It may happen to you that you think this idea of wearing a hat is really fashionable. Therefore, it got really exciting and it oscillates faster. Can that happen? How many of you actually think that is going to happen? OK, one-- you think so? Two. Very good, we have 2. What do you think? Where are the rest? Only 30 of you actually think that is going to happen.

OK, all the rest think of the class think that this one is going to-- pew! Disappear to the moon, OK? So that is actually the opinion. And we have completed the poll. And what we are going to do is that we are going to solve this system and see what is going to happen. And we will do that experiment in front of you, OK? All right, so that's very nice. So now, we have this question of motion.

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And now, I will pretend that I'm from the math department for a bit and help guide you through the solution. So now, I can use this trick. I can actually say theta is actually the real part of the z, which is a complex function. And as we learned before, z of t, and I assume that to be exponential I alpha t. So alpha is actually some kind of constant, which I don't really know what is the constant yet.

OK, I can now actually write the equation of motion in the form of z. Then basically, what I get is z double-dot t plus gamma z dot t plus omega 0 squared z of t. And this is equal to 0, OK? So remember, exponential function cannot be killed by differentiation, right? Therefore, it's really convenient. You can see from here. Now, I can plug in this expression-- which I did this and guessed to this equation of motion.

Then what I am going to get is minus alpha squared. Because you take I alpha I alpha out of this exponential function, right? Because you do double differentiation. So you get minus alpha squared plus i gamma alpha-- because this is only differentiated one time-- plus omega 0 squared. And all those things are actually multiplying this exponential function, exponential i alpha t equal to 0, OK?

So we will write this expression. That is very nice. And we also know that, this expression is going to be valid all the time. No matter what t you put in, it should be valid, right? Because this is the equation of motion. And we hope that this solution will survive this test. So I can easily conclude that this one is actually not equal to 0. It can be some value, not 0. So what is actually equal to 0? This first term is actually equal to 0, OK?

Therefore, I can now solve this equation; minus alpha squared plus i gamma alpha plus omega 0 squared equal to 0. I can solve it, OK? If I do that, then I would get alpha is equal to i gamma plus/minus square root of 4 omega 0 squared minus gamma squared divided by 2. This is actually the second order polynomial. And that is actually equal to 0. Therefore, you can actually solve it easily. And this is actually the solution. And I can write it down in a slightly different form. i gamma over 2 plus/minus square root of omega 0 squared minus gamma squared over 4, OK? Any questions so far? Am I going too fast? Everything's OK?

OK, So you can see that alpha is equal to this expression. And I would like to consider a situation where omega 0 is much, much larger than gamma, OK? Just a reminder of what is gamma, OK? Maybe you've got already a bit confused. What is gamma? Gamma is related to the strength of the direct force, right? It is actually 3b over ml squared, OK? b is actually determining the size of the direct force, OK?

So I would like to consider a situation. The first situation is if omega 0 squared is larger than gamma squared over 4. So in that case, the drag force is small. It is not huge. It's small, OK? If that is the case, this is actually real, right? Because omega 0 squared is larger than gamma squared over 4. Therefore, this is real, OK? So now, I can actually define omega squared, define that as omega 0 squared minus gamma squared over 4, OK?

And this will become i gamma over 2 plus/minus omega, OK? So that means I would have two solutions coming from this exercise. Z plus of t is equal to exponential minus gamma over 2 t exponential i omega t, OK? And the second solution, if I take one of the plus sign and one of the minus sign solutions, then the second solution would be exponential minus i gamma over 2 t exponential minus i omega t, OK? Any questions so far?

OK, so we would like to go back to theta, right? So what would be the theta? So that means I would have a theta 1 of t, which is actually taking the real part. So it's theta plus maybe, taking the real part of z plus. And that will give you exponential minus gamma over 2 t cosine omega t, OK? I'm just plugging in the solution to this equation, OK? Theta minus t would be equal to exponential, and this gamma over 2 t sine omega t, OK?

