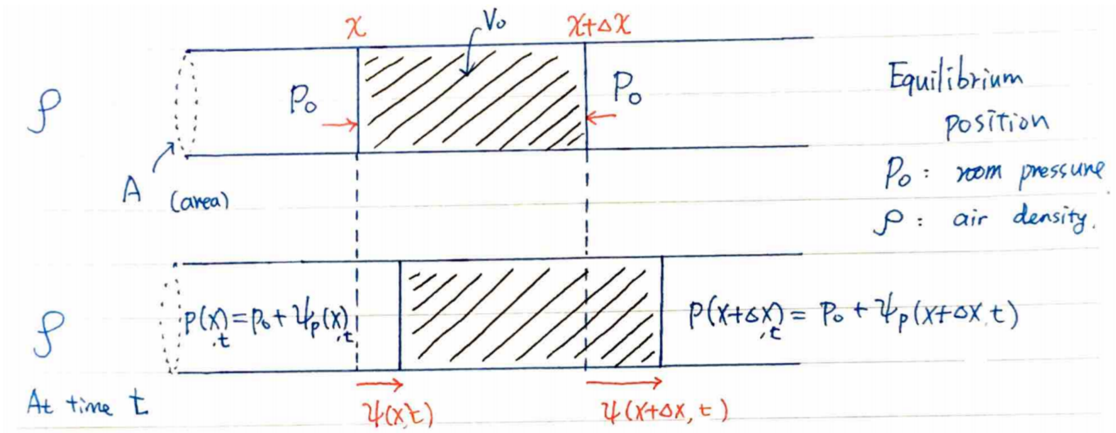
8.03 Lecture 11

We have discussed the motion of a massive string extensively. This time we will give more examples which can be described by the wave equation. "Longitudinal waves":



Change in volume (ΔV):

$$A(\psi(x+\Delta x,t)-\psi(x,t)) \approx A\frac{\partial \psi}{\partial x}\Delta x$$

Pressure difference (ΔP):

$$-\psi_P(x+\Delta x,t)+\psi_P(x,t) \approx -\frac{\partial \psi_P}{\partial x}\Delta x$$

*Question: How we do we relate pressure and volume?
(1) Ideal gas law

$$PV = nRT \quad \Rightarrow \quad V \propto P^{-1}$$

Not quite right because this assume T is constant, which is not true in a sound wave.
(2) The compression is actually adiabatic meaning that there is almost no heat flowing in or out of the volume. The time scale of the heat flow is longer than the scale of the oscillation

$$PV^{\gamma} = \text{ constant}$$

Consider: small vibration

$$\psi_P << P_0 \text{ and } \Delta V << V_0$$

Where ψ_P is the change in pressure with respect to P_0
Before:

$$P_0 V_0^{\gamma} = C$$

After:

$$(P_0+\Delta P)(V_0+\Delta V)^{\gamma} = C$$

$$C = (P_0 \psi_P) V_0^\gamma \left(1 + \frac{\Delta V}{V_0}\right)^\gamma$$
$$C \approx (P_0 \psi_P) V_0^\gamma \left(1 + \frac{\gamma \Delta V}{V_0}\right)$$
$$C \approx P_0 V_0^\gamma + \gamma \Delta V V_0^{\gamma-1} P_0 + \psi_P V_0^\gamma + \gamma \Delta V \psi_P V_0^{\gamma-1}$$

Where we ignore the last term because it is small. Because the first term is also equal to C, the two middle terms should add to zero; rearranging:

$$\psi_P = \frac{-\gamma P_0}{V_0} \Delta V$$

Plug in the expression we got before for

$$\psi_P = \frac{-\gamma P_0 A \Delta x}{V_0} \frac{\partial \psi}{\partial x}$$

Because $\Delta V = A \Delta x \frac{\partial \psi}{\partial x}$ And we get

$$\psi_P = -\gamma P_0 \frac{\partial \psi}{\partial x}$$

Now we know how to related the pressure change ψ_P and the displacement ψ
$\psi(x,t)$: displacement of the air with respect to the equilibrium position x
$\psi_P(x,t)$: “displacement” or change in pressure with respect to the room pressure, P_0
Force acting on this volume of air:

$$F_{total} = \Delta P \cdot A = -A \frac{\partial \psi_P}{\partial x} \Delta x$$

