Chapter 4 One Dimensional Kinematics

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In the first place, what do we mean by time and space? It turns out that these deep philosophical questions have to be analyzed very carefully in physics, and this is not easy to do. The theory of relativity shows that our ideas of space and time are not as simple as one might imagine at first sight. However, for our present purposes, for the accuracy that we need at first, we need not be very careful about defining things precisely. Perhaps you say, "That's a terrible thing—I learned that in science we have to define everything precisely." We cannot define anything precisely! If we attempt to, we get into that paralysis of thought that comes to philosophers, who sit opposite each other, one saying to the other, "You don't know what you are talking about!" The second one says. "What do you mean by know? What do you mean by talking? What do you mean by you?", and so on. In order to be able to talk constructively, we just have to agree that we are talking roughly about the same thing. You know as much about time as you need for the present, but remember that there are some subtleties that have to be discussed; we shall discuss them later.[1]

Richard Feynman

4.1 Introduction

Kinematics is the mathematical description of motion. The term is derived from the Greek word *kinema,* meaning movement. In order to quantify motion, a mathematical coordinate system, called a *reference frame*, is used to describe space and time. Once a reference frame has been chosen, we shall introduce the physical concepts of position, velocity and acceleration in a mathematically precise manner. Figure 4.1 shows a Cartesian coordinate system in one dimension with unit vector $\hat{\mathbf{i}}$ pointing in the direction of increasing x-coordinate.

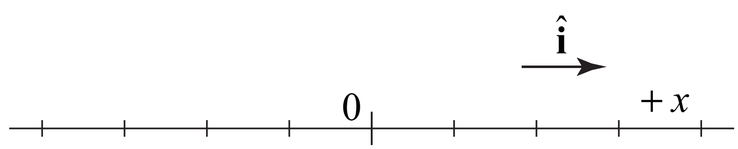

Figure 4.1 A one-dimensional Cartesian coordinate system.

[1] Richard P. Feynman, Robert B. Leighton, Matthew Sands, *The Feynman Lectures on Physics*, Addison-Wesley, Reading, Massachusetts, (1963), p. 12-2.

4.2 Position, Time Interval, Displacement

4.2.1 Position

Consider a point-like object moving in one dimension. We denote the ***position coordinate*** of the object *with respect to the choice of origin* by $x(t)$. The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. The position of the object with respect to the origin has both direction and magnitude, and hence is a vector (Figure 4.2), which we shall denote as the **position vector** (or simply position) and write as

$$\vec{\mathbf{r}}(t) = x(t)\,\hat{\mathbf{i}}\,. \qquad (4.2.1)$$

We denote the position coordinate at $t = 0$ by the symbol $x_0 \equiv x(t = 0)$. The SI unit for position is the meter [m].

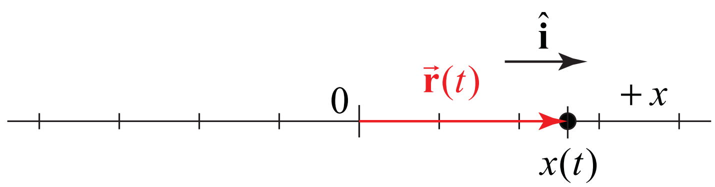

Figure 4.2 The position vector, with reference to a chosen origin.

4.2.2 Time Interval

Consider a closed interval of time $[t_1, t_2]$. We characterize this time interval by the difference in endpoints of the interval,

$$\Delta t = t_2 - t_1\,. \qquad (4.2.2)$$

The SI units for time intervals are seconds [s].

4.2.3 Displacement

The ***displacement*** of a body during a time interval $[t_1, t_2]$ (Figure 4.3) is defined to be the change in the position of the body

$$\Delta\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}(t_2) - \vec{\mathbf{r}}(t_1) = (x(t_2) - x(t_1))\,\hat{\mathbf{i}} \equiv \Delta x(t)\,\hat{\mathbf{i}}\,. \qquad (4.2.3)$$

Displacement is a vector quantity.

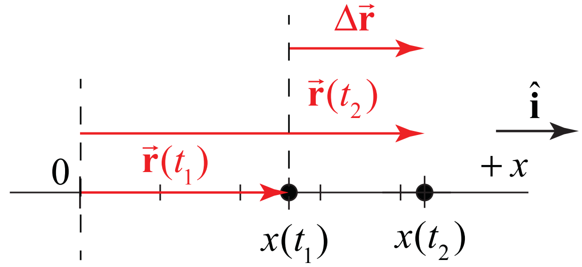

Figure 4.3 The displacement vector of an object over a time interval is the vector difference between the two position vectors

4.3 Velocity

When describing the motion of objects, words like "speed" and "velocity" are used in natural language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

4.3.1 Average Velocity

The x-component of the ***average velocity***, $v_{x,ave}$, for a time interval Δt is defined to be the displacement Δx divided by the time interval Δt,

$$v_{x,ave} \equiv \frac{\Delta x}{\Delta t}. \qquad (4.3.1)$$

Because we are describing one-dimensional motion we shall drop the subscript x and denote

$$v_{ave} = v_{x,ave}. \qquad (4.3.2)$$

When we introduce two-dimensional motion we will distinguish the components of the velocity by subscripts. The average velocity vector is then

$$\vec{\mathbf{v}}_{ave} \equiv \frac{\Delta x}{\Delta t}\hat{\mathbf{i}} = v_{ave}\,\hat{\mathbf{i}}. \qquad (4.3.3)$$

The SI units for average velocity are meters per second $\left[\mathrm{m \cdot s^{-1}}\right]$. The average velocity is not necessarily equal to the distance in the time interval Δt traveled divided by the time interval Δt. For example, during a time interval, an object moves in the positive x-direction and then returns to its starting position, the displacement of the object is zero, but the distance traveled is non-zero.

4.3.3 Instantaneous Velocity

Consider a body moving in one direction. During the time interval $[t, t+\Delta t]$, the average velocity corresponds to the slope of the line connecting the points $(t, x(t))$ and $(t+\Delta t, x(t+\Delta t))$. The slope, the rise over the run, is the change in position divided by the change in time, and is given by

$$v_{ave} \equiv \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t}. \tag{4.3.4}$$

As $\Delta t \to 0$, the slope of the lines connecting the points $(t, x(t))$ and $(t+\Delta t, x(t+\Delta t))$, approach slope of the tangent line to the graph of the function $x(t)$ at the time t (Figure 4.4).

