8.022 (E&M) – Lecture 21

Topics:

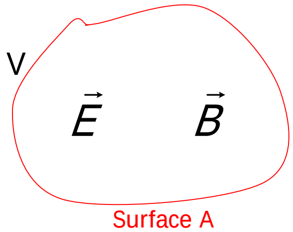
- Energy and momentum carried by EM waves
  - Poynting vector
- Transmission lines
- Scattering of light and sunset demo…

Last time

- Solution of Maxwell's equations in vacuum $\vec{\nabla}^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$
  - Solution of wave equation $f(\vec{r} \pm c\hat{k}t)$ can be expressed as linear combination of plane waves:
  - Properties of plane waves: $\vec{E} = \vec{E}_0 \sin(\vec{k}\bullet\vec{r} - \omega t); \vec{B} = \vec{B}_0 \sin(\vec{k}\bullet\vec{r} - \omega t)$
    - They travel at the speed of light // to k (wave vector)
    - E, B and k are always perpendicular to each other
    - Amplitude of E and B are the same in cgs
  - Polarization of EM waves
    - Linear: when the direction of E_0 is constant in time
    - Circular: when the vector E_0 describes a circle over time
    - Elliptical: all the situations in between these 2 cases
- Today we will complete the study of these properties…

EM Energy

- EM radiation carries energy
  - Obvious if you think about the fact that is the light from the sun that keeps us warm…
- How does this energy propagate?
  - Consider a volume V of surface A containing E and B



Energy density: $u = \dfrac{energy}{volume} = \dfrac{1}{8\pi}(\vec{E}\bullet\vec{E} + \vec{B}\bullet\vec{B})$

Total energy: $U = \int_V u\,dV = \dfrac{1}{8\pi}\int_V (\vec{E}\bullet\vec{E} + \vec{B}\bullet\vec{B})dV$

The Poynting vector

- How does total derivative change over time?

$$\frac{\partial U}{\partial t} = \frac{1}{8\pi}\frac{\partial}{\partial t}\int_V (\vec{E}\bullet\vec{E} + \vec{B}\bullet\vec{B})dV = \frac{1}{4\pi}\int_V \left(\frac{\partial \vec{E}}{\partial t}\bullet\vec{E} + \frac{\partial \vec{B}}{\partial t}\bullet\vec{B}\right)dV$$

Remembering that in vacuum: $\vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla}\times\vec{B} = \dfrac{1}{c}\dfrac{\partial \vec{E}}{\partial t}$

$$\Rightarrow \frac{\partial U}{\partial t} = \frac{c}{4\pi}\int_V (\vec{\nabla}\times\vec{B}\bullet\vec{E} - \vec{\nabla}\times\vec{E}\bullet\vec{B})dV$$

Remembering that $\vec{\nabla}\bullet(\vec{E}\times\vec{B}) = -\vec{E}\bullet(\vec{\nabla}\times\vec{B}) + \vec{B}\bullet(\vec{\nabla}\times\vec{E})$

$$\Rightarrow \frac{\partial U}{\partial t} = -\frac{c}{4\pi}\int_V \vec{\nabla}\bullet\left(\vec{B}\times\vec{E}\right)dV \equiv -\int_V \vec{\nabla}\bullet\vec{S}\;dV$$

where we defined the Poynting vector as $\vec{S} \equiv \dfrac{c}{4\pi}\vec{B}\times\vec{E}$

Interpretation of Poynting vector

- Given: $\frac{\partial U}{\partial t} = -\int_V \vec{\nabla} \bullet \vec{S}\ dV \overset{Stokes}{\rightarrow} \frac{\partial U}{\partial t} = -\int_A \vec{S} \bullet d\vec{a} = -\Phi_{\vec{S}}(A)$

→ The rate of change of EM energy in the volume V is given by the flux of the Poynting vector S through the surface A

  - Minus sign: dA points outward → U increases when S is opposite to dA
- Interpretation of Poynting vector:
  - $\vec{S} \equiv \frac{c}{4\pi} \vec{B} \times \vec{E}$ points in the direction of the EM energy flow
    - Remember that $\vec{E}_0 \times \vec{B}_0 = \left|\vec{E}_0\right|^2 \hat{k}$
  - The flux of S through a surface gives the power through A

Power through A: $\int_A \vec{S} \bullet d\vec{a}$

Poynting vector: dimensional analysis

- What are the units of the Poynting vector?