Finally, the full solution of theta of t would be a linear combination of these two solution, right? Therefore, you will get theta of t equal to exponential minus gamma over 2 t a (is some kind of constant) times cosine omega t plus b sine omega t. And of course, from the last time, as you will know, this can also be written as A cosine omega t plus phi, OK? Any questions so far?

OK, very good. So we have actually already solved this equation. And of course, we can actually plug this back into this equation of motion. And you will see that it really works. And I'm not going to do that now. But you can actually go back home and check. And if you believe me, it works. And also at the same time, it got two undetermined constants, since this is a second order differential equation. Therefore, huh, this thing actually works. It has two arbitrary constants. Therefore, that is actually the one and only one solution in the universe which satisfies the equation of motion or satisfies that differential question, OK?

So this thing actually has dramatic consequences. The first thing which we learn is that, as a function of time, what is going to happen? The amplitude is now becoming exponential minus gamma over 2 t times A. This is actually the amplitude. The amplitude is decreasing exponentially. So that is actually the first prediction coming from this exercise, OK?

The second prediction is that this thing is still oscillating. Because you've got the cosine omega t plus phi there, you see? So the damping motion is going to be like going up and down, up and down, and get tired. Therefore, the amplitude becomes smaller, and smaller, and smaller. But it's never 0, right? It's never 0, OK? It's actually going to be oscillating down, down, down, so small I couldn't see it. But it's still oscillating, OK?

Finally, we actually have also the answer to the original question we posed, OK? So now, you can see that the oscillation frequency is omega, OK? Originally, before we introduced the drag force, omega 0, which is the oscillation frequency, is actually an angular frequency. It's actually the square root of 3g over 2l. And you can see that the new omega, the oscillation frequency with drag force, is the square root of this, omega 0 squared minus gamma squared over 4.

So what this actually tells us is that this is going to be smaller, because of the drag force, OK? So that's a prediction. Let's do the experiment and see what is going to happen. So let's take a look at these two systems. They have the identical mass, which our technical instructor actually carefully prepared. They have the same mass, even though one actually looks a bit funny. The other one looks normal, OK?

Now, what I'm going to do is to really try and see which one is actually oscillating faster, OK? So let's see. I release them at the same time. And you can see that originally they seem to be oscillating at the same frequency. But you can see very clearly that the one with the hat is actually oscillating slower, OK? So you can see that, OK, 17 of you actually got the correct answer.

And the most important thing is that you can see that this simple mass actually describes and predicts what is going to happen in my little experiment. So that is actually really cool. And I think it's time to take a little break. And then, we will come back and look at other solutions. And of course, you are welcome to come to the front to play with those demonstrations.

So there are two small issues which were raised during the break. So the first one is that, if you actually calculate the torque from this equation-- so I made a mistake. The R vector should be actually pointing from the nail to the center of mass, OK? So I think that's a trivial mistake. So if you do this, then you can actually calculate the tau equal to R cross F. Then, you actually get this minus sign, OK? So if I make a mistake in pointing towards the nail, then you will get no minus sign, then that didn't really work, OK? So very good, I'm very happy that you are actually paying very much attention to capture those.

The second issue is that-- so now, I'm saying that, OK, now I have the solution in the complex format. So I have a Z plus and I have a Z minus, OK? And then I would like to go to the real world, right? Because the imaginary thing is actually hidden in some kind of motion in the actual dimension, et cetera, I would like to go back to reality, OK?

And what I said in the class is that I take the real part of one of the solutions. And I can also take a real part of i times one of the solutions. But of course, you can also do this by doing a linear combination of the solutions, right? As we actually discussed last time, the linear combination of the solutions is also a solution to the same equation of motion, since this one is actually linear.

Therefore, what I actually do is actually to sum the two solutions, Z plus and Z minus and divide it by 2. Or actually, I can actually do a minus i/2 times Z plus minus Z minus, OK? And then I can also extract this sign term here, OK? So that should be the correct explanation of the two solutions in the real axis, OK? Any questions so far? Thank you very much for capturing those.