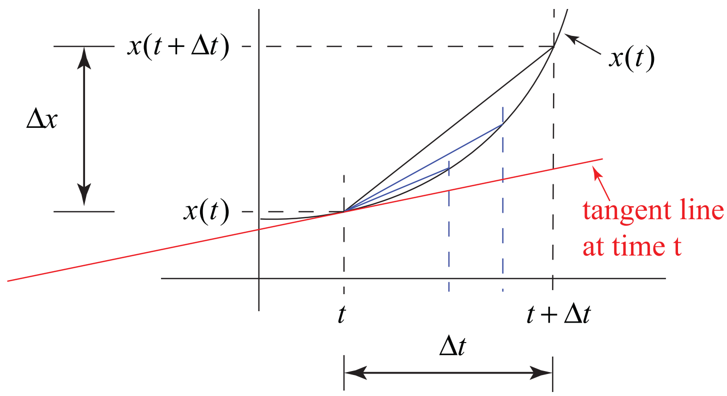

Figure 4.4 Plot of position *vs*. time showing the tangent line at time t.

The limiting value of this sequence is defined to be the x-component of the instantaneous velocity at the time t.

> The x-component of ***instantaneous velocity*** at time t is given by the slope of the tangent line to the graph of the position function at time t:

$$v(t) \equiv \lim_{\Delta t \to 0} v_{ave} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \tag{4.3.5}$$

The instantaneous velocity vector is then

$$\vec{\mathbf{v}}(t) = v(t)\,\hat{\mathbf{i}}. \tag{4.3.6}$$

The component of the velocity, $v(t)$, can be positive, zero, or negative, depending on whether the object is travelling in the positive x-direction, instantaneously at rest, or the negative x-direction.

Example 4.1 Determining Velocity from Position

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + \frac{1}{2}bt^2 \tag{4.3.7}$$

where x_0 is the initial position of the object at $t = 0$. We can explicitly calculate the x-component of instantaneous velocity from Equation (4.3.5) by first calculating the displacement in the x-direction, $\Delta x = x(t + \Delta t) - x(t)$. We need to calculate the position at time $t + \Delta t$,

$$x(t + \Delta t) = x_0 + \frac{1}{2}b(t + \Delta t)^2 = x_0 + \frac{1}{2}b(t^2 + 2t\Delta t + \Delta t^2)\,. \tag{4.3.8}$$

Then the x-component of instantaneous velocity is

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left(x_0 + \frac{1}{2}b(t^2 + 2t\,\Delta t + \Delta t^2)\right) - \left(x_0 + \frac{1}{2}bt^2\right)}{\Delta t}\,. \tag{4.3.9}$$

This expression reduces to

$$v(t) = \lim_{\Delta t \to 0} \left(bt + \frac{1}{2}b\Delta t\right). \tag{4.3.10}$$

The first term is independent of the interval Δt and the second term vanishes because in the limit as $\Delta t \to 0$, the term $(1/2)b\Delta t \to 0$ is zero. Therefore the x-component of instantaneous velocity at time t is

$$v(t) = bt\,. \tag{4.3.11}$$

In Figure 4.5 we plot the instantaneous velocity, $v(t)$, as a function of time t.

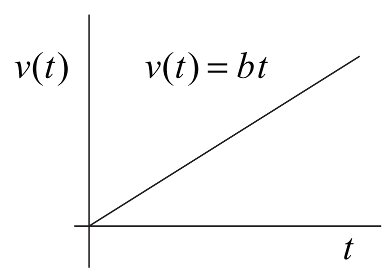

Figure 4.5 Plot of instantaneous velocity instantaneous velocity as a function of time.

Example 4.2 Mean Value Theorem

Consider an object that is moving along the x-coordinate axis with the position function given by

$$x(t) = x_0 + v_0 t + \frac{1}{2} b t^2 . \tag{4.3.12}$$

The graph of $x(t)$ vs. t is shown in Figure 4.6.

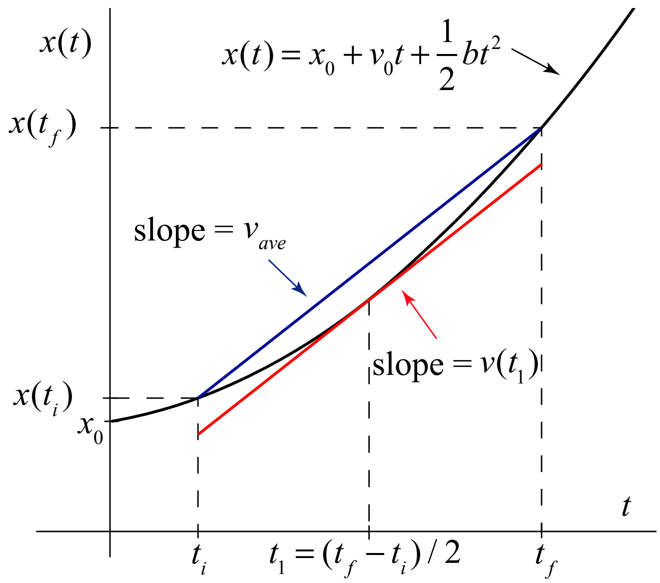

Figure 4.6 Intermediate Value Theorem

The x-component of the instantaneous velocity is

$$v(t) = \frac{dx(t)}{dt} = v_0 + bt . \tag{4.3.13}$$

For the time interval $[t_i, t_f]$, the displacement of the object is

$$x(t_f) - x(t_i) = \Delta x = v_0 (t_f - t_i) + \frac{1}{2} b (t_f{}^2 - t_i^2) = v_0 (t_f - t_i) + \frac{1}{2} b (t_f - t_i)(t_f + t_i) . \tag{4.3.14}$$

Recall that the x-component of the average velocity is defined by the condition that

$$\Delta x = v_{ave} (t_f - t_i) . \tag{4.3.15}$$

We can determine the average velocity by substituting Eq. (4.3.15) into Eq. (4.3.14) yielding

$$v_{ave} = v_0 + \frac{1}{2}b(t_f + t_i)\,. \tag{4.3.16}$$

The Mean Value Theorem from calculus states that there exists an instant in time t_1, with $t_i < t_1 < t_f$, such that the x-component of the instantaneously velocity, $v(t_1)$, satisfies

$$\Delta x = v(t_1)(t_f - t_i)\ . \tag{4.3.17}$$

Geometrically this means that the slope of the straight line (blue line in Figure 4.6) connecting the points $(t_i, x(t_i))$ to $(t_f, x(t_f))$ is equal to the slope of the tangent line (red line in Figure 4.6) to the graph of $x(t)$ vs. t at the point $(t_1, x(t_1))$ (Figure 4.6),

$$v(t_1) = v_{ave}\ . \tag{4.3.18}$$

We know from Eq. (4.3.13) that

$$v(t_1) = v_0 + bt_1\,. \tag{4.3.19}$$

We can solve for the time t_1 by substituting Eqs. (4.3.19) and (4.3.16) into Eq. (4.3.18) yielding

$$t_1 = (t_f + t_i)/2 \tag{4.3.20}$$

This intermediate value $v(t_1)$ is also equal to one-half the sum of the initial velocity and final velocity

$$v(t_1) = \frac{v(t_i) + v(t_f)}{2} = \frac{(v_0 + bt_i) + (v_0 + bt_f)}{2} = v_0 + \frac{1}{2}b(t_f + t_i) = v_0 + bt_1\ . \tag{4.3.21}$$

For any time interval, the quantity $(v(t_i) + v(t_f))/2$, is the arithmetic mean of the initial velocity and the final velocity (but unfortunately is also sometimes referred to as the average velocity). The average velocity, which we defined as $v_{ave} = (x_f - x_i)/\Delta t$, and the arithmetic mean, $(v(t_i) + v(t_f))/2$, are only equal in the special case when the velocity is a linear function in the variable t as in this example, (Eq. (4.3.13)). We shall only use the term average velocity to mean displacement divided by the time interval.

