# Chapter 25 Celestial Mechanics

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> *...and if you want the exact moment in time, it was conceived mentally on 8th March in this year one thousand six hundred and eighteen, but submitted to calculation in an unlucky way, and therefore rejected as false, and finally returning on the 15th of May and adopting a new line of attack, stormed the darkness of my mind. So strong was the support from the combination of my labour of seventeen years on the observations of Brahe and the present study, which conspired together, that at first I believed I was dreaming, and assuming my conclusion among my basic premises. But it is absolutely certain and exact that "the proportion between the periodic times of any two planets is precisely the sesquialterate proportion of their mean distances ..."* [1]
>
> *Johannes Kepler*

## 25.1 Introduction: The Kepler Problem

Johannes Kepler first formulated the laws that describe planetary motion,

I. Each planet moves in an ellipse with the sun at one focus.

II. The radius vector from the sun to a planet sweeps out equal areas in equal time.

III. The period of revolution $T$ of a planet about the sun is related to the semi-major axis $a$ of the ellipse by $T^2 = k\,a^3$ where $k$ is the same for all planets.[2]

The third law was published in 1619, and efforts to discover and solve the equation of motion of the planets generated two hundred years of mathematical and scientific discovery. In his honor, this problem has been named ***the Kepler Problem*.**

When there are more than two bodies, the problem becomes impossible to solve exactly. The most important "three-body problem" in the 17th and 18th centuries involved finding the motion of the moon, due to gravitational interaction with both the sun and the earth. Newton realized that if the exact position of the moon were known, the longitude of any observer on the earth could be determined by measuring the moon's position with respect to the stars.

In the eighteenth century, Leonhard Euler and other mathematicians spent many years trying to solve the three-body problem, and they raised a deeper question. Do the small contributions from the gravitational interactions of all the planets make the planetary system unstable over long periods of time? At the end of 18th century, Pierre

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[1] Kepler, Johannes, *Harmonice mundi* Book 5, Chapter 3, trans. Aiton, Duncan and Field, p. 411

[2] As stated in *An Introduction to Mechanics*, Daniel Kleppner and Robert Kolenkow, McGraw-Hill, 1973, p 401.

Simon Laplace and others found a series solution to this stability question, but it was unknown whether or not the series solution converged after a long period of time. Henri Poincaré proved that the series actually diverged. Poincaré went on to invent new mathematical methods that produced the modern fields of differential geometry and topology in order to answer the stability question using geometric arguments, rather than analytic methods. Poincaré and others did manage to show that the three-body problem was indeed stable, due to the existence of periodic solutions. Just as in the time of Newton and Leibniz and the invention of calculus, unsolved problems in celestial mechanics became the experimental laboratory for the discovery of new mathematics.

## 25.2 Planetary Orbits and the Center of Mass Reference Frame

We now commence a study of the Kepler Problem. We shall determine the equation of motion for the motions of two bodies interacting via a gravitational force (two-body problem) using both force methods and conservation laws.

### 25.2.1 Reducing the Two-Body Problem into a One-Body Problem

We shall begin by showing how the motion of two bodies interacting via a gravitational force (two-body problem) is mathematically equivalent to the motion of a single body acted on by an external central gravitational force, where the mass of the single body is the *reduced mass* $\mu$,

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2}\Rightarrow \mu=\frac{m_1 m_2}{m_1+m_2}. \qquad (25.2.1)$$

Once we solve for the motion of the reduced body in this ***equivalent one-body problem***, we can then return to the real two-body problem and solve for the actual motion of the two original bodies. The reduced mass was introduced in Chapter 13 Appendix A of these notes. That appendix used similar but slightly different notation from that used in this chapter.

Consider a system consisting of two bodies with masses $m_1$ and $m_2$ interacting via a gravitational force as shown in Figure 25.1.

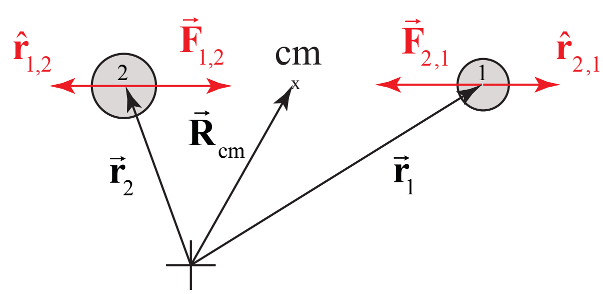



**Figure 25.1** Two bodies interacting via a gravitational force

Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$. The location of the center of mass is given by the vector

$$\vec{\mathbf{R}}_{cm} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{25.2.2}$$

Newton's Second Law can be applied to the motion of center of mass:

$$\vec{\mathbf{F}}_{ext} = (m_1 + m_2)\vec{\mathbf{A}}_{cm}. \tag{25.2.3}$$

For our two-body system the gravitational forces are internal forces and by Newton's Third Law sum to zero. If we assume that there are no external forces acting on our two-body system, then the center of mass is either at rest of moving with a constant velocity. We can choose as our reference frame, the center of mass reference frame, that is a reference frame moving with the velocity of the center of mass with our origin located at the center of mass.