$$\left[\vec{S}\right] = \left[\frac{c}{4\pi}\vec{E} \times \vec{B}\right] = [c][B][E] \overset{cgs}{=} [c][E]^2$$

$$[c] = \frac{Lenght}{Time}$$

From $u = \frac{1}{8\pi}(\vec{E} \bullet \vec{E} + \vec{B} \bullet \vec{B}) \Rightarrow [E]^2 = \frac{Energy}{Volume}$

$$\Rightarrow \left[\vec{S}\right] = \frac{Lenght}{Time}\frac{Energy}{Volume} = \frac{Energy}{Time\ \ Area} = \frac{Power}{Area}$$

- Expected if the flux of S is the power through area A
- In cgs: $[S]=\text{erg s}^{-1}\text{ cm}^{-2}$
- NB: Magnitude of S is know as Intensity I
  - Intense source of radiations emit a lot of power per unit area

Applications: plane waves

- Consider a linearly polarized plane wave: $\begin{cases} \vec{E} = E_0 \cos(kz - \omega t)\hat{x} \\ \vec{B} = B_0 \cos(kz - \omega t)\hat{y} \end{cases}$
- Poynting vector associated with it:

$$\vec{S} \equiv \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi} E_0^2 \sin^2(kz - wt)\hat{k}$$

- This can be compared to the energy density of the wave:

$$u = \frac{1}{8\pi}(\vec{E}\bullet\vec{E} + \vec{B}\bullet\vec{B}) = \frac{1}{4\pi}E_0^{\,2}\sin^2(kz - \omega t)$$

$$\Rightarrow \boxed{\vec{S} = u\vec{c} = uc\hat{k}}$$

- This is similar to $\vec{J} = \rho\vec{v}$

→ another way to show that S tells us about the flow of energy!

- Usually the oscillation is very fast (e.g.: visible~10^{14} Hz) → all that matters is the average energy density <S> and intensity <I>:

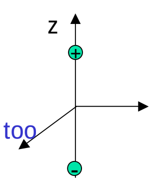
$$\langle \vec{S} \rangle = \frac{c\hat{k}}{8\pi}E_0^{\,2}; \qquad \langle I \rangle = \frac{c}{8\pi}E_0^{\,2}$$

Application 2: Dipole radiation

- Radiation emitted by a dipole oriented along the z axis in spherical coordinates:

$$\begin{cases} \vec{E} = \dfrac{\omega^2 p}{c^2}\sin\theta \dfrac{\sin(kr - \omega t)}{r}\hat{\theta} \\ \vec{B} = \dfrac{\omega^2 p}{c^2}\sin\theta \dfrac{\sin(kr - \omega t)}{r}\hat{\phi} \end{cases}$$

This is 8.07 stuff:
just trust me for the moment



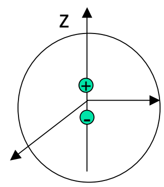
  - NB: this solution is only valid for $r >> \lambda = 2\pi/k$
- This is the Radiation propagates radially, some angular dependence too
- Poynting vector: $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{1}{4\pi c^3}\omega^4 p^2 \sin^2\theta \frac{\sin^2(kr - \omega t)}{r^2}\hat{r}$

$$\Rightarrow \langle \vec{S} \rangle = \frac{\omega^4 p^2}{8\pi r^2 c^3}\sin^2\theta\hat{r}$$

- NB: Poynting vector (and I) falls as $1/r^2$: this should be intuitive. Why?

Dipole radiation: cont.

- Draw a sphere of radius R around the dipole centered in origin:
  - NB: R >> d
- Compute power radiated through the sphere:



$$\left\langle \frac{\partial U}{\partial t} \right\rangle = \int_R \left\langle \vec{S} \right\rangle \cdot d\vec{a} = \int_R \frac{\omega^4 p^2}{8\pi r^2 c^3} \sin^2\theta \hat{r} \cdot d\vec{a}$$

Since $d\vec{a} = R^2 \sin\theta d\phi \hat{r}$:

$$\left\langle \frac{\partial U}{\partial t} \right\rangle = \frac{\omega^4 p^2}{8\pi R^2 c^3} R^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta d\theta$$

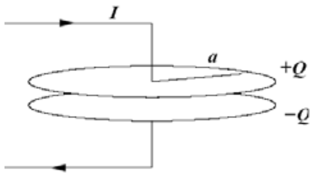
Since $\int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3} \Rightarrow \left\langle \frac{\partial U}{\partial t} \right\rangle = \frac{\omega^4 p^2}{3c^3}$ (Larmor formula)

- NB: power through sphere does not depend on R
  - Why? S falls as $1/r^2$, area increases as r^2

→ Power through S (flux through S) is constant: Energy is conserved

Application 3: capacitor

- The Poynting vector applies to ANY situation in which both E and B appear, not just when we have radiation
- Example: charging capacitor



$$\vec{E} = -\frac{4\pi Q}{A}\hat{z} = -\frac{4Q}{a^2}\hat{z}$$

From generalized Ampere law: $\vec{B}(r) = \frac{2Ir}{ca^2}\hat{\phi}$

- Calculate Poynting vector:

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi}\frac{4Q}{a^2}\frac{2Ir}{ca^2}\hat{z} \times \hat{\phi} = \frac{2IQr}{\pi a^4}(-\hat{r})$$

- NB: what is important here is the direction of S:
  - S points into the center of the capacitor as it should: the plates are charging up!