4.4 Acceleration

We shall apply the same physical and mathematical procedure for defining acceleration, as the rate of change of velocity with respect to time. We first consider how the instantaneous velocity changes over a fixed time interval of time and then take the limit as the time interval approaches zero.

4.4.1 Average Acceleration

Average acceleration is the quantity that measures a change in velocity over a particular time interval. Suppose during a time interval Δt a body undergoes a change in velocity

$$\Delta \vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t) . \tag{4.3.22}$$

The change in the x-component of the velocity, Δv, for the time interval $[t, t+\Delta t]$ is then

$$\Delta v = v(t+\Delta t) - v(t) . \tag{4.3.23}$$

The x***-component of the average acceleration*** for the time interval Δt is defined to be

$$\vec{\mathbf{a}}_{ave} = a_{ave}\,\hat{\mathbf{i}} \equiv \frac{\Delta v}{\Delta t}\,\hat{\mathbf{i}} = \frac{(v(t+\Delta t) - v(t))}{\Delta t}\,\hat{\mathbf{i}} . \tag{4.3.24}$$

The SI units for average acceleration are meters per second squared, $[\mathrm{m \cdot s^{-2}}]$.

4.4.2 Instantaneous Acceleration

Consider the graph of the x-component of velocity, $v(t)$, (Figure 4.7).

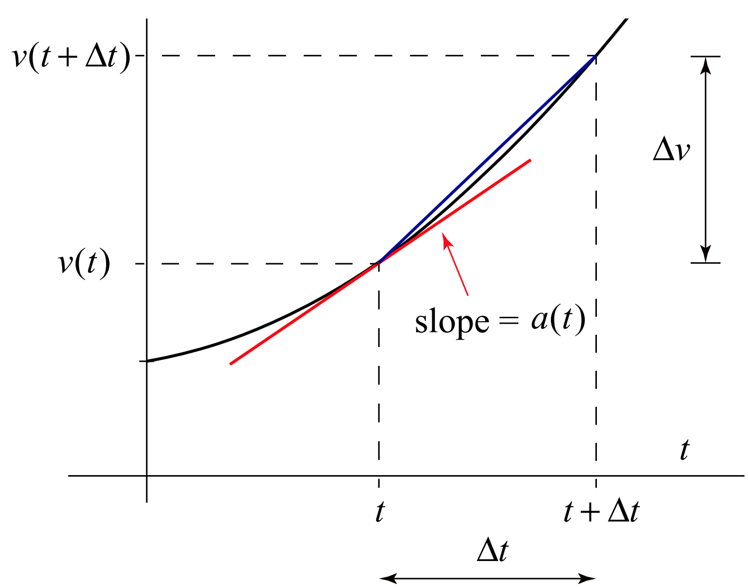

Figure 4.7 Graph of velocity *vs*. time showing the tangent line at time t.

The average acceleration for a fixed time interval Δt is the slope of the straight line connecting the two points $(t, v(t))$ and $(t+\Delta t, v(t+\Delta t))$. In order to define the x-component of the instantaneous acceleration at time t, we employ the same limiting argument as we did when we defined the instantaneous velocity in terms of the slope of the tangent line.

> The x***-component of the instantaneous acceleration*** *at time* t is the slope of the tangent line at time t of the graph of the x-component of the velocity as a function of time,

$$a(t) \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{(v(t+\Delta t) - v(t))}{\Delta t} \equiv \frac{dv}{dt}. \qquad (4.3.25)$$

The instantaneous acceleration vector at time t is then

$$\vec{\mathbf{a}}(t) = a(t)\,\hat{\mathbf{i}}. \qquad (4.3.26)$$

Because the velocity is the derivative of position with respect to time, the x-component of the acceleration is the second derivative of the position function,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \qquad (4.3.27)$$

Example 4.3 Determining Acceleration from Velocity

Let's continue Example 4.1, in which the position function for the body is given by $x = x_0 + (1/2)bt^2$, and the x-component of the velocity is $v = bt$. The x-component of the instantaneous acceleration is the first derivative (with respect to time) of the x-component of the velocity:

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{bt + b\Delta t - bt}{\Delta t} = b. \qquad (4.3.28)$$

Note that in Eq. (4.3.28), the ratio $\Delta v / \Delta t$ is independent of t, consistent with the constant slope as shown in Figure 4.5.

4.5 Constant Acceleration

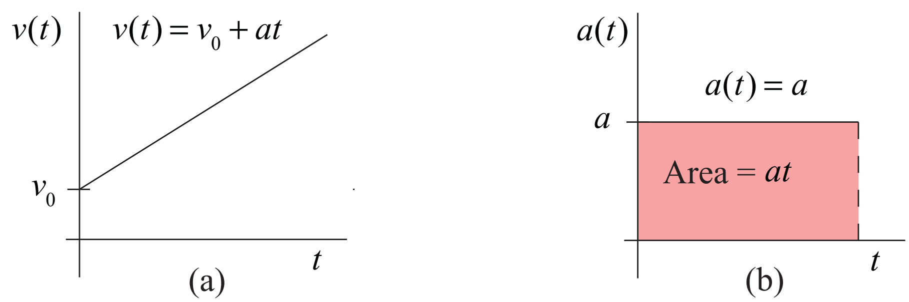

Figure 4.8 Constant acceleration: (a) velocity, (b) acceleration

When the x-component of the velocity is a linear function (Figure 4.8(a)), the average acceleration, $\Delta v / \Delta t$, is a constant and hence is equal to the instantaneous acceleration (Figure 4.8(b)). Let's consider a body undergoing constant acceleration for a time interval $[0, t]$, where $\Delta t = t$. Denote the x-component of the velocity at time $t = 0$ by $v_0 \equiv v(t = 0)$. Therefore the x-component of the acceleration is given by

$$a(t) = \frac{\Delta v}{\Delta t} = \frac{v(t) - v_0}{t} . \tag{4.4.1}$$

Thus the x-component of the velocity is a linear function of time given by

$$v(t) = v_0 + at . \tag{4.4.2}$$

4.5.1 Velocity: Area Under the Acceleration *vs.* Time Graph

In Figure 4.8(b), the area under the acceleration *vs*. time graph, for the time interval $\Delta t = t - 0 = t$, is

$$\text{Area}(a(t), t) = a\,t . \tag{4.4.3}$$

From Eq. (4.4.2), the area is the change in the x-component of the velocity for the interval $[0, t]$:

$$\text{Area}(a(t), t) = a\,t = v(t) - v_0 = \Delta v . \tag{4.4.4}$$

4.5.2 Displacement: Area Under the Velocity *vs.* Time Graph

In Figure 4.9 shows a graph of the x-component of the velocity *vs*. time for the case of constant acceleration (Eq. (4.4.2)).