Ok, so now, you can see that we have been discussing the equation of motion of this functional form. And the one thing which is really, really interesting is that the solution, when we take a small drag force limit, actually we arrive at a beautiful solution that looks like this, A exponential minus gamma over 2 t cosine omega t plus phi. That actually predicts the oscillation, OK?

At the same time, it also says that the amplitude is actually going to drop exponentially, but never 0, OK? Finally, we also know that this solution actually tells us that, if we have a spring mass system oscillating up and down, if we have a rod like what we actually solve in a class, this object is going to pass through 0, the equilibrium position, an infinite number of times, right? Because the cosine is always there. Therefore, although the amplitude will become very small, but it's still oscillating forever until the end of the universe, OK?

All right, so that's actually what we have learned. And also, one thing which we learned last time is that simple harmonic motion, like this one, which we were just showing here, or this one, which is actually a mass oscillating back and forth on the track, is actually just a projection of a circular motion in a complex plane, OK? And what we are really seeing here in front of you is actually a projection to the real axis, OK? So that's actually a really remarkable result and a beautiful picture.

And of course, we can actually also plug in the solution with damping. So what is actually the picture in this language, in this exact same language? If we actually follow the locus, then basically what you are going to see is that this thing actually spirals. And the amplitude is actually getting smaller and smaller and is sucked into this black hole in the 0, 0, OK? So you can see that now the picture looks as if there is something really rotating in the complex plane. And it's actually approaching 0. Because the amplitude is actually getting smaller and smaller. But this whole thing is still rotating, OK? OK, that's really nice.

All right, so now, this is actually a special case. When we actually assume that gamma is actually pretty small. So you have very small drag force, OK? So let's actually check what would happen. If I now start to increase the drag force, make this hat larger, larger, and larger, introducing more and more drag, what is going to happen?

OK, so now, I consider the second situation, omega 0 squared equal to gamma squared over 4. OK, so when the gamma is very small, what we see is that this is actually underdamped, right? So the damping is really small. But if I increase the gamma to a critical value, now omega 0 squared happens to be equal to gamma squared over 4, OK? I call this a critically damped oscillator, OK?

So what does that mean? That means omega is equal to 0, you see? This is our definition of omega, right? If omega 0 squared is equal to gamma squared of over 4, then omega is equal to 0. That is actually the critical moment the system stops oscillating, OK? So it is not oscillating anymore. So now, I can actually start from the solution I obtained from 1, OK? Then, I can actually now make use of these two solutions, the theta plus and the theta minus.

Theta plus t would be equal to exponential minus gamma over 2 t cosine omega t. When omega goes to 0, what is going to happen is that this is actually becoming, which value? Anybody know? If omega is 0, what is going to happen? 1, yeah. OK, 1, right? So that will give me exponential minus gamma over 2 t.

Theta minus t-- OK, I can do the same trick and see what will happen. So I take theta minus t, which is actually obtained from the exercise number one when we discussed the underdamped system. Then, you actually get exponential minus gamma over 2 t sine omega t. When omega goes to 0, actually then I get 0 this time, OK? So that doesn't really work, right? Because if I have a solution which is 0, then it's not describing anything, right? I can always add 0 to the solution. But that doesn't help you.

OK, so instead of taking the limit of this function, actually we choose to actually do theta minus t divided by omega. And then, we actually make this omega approaching 0. Then basically, I get exponential minus gamma over 2 t sine omega t divided by omega, OK? If I have this omega approaching to 0, then this is actually roughly just exponential minus gamma over 2 t omega t over omega. And this is actually giving you t times exponential minus gamma over 2 t. Any questions so far? Yes.

AUDIENCE: Completely unrelated, but is that a negative sign in front of the theta minus negative 1/2?

YEN-JIE LEE: This one?

AUDIENCE: Yeah.