Momentum carried by EM wave

- Since EM carry energy it's not surprising that they carry momentum as well
- In relativity, E and p are related by $E^2 = |\vec{p}^2| c^2 + m^2 c^4$
- For EM radiation, m=0:

$$E^2 = |\vec{p}^2| c^2 \Rightarrow \boxed{p = \frac{U}{c}}$$

- Remember that

$$\vec{S} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Energy}}{\text{Time Area}} \Rightarrow \frac{\vec{S}}{c} = \frac{\text{Energy/c}}{\text{Time Area}} = \frac{\text{Momentum}}{\text{Time Area}}$$

- Dimensional analysis will also tell us that:

$$\frac{\vec{S}}{c} = \frac{\text{Momentum}}{\text{Time Area}} = \frac{\text{Force}}{\text{Area}} = \textit{Pressure}$$

→ Radiation exerts pressure

Demo

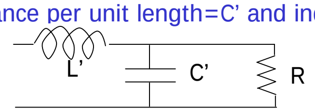
Summary on Poynting vector

- Energy flux: Energy / area / unit time
- Energy density u: Energy / unit volume
- Momentum flux: Momentum / area / unit time
- Momentum density: Momentum/ unit volume

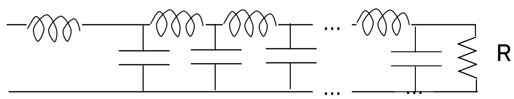
	Energy	Momentum
Flux X/(Area sec)	$\vec{S} \equiv \frac{c}{4\pi} \vec{B} \times \vec{E}$	$\frac{\vec{S}}{c}$ (same as pressure)
Density X/Volume	$\frac{\|\vec{S}\|}{c}$	$\frac{\|\vec{S}\|}{c^2}$

Transmission line

- Transmission line = a pair of (twisted) cables used to transmit a signal
  - Current flows in one direction on one cable and comes back on the other cable
- If terminated correctly, Z is purely real: $Z \sim R_{termination}$
- Find R when capacitance per unit length=C' and inductance per unit length=L'
  - In theory:



  - In practice infinite sum of infinitesimal elements C and L:



  - Calculate Z of the last piece and impose that it's purely real.

$$Z_{eq} = i + \left(\frac{1}{R} + i\omega C\right)^{-1} = i\omega L + \frac{R}{1+i\omega RC} = \frac{i\omega L' - \omega^2 RL'C' + R}{1+i\omega RC'} \overset{impose}{=} R$$

$i\omega L - \omega^2 LCR + R = R + i\omega CR^2$. Ignoring term with LC (small): $\Rightarrow$ $R=\sqrt{\frac{L}{C}}=\sqrt{\frac{L'}{C'}}$

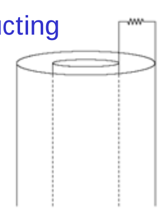
Transmission line (2)

- What happens when transmission line is terminated correctly?
- Z is purely real: $Z \sim R_{termination}$ → Z is a constant of the cable:
  - Z does not depend on how long the cable is!
- If $R \neq \sqrt{L'/C}$:

  → Z will depend on how long the cable is and on the frequency of the signal

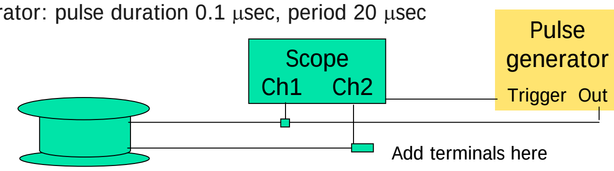
  → Distortions of the signal!
- Example of transmission line: coaxial cable, a pair of conducting tubes nested in one another
  - Homework: prove that for a cylindrical coaxial cable

    Z=2 ln(b/a) /c and the velocity of propagation is c.
- Typical $R_{termination}$: 50 Ohm



Transmission line: demo

- Coaxial cable (127.4 m long)
- Pulse generator: pulse duration 0.1 μsec, period 20 μsec