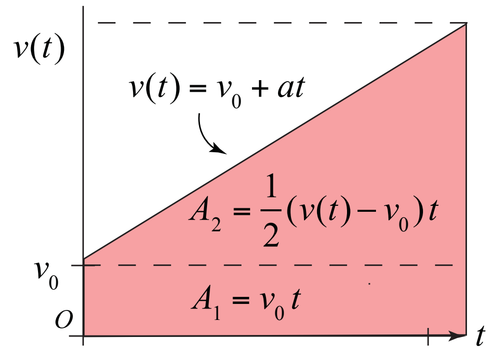

Figure 4.9 Graph of velocity as a function of time for a constant.

The region under the velocity *vs*. time curve is a trapezoid, formed from a rectangle with area $A_1 = v_0 t$, and a triangle with area $A_2 = (1/2)(v(t) - v_0)t$. The total area of the trapezoid is given by

$$\text{Area}(v(t),t) = A_1 + A_2 = v_0 t + \frac{1}{2}(v(t) - v_0)t\,. \tag{4.4.5}$$

Substituting for the velocity (Eq. (4.4.2)) yields

$$\text{Area}(v(t),t) = v_0 t + \frac{1}{2}at^2\,. \tag{4.4.6}$$

Recall that from Example 4.2 (setting $b = a$ and $\Delta t = t$),

$$v_{ave} = v_0 + \frac{1}{2}at = \Delta x / t\,, \tag{4.4.7}$$

therefore Eq. (4.4.6) can be rewritten as

$$\text{Area}(v(t),t) = (v_0 + \frac{1}{2}at)t = v_{ave}t = \Delta x \tag{4.4.8}$$

The displacement is equal to the area under the graph of the x-component of the velocity *vs*. time. The position as a function of time can now be found by rewriting Equation (4.4.8) as

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2\,. \tag{4.4.9}$$

Figure 4.10 shows a graph of this equation. Notice that at $t = 0$ the slope is non-zero, corresponding to the initial velocity component v_0.

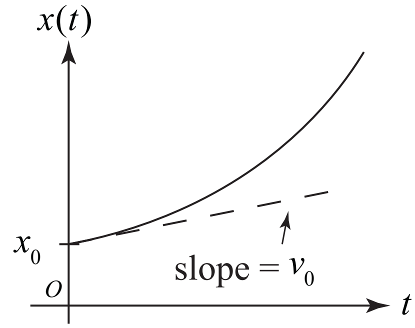

Figure 4.10 Graph of position vs. time for constant acceleration.

Example 4.4 Accelerating Car

A car, starting at rest at $t = 0$, accelerates in a straight line for $100\ \text{m}$ with an unknown constant acceleration. It reaches a speed of $20\ \text{m}\cdot\text{s}^{-1}$ and then continues at this speed for another $10\ \text{s}$. (a) Write down the equations for position and velocity of the car as a function of time. (b) How long was the car accelerating? (c) What was the magnitude of the acceleration? (d) Plot speed *vs.* time, acceleration *vs.* time, and position *vs.* time for the entire motion. (e) What was the average velocity for the entire trip?

Solutions: (a) For the acceleration a, the position $x(t)$ and velocity $v(t)$ as a function of time t for a car starting from rest are

$$\begin{aligned} x(t) &= (1/2)\,a\,t^2 \\ v_x(t) &= a\,t. \end{aligned} \tag{4.4.10}$$

b) Denote the time interval during which the car accelerated by t_1. We know that the position $x(t_1) = 100\,\text{m}$ and $v(t_1) = 20\ \text{m}\cdot\text{s}^{-1}$. Note that we can eliminate the acceleration a between the Equations (4.4.10) to obtain

$$x(t) = (1/2)\,v(t)\,t\,. \tag{4.4.11}$$

We can solve this equation for time as a function of the distance and the final speed giving

$$t = 2\frac{x(t)}{v(t)}\,. \tag{4.4.12}$$

We can now substitute our known values for the position $x(t_1) = 100\,\text{m}$ and $v(t_1) = 20\ \text{m}\cdot\text{s}^{-1}$ and solve for the time interval that the car has accelerated

$$t_1 = 2\frac{x(t_1)}{v(t_1)} = 2\frac{100\text{ m}}{20\text{ m}\cdot\text{s}^{-1}} = 10\,\text{s}. \tag{4.4.13}$$

c) We can substitute into either of the expressions in Equation (4.4.10); the second is slightly easier to use,

$$a = \frac{v(t_1)}{t_1} = \frac{20\text{ m}\cdot\text{s}^{-1}}{10\,\text{s}} = 2.0\,\text{m}\cdot\text{s}^{-2}. \tag{4.4.14}$$

d) The x-component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are piece-wise functions given by

$$a(t) = \begin{cases} 2\text{ m}\cdot\text{s}^{-2}; & 0 < t \le 10\text{ s} \\ 0; & 10\text{ s} < t < 20\text{ s} \end{cases},$$

$$v(t) = \begin{cases} (2\text{ m}\cdot\text{s}^{-2})t; & 0 < t \le 10\text{ s} \\ 20\text{ m}\cdot\text{s}^{-1}; & 10\text{ s} \le t \le 20\text{ s} \end{cases},$$

$$x(t) = \begin{cases} (1/2)(2\text{ m}\cdot\text{s}^{-2})t^2; & 0 < t \le 10\text{ s} \\ 100\text{ m} + (20\text{ m}\cdot\text{s}^{-2})(t - 10\text{ s}); & 10\text{ s} \le t \le 20\text{ s} \end{cases}.$$

The graphs of the x-component of acceleration vs. time, x-component of the velocity vs. time, and the position vs. time are shown in Figure 4.11.