YEN-JIE LEE: Yeah. So actually, OK, yeah.

AUDIENCE: In front of the 1/2 is that a negative sign?

YEN-JIE LEE: Yes, this is a negative sign. OK, any other questions?

OK, so you can see that now I arrive at two solutions. One is actually proportional to exponential minus gamma over 2 t. The other one is actually proportional to t times exponential minus gamma over 2 t, OK? So you can see that the cosine or sine term disappeared, right? So that means you are never oscillating, OK? So this is actually what we see in this slide, this so-called critically damped, OK? When actually, omega 0 squared is equal to gamma squared over 4. And you can see that what is going to happen is that this mass or this rod is going to pass 0 only one time at most, OK?

And it could actually never passed 0, if you actually set up the initial condition correctly, OK? So one thing which I can do is I really shoot this mass really, really, very forcefully, so that I have a very large initial velocity. And what it actually is going to do, like the right-hand side diagram, is that, oh, you overshoot the 0 a bit. Then, it goes back almost exponentially, OK? So at most, you can only pass through 0 one time, if you do this kind of initial condition, OK?

So that is actually pretty interesting. And there are practical applications of this solution, actually. So for example, we have the door closed. So it's also here, right? The door closed, you would like to have the door go back to the original closed mold, the position of equilibrium position actually really fast, OK? So what you can do is really design this door close so that it actually matches with the critical dampness situation, of your condition, so that actually you would go back to 0 really quick, OK? Any questions?

OK, so now, what we could do is that, instead of having a very small drag force, or we'll a slightly larger drag force, so that actually reach the critically damped situation, what we could do is that we put the whole system into water, right? Then, the drag force will be very big, OK? And we would like to see what is going to happen, OK? So in this case is the third situation. The third situation is that omega 0 squared is actually smaller than gamma squared over 4. So you have huge drag force, OK? So that would give you a situation which is called overdamped oscillator.

Now, I have, again, alpha is equal to i gamma over 2 plus/minus square root of omega 0 squared minus gamma squared over 4, right? I'm just copying from here, OK? And that will be equal to i-- I can take out the i, OK?-- gamma over 2 plus/minus square root of gamma squared over 4 minus omega 0 squared. Now, I can actually define gamma plus/minus equal to gamma over 2 plus/minus square root of gamma squared over 4 minus omega 0 squared, OK?

Then basically, the solution-- actually now, I already have the solution. So basically, the two solutions would be looking like this. Theta of t would be equal to A plus some kind of constant exponential minus gamma plus t plus A minus exponential minus gamma minus t, OK? Because this is actually becoming already-- OK, so alpha is actually i times gamma plus/minus. Therefore, if you put it back into this, then basically what you are getting is exponential minus gamma plus t or exponential minus gamma minus t, OK? So that's already a real function. And the linear combination of these two solutions is our final, full solution to the equation of motion.

OK, again, what we are going to see is that actually the drag force is huge. I just throw the whole system into water. And the water is really trying to stop the oscillation, really very much. Therefore, you can see that, huh, again, I don't have any oscillation, OK? If I am very, very strong, I really start the initial velocity or initial angular velocity really high, I actually give a huge amount of energy into the system, then, at most again, I can actually have the system to pass through the equilibrium position only one time. Then, this whole system will slowly recover, because exponential function we see here. The amplitude is going to be decaying exponentially, OK? Any questions?

So let's actually do a quick demonstration here, OK? So here, this is actually the original little ball here, a metal one, which actually you can see that this is really going to go back and forth really nicely. And you can see that, because of the friction, actually the amplitude is becoming smaller and smaller, OK? So that actually matches with this situation, right? So it's actually an underdamped situation. This ball, in an idealized situation, is going to go through 0 infinite number of times, OK?