Where we have used our expression for the change in pressure ΔP from page 1.
Mass: $\Delta m = \rho \cdot A \cdot \Delta x$
*Newton’s law $F = ma$

$$\rho A \Delta x \ddot{\psi} = -A \Delta x \frac{\partial \psi_P}{\partial x}$$
$$\rho \ddot{\psi} = \frac{\partial \psi_P}{\partial x}$$
$$= \gamma P_0 \frac{\partial^2 \psi}{\partial x^2}$$
$$\Rightarrow \quad \ddot{\psi}(x,t) = \frac{\gamma P_0}{\rho} \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

This is the wave equation with velocity

$$v_p = \sqrt{\frac{\gamma P_0}{\rho}}$$

Adiabatic index: γ
*First law of thermodynamics:

$$dU + \delta W = \delta Q$$

Where U is the internal energy, W is the work done by the system and Q is the heat supplied to the system. For an adiabatic process we have

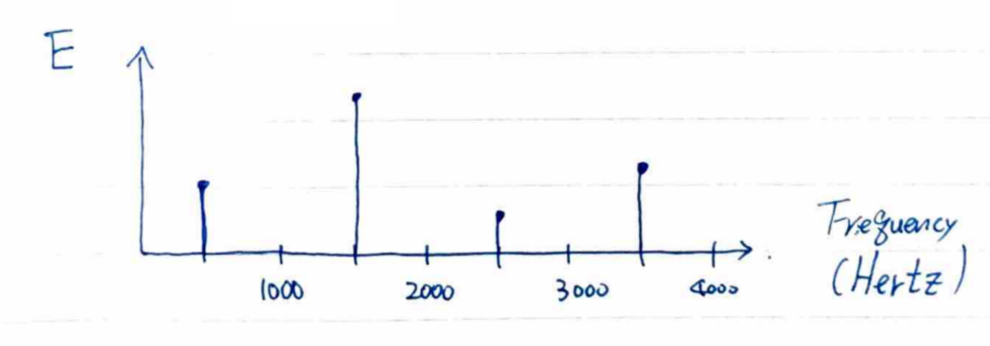
$$
\begin{aligned}
& dU + \delta W = 0 \\
& \delta W = PdV \\
& U = \alpha nRT \\
& \;= \alpha PV \\
& dU = \alpha(VdP + PdV) = -\delta W = -PdV \\
& (\alpha+1)PdV = -\alpha VdP \\
& \frac{dP}{P} = -\left(\frac{\alpha+1}{\alpha}\right)\frac{dV}{V} = -\gamma\frac{dV}{V} \quad \gamma \equiv \frac{\alpha+1}{\alpha} \\
& PV^{\gamma} = \text{ constant}
\end{aligned}
$$

Where α is defined as the the number of degrees of freedom divided by 2. For a monatomic gas (which has 3 translational degrees of freedom) $\alpha = 3/2$ and $\gamma = 5/3$. For a diatomic gas (which as 3 translational and 2 rotational degrees of freedom) $\alpha = 5/2$ and $\gamma = 7/5$