(e) After accelerating, the car travels for an additional ten seconds at constant speed and during this interval the car travels an additional distance $\Delta x = v(t_1)\times 10\,\text{s} = 200\,\text{m}$ (note that this is twice the distance traveled during the $10\,\text{s}$ of acceleration), so the total distance traveled is $300\,\text{m}$ and the total time is $20\,\text{s}$, for an average velocity of

$$v_{\text{ave}} = \frac{300\,\text{m}}{20\,\text{s}} = 15\,\text{m}\cdot\text{s}^{-1}. \tag{4.4.15}$$

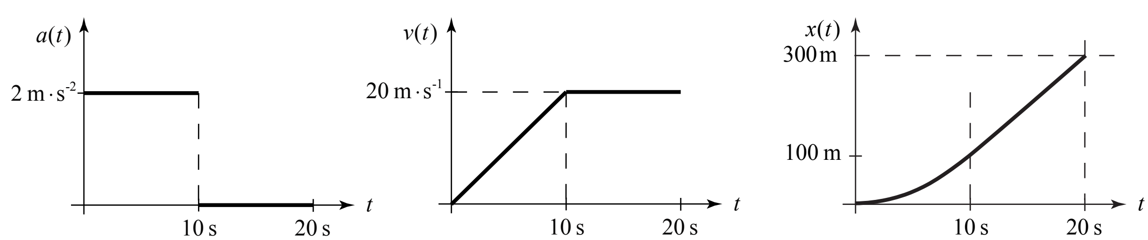

Figure 4.11 Graphs of the x-components of acceleration, velocity and position as piece-wise functions

Example 4.5 Catching a Bus

At the instant a traffic light turns green, a car starts from rest with a given constant acceleration, $3.0\ \text{m}\cdot\text{s}^{-2}$. Just as the light turns green, a bus, traveling with a given constant velocity, $1.6\times10^{1}\ \text{m}\cdot\text{s}^{-1}$, passes the car. The car speeds up and passes the bus some time later. How far down the road has the car traveled, when the car passes the bus?

Solution:

There are two moving objects, bus and the car. Each object undergoes one stage of one-dimensional motion. We are given the acceleration of the car, the velocity of the bus, and infer that the position of the car and the bus are equal when the bus just passes the car. Figure 4.12 shows a qualitative sketch of the position of the car and bus as a function of time.

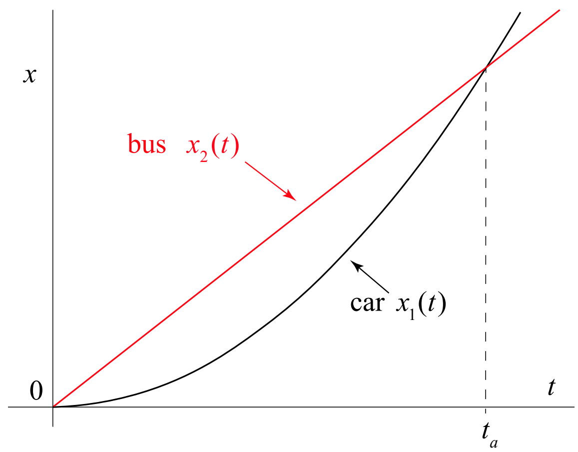

Figure 4.12 Position vs. time of the car and bus

Choose a coordinate system with the origin at the traffic light and the positive x-direction such that car and bus are travelling in the positive x-direction. Set time $t=0$ as the instant the car and bus pass each other at the origin when the light turns green. Figure 4.13 shows the position of the car and bus at time t.

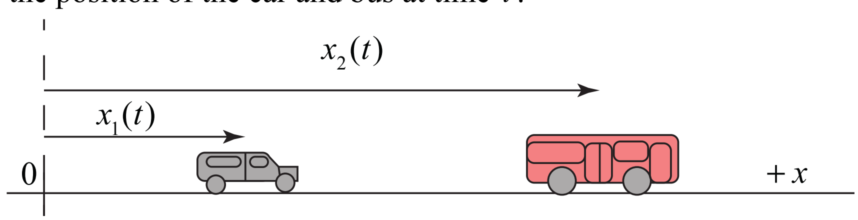

Figure 4.13 Coordinate system for car and bus

Let $x_1(t)$ denote the position function of the car, and $x_2(t)$ the position function for the bus. The initial position and initial velocity of the car are both zero, $x_{1,0}=0$ and $v_{1,0}=0$, and the acceleration of the car is non-zero $a_1 \neq 0$. Therefore the position and velocity functions of the car are given by

$$x_1(t) = \frac{1}{2} a_1 t^2 ,$$
$$v_1(t) = a_1 t .$$

The initial position of the bus is zero, $x_{2,0} = 0$, the initial velocity of the bus is non-zero, $v_{2,0} \neq 0$, and the acceleration of the bus is zero, $a_2 = 0$. Therefore the velocity is constant, $v_2(t) = v_{2,0}$, and the position function for the bus is given by $x_2(t) = v_{2,0} t$.

Let $t = t_a$ correspond to the time that the car passes the bus. Then at that instant, the position functions of the bus and car are equal, $x_1(t_a) = x_2(t_a)$. We can use this condition to solve for t_a:

$$(1/2) a_1 t_a^{\ 2} = v_{2,0} t_a \Rightarrow t_a = \frac{2v_{2,0}}{a_1} = \frac{(2)(1.6 \times 10^1 \,\mathrm{m \cdot s^{-1}})}{(3.0 \,\mathrm{m \cdot s^{-2}})} = 1.1 \times 10^1 \,\mathrm{s}.$$

Therefore the position of the car at t_a is

$$x_1(t_a) = \frac{1}{2} a_1 t_a^{\ 2} = \frac{2 v_{2,0}^{\ \ 2}}{a_1} = \frac{(2)(1.6 \times 10^1 \,\mathrm{m \cdot s^{-1}})^2}{(3.0 \,\mathrm{m \cdot s^{-2}})} = 1.7 \times 10^2 \,\mathrm{m}.$$

4.6 One Dimensional Kinematics and Integration

When the acceleration $a(t)$ of an object is a non-constant function of time, we would like to determine the time dependence of the position function $x(t)$ and the x-component of the velocity $v(t)$. Because the acceleration is non-constant we no longer can use Eqs. (4.4.2) and (4.4.9). Instead we shall use integration techniques to determine these functions.

4.6.1 Change of Velocity as the Indefinite Integral of Acceleration

Consider a time interval $t_1 < t < t_2$. Recall that by definition the derivative of the velocity $v(t)$ is equal to the acceleration $a(t)$,

$$\frac{dv(t)}{dt} = a(t) \quad . \tag{4.5.1}$$

Integration is defined as the inverse operation of differentiation or the 'anti-derivative'. For our example, the function $v(t)$ is called the ***indefinite integral*** of $a(t)$ with respect to t, and is unique up to an additive constant C. We denote this by writing

$$v(t)+C=\int a(t)\,dt \quad . \tag{4.5.2}$$

The symbol $\int ...dt$ means the 'integral, with respect to t, of …", and is thought of as the inverse of the symbol $\frac{d}{dt}$..... Equivalently we can write the differential $dv(t)=a(t)dt$, called the ***integrand***, and then Eq. (4.5.2) can be written as

$$v(t)+C=\int dv(t) \quad , \tag{4.5.3}$$

which we interpret by saying that the integral of the differential of function is equal to the function plus a constant.

Example 4.6 Non-constant acceleration

Suppose an object at time $t=0$ has initial non-zero velocity v_0 and acceleration $a(t)=bt^2$, where b is a constant. Then $dv(t)=bt^2dt=d(bt^3/3)$. The velocity is then $v(t)+C=\int d(bt^3/3)=bt^3/3$. At $t=0$, we have that $v_0+C=0$. Therefore $C=-v_0$ and the velocity as a function of time is then $v(t)=v_0+(bt^3/3)$.