So now, what I am going to do is now I change this ball to something which is different, OK? This is actually made of magnets, OK? And let's see what is going to happen. So now, you can see that, because this is actually made of magnets, therefore, the drag force will be colossal, will be very, very big. And let's see what will happen. You see that the drag force is huge. Therefore, you see I put it here so that it has big initial velocity. It only passes through 0 once, right?

Of course, it now is actually approaching the zero really, really slowly, exponentially. But it is not 0, OK? So it only passes through the 0 if you believe the math, only once, OK? Just to show that this is a real deal-- OK, now, whoa, right? Oh, I'm not trying to destroy the classroom, OK? So you can actually play with this after we finish your lecture, OK?

I would like to ask you a question. After we learned this from this lecture, there are three situations, underdamped, critically damped, and overdamped, OK? I would like to ask you two questions. The first one is through this demonstration, OK? So, now I have a system which is nicely constructed. I hope you can see it, OK? You can see it.

And this system is made of a torsional spring. And also, there's a pad here, which is connected to the spring, OK? If I actually perturb this thing, it's going to be oscillating back and forth before I turn on the power, so that the lower part is actually you have a magnet, OK? It's not turned on yet, OK? And this magnet is going to provide a drag force to actually change the behavior of the system, OK? So you can see that, before I turn on the magnetic field, the whole system is actually oscillating back and forth really nicely. As we predicted, small amplitude vibration is harmonic oscillation, OK? So that's very nice.

So now, what am I going to do is to turn on the power and see what is going to happen. After I turn on the power, there's an electric field, OK? And this is actually going to be-- OK, so the magnetic field is actually turned down. Therefore, it is actually acting like a drag force to this system, OK? So let's actually see what is going to happen. Now, I release this. The behavior of the system looks like this. It first oscillates, and then it stops.

So the question is, is this a critically damped, underdamped, or overdamped system? Anybody knows? Yeah?

AUDIENCE: Underdamped.

YEN-JIE LEE: Yes, this is underdamped. How do I see that? That is because, when I do this experiment, you would pass through 0s multiple times. Therefore, there are oscillations coming into play. Therefore, I can conclude that the drag force is not large enough. So that is actually an underdamped situation, OK? And the next time, we are going to drag this system.

I have a second question for you. So now, your friends know that you took 8.03. Therefore, they will wonder if you can actually design a car suspension system, to see if you can actually make this design for them. When you design this car, which condition will you consider to set up the car? Will you set it up as underdamped, critically damped, or overdamped? How many of you actually think it should be underdamped? No, nobody? How many of you actually think it should be overdamped? 1, 2, 3, 4, OK. How many of you actually think it should be critically damped? OK, the majority of you think that should be the correct design.

So if you have the car designed as an underdamped situation, then, when you drive the car, you are going to have very funny style. You are going to have this. This is the style. So the car is going to be oscillating all the time, OK? Because it's going to be there. And it's really damping really slowly, OK? If you design it to be overdamped, it would become very bumpy, right? So let's take a limit of infinitely large drag force constant, OK? Then, it's like, when you hit some bump, you go woo! Wow! It doesn't really help you to reduce the amplitude, OK? So the correct answer is you would give the advice that you would do it critically damped, OK?

So before we end the section today, I would like to pose a question to you. The thing which we have learned from simple harmonic motion is that the energy is conserved in a simple harmonic motion, OK? I have the Fs, the spring force, proportional to minus k times x. And the energy is conserved, OK? But if I add a drag force in the form or minus b times v, energy is not conserved, right? So you can see that it was actually oscillating. Now, it's not oscillating, right? This thing has stopped oscillating, OK? Why is that the case mathematically?

OK, we know what is happening physically in this physical system. Because OK, this Mexican hat is trying to push the air away. So what is going to happen is that it's transferring the energy from this system to the molecules of the air, OK? So it's accelerating the air. So the energy goes away. But why the mathematical form looks so similar and it does different things? And think about it. And I'm not going to talk about the answer today. And thank you very much. And we will continue next time to see what we can learn if I start to drive the oscillator. Bye-bye.