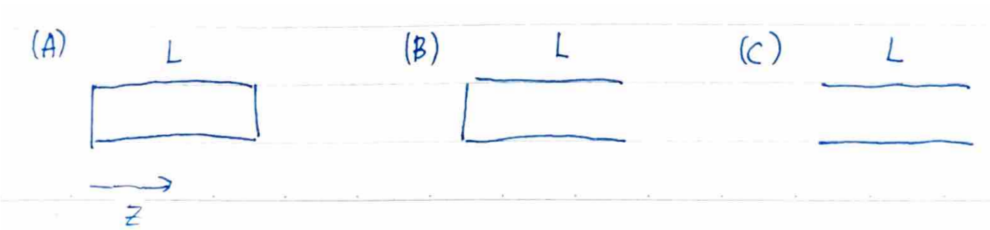
Air at sea level: $P_0 \approx 10^5\ kg/ms^2$
Air density: $\rho = 1.2\ kg/m^3$
$\Rightarrow$ speed of sound: $v_p = 342\ m/s$
Experimentally: $v_p = 343\ m/s$ at 70° F $\longrightarrow$ very nice agreement!
What have we learned?

1. The speed of sound increases if we use a monatomic gas to replace a diatomic gas (air) because γ increases. If the wavelength is fixed, there is a higher frequency.
2. The fact that they are described by wave equation:
   (a) $\omega = v_p k$
   (b) Normal modes:
   $$\psi(x) = \sum_{m=1}^{\infty} A_m \sin(k_m x + \alpha_m)\sin(\omega_m t + \beta_m)$$
   (c) k_m, α_m: determined by boundary conditions
   (d) A_m, β_m: determined by initial conditions

Example: an audio analyzer recorded the following energy versus frequency:



(i) Which configuration gave rise to this power spectrum?



Boundary condition for a closed end: $\psi = 0$ air can go nowhere :)
Boundary condition for an open end:

$$\frac{\partial \psi}{\partial z} = 0$$

Pressure has to be equal to the room pressure

Recall

$$\psi(x) = A_m \sin(k_m x + \alpha_m)\sin(\omega_m t + \beta_m)$$

(A):

$$\psi(0) = 0 \quad , \quad \psi(L) = 0$$

From the boundary conditions we get:

$$\begin{aligned} &\sin(\alpha_m) = 0 \\ \Rightarrow \ \ &\alpha_m = 0 \\ &\sin(k_m L) = 0 \\ \Rightarrow \ \ &k_m = \frac{m\pi}{L} \\ \Rightarrow \ \ &\omega_m = \frac{m\pi v}{L} \end{aligned}$$

(B):

$$\psi(0) = 0 \quad , \quad \frac{\partial \psi(L)}{\partial z} = 0$$

Again we get $\alpha_m = 0$ and from the second condition:

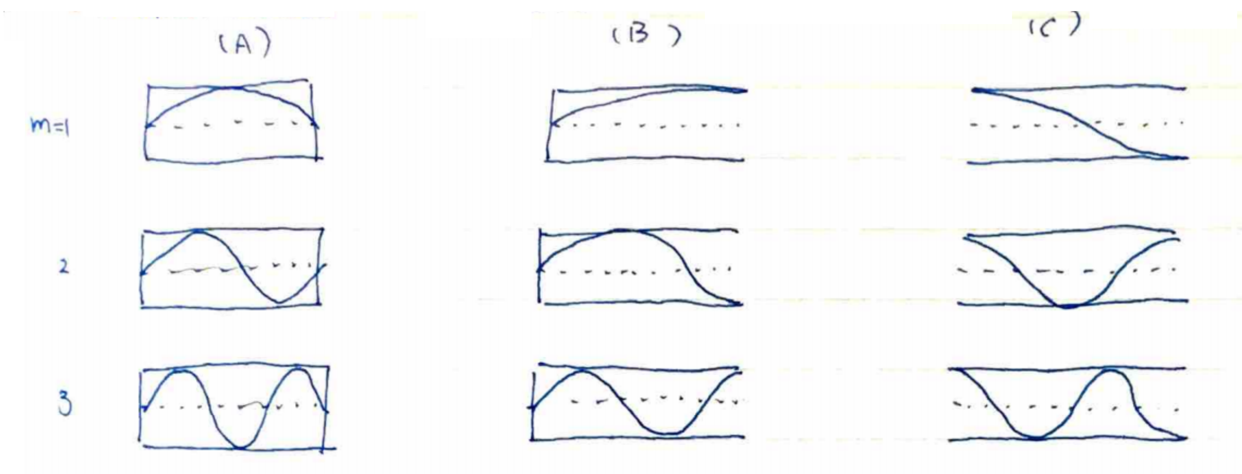
$$\cos(k_m L) = 0$$
$$\Rightarrow \ k_m = \frac{(m-1/2)\pi}{L}$$
$$\Rightarrow \ \omega_m = \frac{(m-1/2)\pi v}{L}$$