4.6.2 Area as the Indefinite Integral of Acceleration

Consider the graph of a positive-valued acceleration function $a(t)$ vs. t for the interval $t_1 \le t \le t_2$, shown in Figure 4.14a. Denote the area under the graph of $a(t)$ over the interval $t_1 \le t \le t_2$ by $A_{t_1}^{t_2}$.

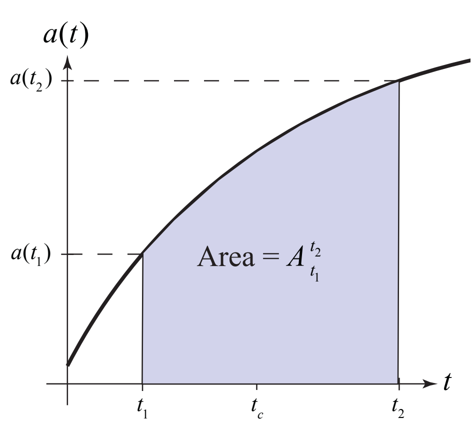

Figure 4.14a: Area under the graph of acceleration over an interval $t_1 \le t \le t_2$

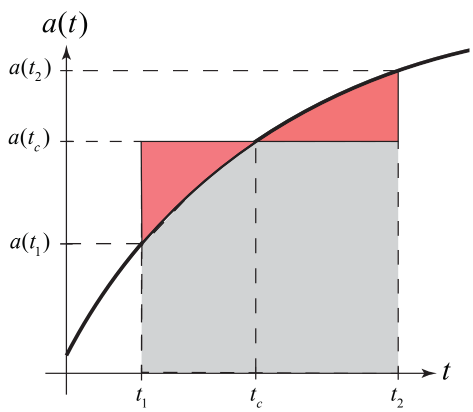

Figure 4.14b: Intermediate value Theorem. The shaded regions above and below the curve have equal areas.

The ***Intermediate Value Theorem*** states that there is at least one time t_c such that the area $A_{t_1}^{t_2}$ is equal to

$$A_{t_1}^{t_2} = a(t_c)(t_2 - t_1) \quad . \tag{4.5.4}$$

In Figure 4.14b, the shaded regions above and below the curve have equal areas, and hence the area $A_{t_1}^{t_2}$ under the curve is equal to the area of the rectangle given by $a(t_c)(t_2 - t_1)$.

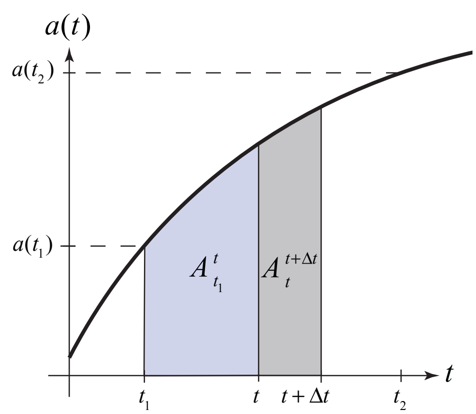

Figure 4.15 Area function is additive

We shall now show that the derivative of the area function is equal to the acceleration and thererfore we can write the area function as an indefinite integral. From Figure 4.15, the area function satisfies the condition that

$$A_{t_1}^{t} + A_{t}^{t+\Delta t} = A_{t_1}^{t+\Delta t} \quad . \tag{4.5.5}$$

Let the small increment of area be denoted by $\Delta A_{t_1}^{t} = A_{t_1}^{t+\Delta t} - A_{t_1}^{t} = A_{t}^{t+\Delta t}$. By the Intermediate Value Theorem

$$\Delta A_{t_1}^{t} = a(t_c)\Delta t \ , \qquad (4.5.6)$$

where $t \le t_c \le t + \Delta t$. In the limit as $\Delta t \to 0$,

$$\frac{dA_{t_1}^{t}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A_{t_1}^{t}}{\Delta t} = \lim_{t_c \to t} a(t_c) = a(t) \ , \qquad (4.5.7)$$

with the initial condition that when $t = t_1$, the area $A_{t_1}^{t_1} = 0$ is zero. Because $v(t)$ is also an integral of $a(t)$, we have that

$$A_{t_1}^{t} = \int a(t)\,dt = v(t) + C \ . \qquad (4.5.8)$$

When $t = t_1$, the area $A_{t_1}^{t_1} = 0$ is zero, therefore $v(t_1) + C = 0$, and so $C = -v(t_1)$. Therefore Eq. (4.5.8) becomes

$$A_{t_1}^{t} = v(t) - v(t_1) = \int a(t)\,dt \ . \qquad (4.5.9)$$

When we set $t = t_2$, Eq. (4.5.9) becomes

$$A_{t_1}^{t_2} = v(t_2) - v(t_1) = \int a(t)\,dt \ . \qquad (4.5.10)$$

The area under the graph of the positive-valued acceleration function for the interval $t_1 \le t \le t_2$ can be found by integrating $a(t)$.

4.6.3 Change of Velocity as the Definite Integral of Acceleration

Let $a(t)$ be the acceleration function over the interval $t_i \le t \le t_f$. Recall that the velocity $v(t)$ is an integral of $a(t)$ because $dv(t)/dt = a(t)$. Divide the time interval $[t_i, t_f]$ into n equal time subintervals $\Delta t = (t_f - t_i)/n$. For each subinterval $[t_j, t_{j+1}]$, where the index $j = 1, 2, \ldots, n$, $t_1 = t_i$ and $t_{n+1} = t_f$, let t_{c_j} be a time such that $t_j \le t_{c_j} \le t_{j+1}$. Let

$$S_n = \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t \ . \qquad (4.5.11)$$

S_n is the sum of the blue rectangle shown in Figure 4.16a for the case $n = 4$. The **Fundamental Theorem of Calculus** states that in the limit as $n \to \infty$, the sum is equal to the change in the velocity during the interval $[t_i, t_f]$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i) \ . \tag{4.5.12}$$