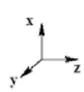


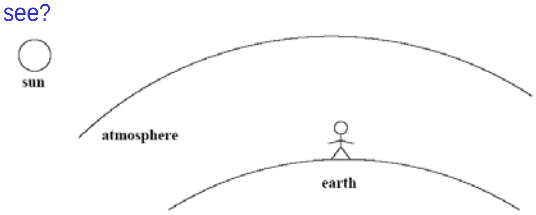
- Simultaneously send pulse from pulse generator (splitter)
  - to Ch 1 of scope
  - to transmission line (back and forth and display on Ch 2)
  - Measure speed of propagation: Time difference: 656 ns → v=L/T~2/3c
- What happens if:
  - Open: signal will bounce back but nasty reflections
  - Short: signal will be reversed on the same cable, nothing on the other cable
  - If I terminate it with 50Ω resistor: signal comes back on return cable with no reflections

Scattering of light

(Logically this topic belongs to last lecture, but we did not have time…)

- When we send light into a medium, the light is scattered in many directions
- Example: light from Sun (unpolarized) passing through atmosphere
  - Propagation of light //z
  - We look up in x direction
  - What kind of light do we see?





Scattering of light (2)

- Since light propagates //z: no polarization // z
- We measure the light (with our eyes!) along the x direction: no polarization // x

  → The light we see must be polarized along the y direction

  - This is actually not really true because the light scatters multiple times, but it suggests the general tendency
- What if the put a giant polaroid in front of the Sun?
  - Scattered light would be more intense in direction perpendicular to polarization direction
  - Rotating the polaroid would allow us to change intensity of the light:
    - Max intensity when polarization direction is // y axis
    - Dark when polarization direction is // x axis

Scattering of light (3)

- How is light scattered?
  - Light hits a molecule; the E shakes the molecule's charges with frequency w; the molecule re-radiates the light often changing the direction → changes in polarization
- Are all frequencies scattered in the same way?
  - Electric fields of scattered radiation depend on acceleration of (dipole) charges

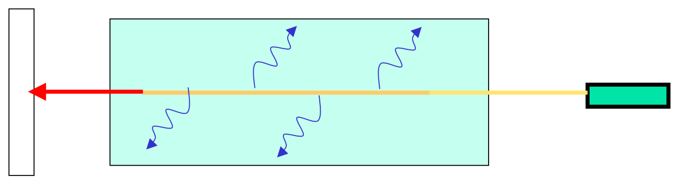
    $E_{scattered} \propto \frac{\partial^2 d}{\partial t^2} \propto \omega^2$ if dipole moment of the shaken molecule goes as d~cosωt
  - Intensity of scattered radiation: $I \propto E^2_{scattered} \propto \omega^4 \propto \lambda^{-4}$
  - Since $\lambda_{red} \sim 2\,\lambda_{blue}$ → Blue is scattered 16 times more than red
    - This explains why the sky is blue during the day and why it's red at sunset

Summary and outlook

- Today:
  - Energy and momentum carried by EM waves
    - Poynting vector and some of their applications
  - Transmission lines
  - Scattering of light
    - What happens at sunset?
- Next Thursday:
  - Magnetic fields through matter? Or review problems?

Sunset experiment

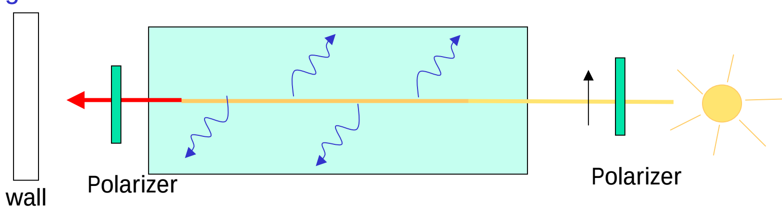
- Solution of distilled water and salt.
  - Unpolarized light is shining through it to the wall



- Add $Na_2S_2O_3 5H_2O$ (Na thiosulfate)
  - Lights starts scattering: fog; light on the wall becomes red first and then dark as all the light is scattered toward the audience (as in sunset)
  - What happened?
    - Chemical reaction creates bigger and bigger molecules that scatter more and more light. Blue light is scattered first. Red makes it for a while but eventually scatters too.
  - NB: light is polarized!

Sugar solution experiment (T8)

- Light goes through a polarizer and then through an optically active sugar solution



- The first polarizer creates a linearly polarized wave, overlap of right-handed and left-handed circularly polarized waves which propagate at different speeds in the solution. This causes linear polarization direction to change slowly. Since the effect depends on λ, different colors are rotated differently.
- The second polarizer check polarization direction at exit