(C):

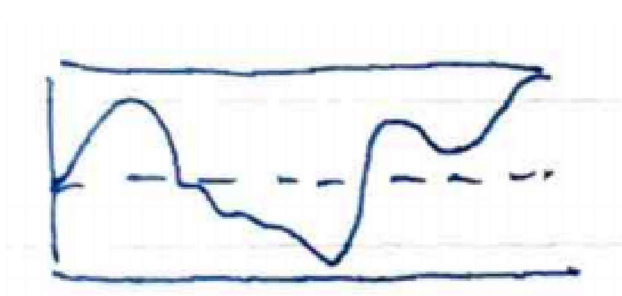
$$\frac{\partial\psi(0)}{\partial z} = 0 \ \ , \ \ \frac{\partial\psi(L)}{\partial z} = 0$$

$$\cos(\alpha_m) = 0$$
$$\Rightarrow \ \alpha_m = \frac{\pi}{2}$$
$$\sin(k_m L + \pi/2) = 0$$
$$\Rightarrow \ k_m = \frac{m\pi}{L}$$
$$\Rightarrow \ \omega_m = \frac{m\pi v}{L}$$

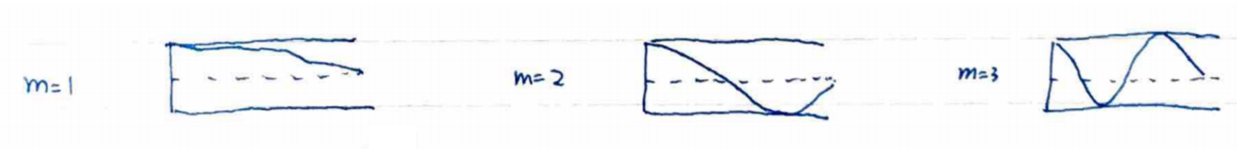
(ii) Normal modes: Amplitude:



(iii) How long does it take for this (or any arbitrary) amplitude to reappear? $\Rightarrow 2\pi/\omega_1$

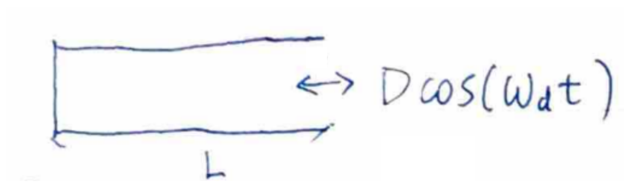


(iv) What about pressure? In each normal mode?



$$\psi_P = \gamma P_0 \frac{\partial \psi}{\partial x}$$

(v) Drive the organ



$$\psi(0) = 0 \quad , \quad \psi(L) = D\cos(\omega_d t)$$

$$k_d = \frac{\omega_d}{v_p} \text{ decided by the dispersion relation } \omega = v \cdot k$$

$$\psi(x) = A_d \sin(k_d x + \alpha)\cos(\omega_d t)$$

$$\begin{aligned} &\psi(0) = 0 \;\Rightarrow\; \alpha = 0 \\ &\psi(L) = D\cos(\omega_d t) \\ &\Rightarrow\; A_d \sin(k_d L) = D \\ &A_d = \frac{D}{\sin(k_d L)} \end{aligned}$$

$$\psi(x) = \frac{D}{\sin(k_d L)} \sin(k_d x + \alpha)\cos(\omega_d t)$$

When $k_d = (m - 1/2)\pi/L \;\Rightarrow\;$ resonance! (Huge amplitude!)

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