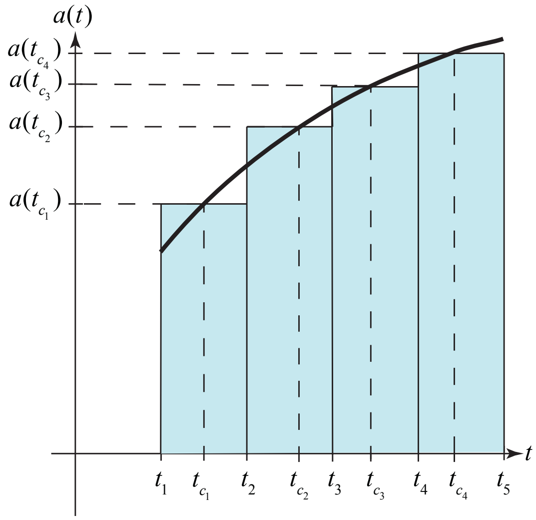

Figure 4.16a Graph of $a(t)$ vs. t

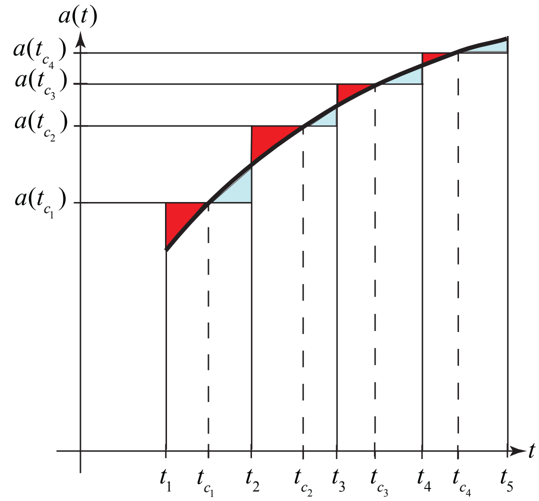

Figure 4.16b Graph of $a(t)$ vs. t

The limit of the sum in Eq. (4.5.12) is a number, which we denote by the symbol

$$\int_{t_i}^{t_f} a(t)\,dt \equiv \lim_{n\to\infty} \sum_{j=1}^{j=n} a(t_{c_j})\,\Delta t = v(t_f) - v(t_i) \ , \tag{4.5.13}$$

and is called the ***definite integral*** of $a(t)$ from t_i to t_f. The times t_i and t_f are called the limits of integration, t_i the lower limit and t_f the upper limit. The definite integral is a linear map that takes a function $a(t)$ defined over the interval $[t_i, t_f]$ and gives a number. The map is linear because

$$\int_{t_i}^{t_f} (a_1(t) + a_2(t))\,dt = \int_{t_i}^{t_f} a_1(t)\,dt + \int_{t_i}^{t_f} a_2(t)\,dt \ , \tag{4.5.14}$$

Suppose the times t_{c_j}, $j = 1,...,n$, are selected such that each t_{c_j} satisfies the Intermediate Value Theorem,

$$\Delta v_j \equiv v(t_{j+1}) - v(t_j) = \frac{dv(t_{c_j})}{dt}\Delta t = a(t_{c_j})\Delta t \ , \tag{4.5.15}$$

where $a(t_{c_j})$ is the instantaneous acceleration at t_{c_j}, (Figure 4.16b). Then the sum of the changes in the velocity for the interval $[t_i, t_f]$ is

$$\sum_{j=1}^{j=n} \Delta v_j = (v(t_2) - v(t_1)) + (v(t_3) - v(t_2)) + \cdots + (v(t_{n+1}) - v(t_n)) = v(t_{n+1}) - v(t_1) \\ = v(t_f) - v(t_i). \tag{4.5.16}$$

where $v(t_f) = v(t_{n+1})$ and $v(t_i) = v(t_1)$. Substituting Eq. (4.5.15) into Equation (4.5.16) yields the exact result that the change in the x-component of the velocity is give by this finite sum.

$$v(t_f) - v(t_i) = \sum_{j=1}^{j=n} \Delta v_j = \sum_{j=1}^{j=n} a(t_{c_j})\, \Delta t\,. \tag{4.5.17}$$

We do not specifically know the intermediate values $a(t_{c_j})$ and so Eq. (4.5.17) is not useful as a calculating tool. The statement of the Fundamental Theorem of Calculus is that the limit as $n \to \infty$ of the sum in Eq. (4.5.12) is independent of the choice of the set of t_{c_j}. Therefore the exact result in Eq. (4.5.17) is the limit of the sum.

Thus we can evaluate the definite integral if we know any indefinite integral of the integrand $a(t)dt = dv(t)$.

Additionally, provided the acceleration function has only non-negative values, the limit is also equal to the area under the graph of $a(t)$ vs. t for the time interval, $[t_i, t_f]$:

$$A_{t_i}^{t_f} = \int_{t_i}^{t_f} a(t)\, dt\ . \tag{4.5.18}$$

In Figure 4.14, the red areas are an overestimate and the blue areas are an underestimate. As $N \to \infty$, the sum of the red areas and the sum of the blue areas both approach zero. If there are intervals in which $a(t)$ has negative values, then the summation is a sum of signed areas, positive area above the t-axis and negative area below the t-axis.

We can determine both the change in velocity for the time interval $[t_i, t_f]$ and the area under the graph of $a(t)$ vs. t for $[t_i, t_f]$ by integration techniques instead of limiting arguments. We can turn the linear map into a function of time, instead of just giving a number, by setting $t_f = t$. In that case, Eq. (4.5.13) becomes

$$v(t) - v(t_i) = \int_{t'=t_i}^{t'=t} a(t')dt' . \qquad (4.5.19)$$

Because the upper limit of the integral, $t_f = t$, is now treated as a variable, we shall use the symbol t' as the integration variable instead of t.

4.6.4 Displacement as the Definite Integral of Velocity

We can repeat the same argument for the definite integral of the x-component of the velocity $v(t)$ *vs*. time t. Because $x(t)$ is an integral of $v(t)$ the definite integral of $v(t)$ for the time interval $[t_i, t_f]$ is the displacement

$$x(t_f) - x(t_i) = \int_{t'=t_i}^{t'=t_f} v(t')dt' . \qquad (4.5.20)$$

If we set $t_f = t$, then the definite integral gives us the position as a function of time

$$x(t) = x(t_i) + \int_{t'=t_i}^{t'=t} v(t')dt' . \qquad (4.5.21)$$

Summarizing the results of these last two sections, for a given acceleration $a(t)$, we can use integration techniques, to determine the change in velocity and change in position for an interval $[t_i, t]$, and given initial conditions (x_i, v_i), we can determine the position $x(t)$ and the x-component of the velocity $v(t)$ as functions of time.

Example 4.5 Non-constant Acceleration

Let's consider a case in which the acceleration, $a(t)$, is not constant in time,

$$a(t) = b_0 + b_1 t + b_2 t^2 . \qquad (4.5.22)$$

The graph of the x-component of the acceleration *vs*. time is shown in Figure 4.16

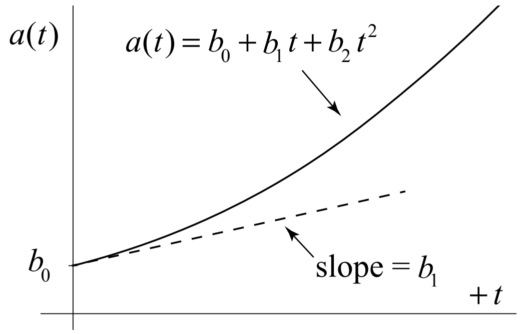

Figure 4.16 Non-constant acceleration *vs*. time graph.

Denote the initial velocity at $t = 0$ by v_0. Then, the change in the x-component of the velocity as a function of time can be found by integration:

$$v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')\,dt' = \int_{t'=0}^{t'=t} (b_0 + b_1 t' + b_2 t'^2)\,dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.23)$$

The x-component of the velocity as a function in time is then

$$v(t) = v_0 + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \qquad (4.5.24)$$

Denote the initial position at $t = 0$ by x_0. The displacement as a function of time is

$$x(t) - x_0 = \int_{t'=0}^{t'=t} v(t')\,dt'. \qquad (4.5.25)$$

Use Equation (4.5.24) for the x-component of the velocity in Equation (4.5.25) and then integrate to determine the displacement as a function of time:

$$\begin{aligned} x(t) - x_0 &= \int_{t'=0}^{t'=t} v(t')\,dt' \\ &= \int_{t'=0}^{t'=t} \left(v_0 + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) dt' = v_0 t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \end{aligned} \qquad (4.5.26)$$

Finally the position as a function of time is then

$$x(t) = x_0 + v_{x,0} t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \qquad (4.5.27)$$

Example 4.6 Bicycle and Car

A car is driving through a green light at $t=0$ located at $x=0$ with an initial speed $v_{c,0}=12\text{ m}\cdot\text{s}^{-1}$. At time $t_1=1\text{ s}$, the car starts braking until it comes to rest at time t_2. The acceleration of the car as a function of time is given by the piecewise function

$$a_c(t)=\begin{cases}0; & 0<t<t_1=1\text{ s}\\ b(t-t_1); & 1\text{ s}<t<t_2\end{cases},$$

where $b=-(6\text{ m}\cdot\text{s}^{-3})$.

(a) Find the x-component of the velocity and the position of the car as a function of time.
(b) A bicycle rider is riding at a constant speed of $v_{b,0}$ and at $t=0$ is 17 m behind the car. The bicyclist reaches the car when the car just comes to rest. Find the speed of the bicycle.

Solution: a) In order to apply Eq. (4.5.19), we shall treat each stage separately. For the time interval $0<t<t_1$, the acceleration is zero so the x-component of the velocity is constant. For the second time interval $\text{t}_1<t<t_2$, the definite integral becomes

$$v_c(t)-v_c(t_1)=\int_{t'=t_1}^{t'=t} b(t'-t_1)\,dt'$$

Because $v_c(t_1)=v_{c0}$, the x-component of the velocity is then

$$v_c(t)=\begin{cases}v_{c0}; & 0<t\le t_1\\ v_{c0}+\displaystyle\int_{t'=t_1}^{t'=t} b(t'-t_1)\,dt'; & t_1\le t<t_2\end{cases}.$$

Integrate and substitute the two endpoints of the definite integral, yields

$$v_c(t)=\begin{cases}v_{c0}; & 0<t\le t_1\\ v_{c0}+\dfrac{1}{2}b(t-t_1)^2; & t_1\le t<t_2\end{cases}.$$

In order to use Eq. (4.5.25), we need to separate the definite integral into two integrals corresponding to the two stages of motion, using the correct expression for the velocity for each integral. The position function is then

$$x_c(t) = \begin{cases} x_{c0} + \int\limits_{t'=0}^{t'=t_1} v_{c0}\, dt'; & 0 < t \le t_1 \\ x_c(t_1) + \int\limits_{t'=t_1}^{t'=t} \left(v_{c0} + \frac{1}{2} b (t' - t_1)^2 \right) dt; & t_1 \le t < t_2 \end{cases}.$$

Upon integration we have

$$x_c(t) = \begin{cases} x_c(0) + v_{c0} t; & 0 < t \le t_1 \\ x_c(t_1) + \left(v_{c0}(t' - t_1) + \frac{1}{6} b (t' - t_1)^3 \right) \Big|_{t'=t_1}^{t'=t}; & t_1 \le t < t_2 \end{cases}.$$

We chose our coordinate system such that the initial position of the car was at the origin, $x_{c0} = 0$, therefore $x_c(t_1) = v_{c0} t_1$. So after substituting in the endpoints of the integration interval we have that

$$x_c(t) = \begin{cases} v_{c0} t; & 0 < t \le t_1 \\ v_{c0} t_1 + v_{c0}(t - t_1) + \frac{1}{6} b (t - t_1)^3; & t_1 \le t < t_2 \end{cases}.$$

(b) We are looking for the instant t_2 that the car has come to rest. So we use our expression for the x-component of the velocity the interval $t_1 \le t < t_2$, where we set $t = t_2$ and $v_c(t_2) = 0$:

$$0 = v_c(t_2) = v_{c0} + \frac{1}{2} b (t_2 - t_1)^2 .$$

Solving for t_2 yields

$$t_2 = t_1 + \sqrt{-\frac{2 v_{c0}}{b}} ,$$

where we have taken the positive square root. Substitute the given values then yields

$$t_2 = 1\,\mathrm{s} + \sqrt{-\frac{2(12\,\mathrm{m}\cdot\mathrm{s}^{-1})}{(-6\,\mathrm{m}\cdot\mathrm{s}^{-3})}} = 3\,\mathrm{s} .$$

The position of the car at t_2 is then given by

$$x_c(t_2) = v_{c0}\,t_1 + v_{c0}(t_2 - t_1) + \frac{1}{6}b(t_2 - t_1)^3$$

$$x_c(t_2) = v_{c0}\,t_1 + v_{c0}\sqrt{-2v_{c0}/b} + \frac{1}{6}b(-2v_{c0}/b)^{3/2}$$

$$x_c(t_2) = v_{c0}\,t_1 + \frac{2\sqrt{2}(v_{c0}{}^{3/2})}{3(-b)^{1/2}}$$

where we used the condition that $t_2 - t_1 = \sqrt{-2v_{c0}/b}$. Substitute the given values then yields

$$x_c(t_2) = v_{c0}\,t_1 + 2\frac{4\sqrt{2}(v_{c0})^{3/2}}{3(-b)^{1/2}} = (12\text{ m}\cdot\text{s}^{-1})(1\text{ s}) + \frac{4\sqrt{2}((12\text{ m}\cdot\text{s}^{-1})^{3/2}}{3((6\text{ m}\cdot\text{s}^{-3}))^{1/2}} = 28\text{ m}.$$

b) Because the bicycle is traveling at a constant speed with an initial position $x_{b0} = -17\text{ m}$, the position of the bicycle is given by $x_b(t) = -17\text{ m} + v_b t$. The bicycle and car intersect at time $t_2 = 3\text{ s}$, where $x_b(t_2) = x_c(t_2)$. Therefore $-17\text{ m} + v_b(3\text{ s}) = 28\text{ m}$. So the speed of the bicycle is $v_b = 15\text{ m}\cdot\text{s}^{-1}$.

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