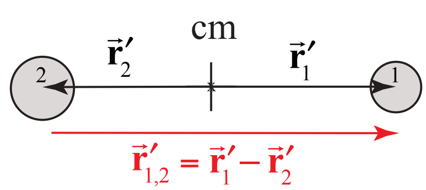



**Figure 25.2** Center of mass reference frame

Let $\vec{\mathbf{r}}_1{}'$ be the vector from the center of mass to body 1 and $\vec{\mathbf{r}}_2{}'$ be the vector from the center of mass to body 2. Then, by the geometry in Figure 25.2, the *relative position vector* $\vec{\mathbf{r}}'_{1,2}$ pointing from body 2 to body 1 is equal to

$$\vec{\mathbf{r}}'_{1,2} = \vec{\mathbf{r}}'_1 - \vec{\mathbf{r}}'_2. \tag{25.2.4}$$

Because the center of mass is located at the origin, $\vec{\mathbf{R}}'_{cm} = \vec{\mathbf{0}}$, therefore

$$m_1 \vec{\mathbf{r}}_1{}' = -m_2 \vec{\mathbf{r}}'_2 , \tag{25.2.5}$$

The position vector of body 1 can be found in terms of the relative position vector $\vec{\mathbf{r}}'_{1,2}$,

$$\begin{aligned} &\vec{\mathbf{r}}'_{1,2} = \vec{\mathbf{r}}'_1 - \vec{\mathbf{r}}'_2 = \vec{\mathbf{r}}'_1\left(1 + \frac{m_1}{m_2}\right) \Rightarrow \\ &\vec{\mathbf{r}}'_1 = \frac{m_2}{m_1 + m_2}\vec{\mathbf{r}}'_{1,2}. \end{aligned} \tag{25.2.6}$$

A similar calculation yields

$$\vec{\mathbf{r}}_2' = -\frac{m_1}{m_1 + m_2}\vec{\mathbf{r}}_{1,2}' . \quad (25.2.7)$$

In what follows we shall drop the prime indices in the center of mass reference frame

The force on body 2 due to the gravitational interaction between the two bodies can be described by Newton's Universal Law of Gravitation

$$\vec{\mathbf{F}}_{1,2} = -G\frac{m_1 m_2}{r_{1,2}^{\ 2}}\hat{\mathbf{r}}_{1,2} . \quad (25.2.8)$$

where $r_{1,2}$ is the relative distance between bodies 1 and 2. Newton's Second Law can be applied individually to the two bodies:

$$\vec{\mathbf{F}}_{2,1} = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2} , \quad (25.2.9)$$

$$\vec{\mathbf{F}}_{1,2} = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2} . \quad (25.2.10)$$

Dividing through by the mass in each of Equations (25.2.9) and (25.2.10) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} , \quad (25.2.11)$$

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2} . \quad (25.2.12)$$

Subtracting the expression in Equation (25.2.12) from that in Equation (25.2.11) yields

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_1} - \frac{\vec{\mathbf{F}}_{1,2}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} . \quad (25.2.13)$$

Newton's Third Law requires that the force on body 2 is equal in magnitude and opposite in direction to the force on body 1,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1} . \quad (25.2.14)$$

Using Newton's Third Law, Equation (25.2.14), Equation (25.2.13) becomes

$$\vec{\mathbf{F}}_{2,1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} . \quad (25.2.15)$$

Define the *reduced mass* $\mu$, by

$$\frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2} \tag{25.2.16}$$

Rewrite Eq. (25.2.15) as

$$\vec{\mathbf{F}}_{2,1}=\mu\frac{d^2\vec{\mathbf{r}}_{1,2}}{dt^2} \tag{25.2.17}$$

where $\vec{\mathbf{F}}_{2,1}$ is given by Equation (25.2.8). Using the reduced mass, the position vector of body 1 can be written as

$$\vec{\mathbf{r}}_1=\frac{\mu}{m_1}\vec{\mathbf{r}}\,. \tag{25.2.18}$$

and

$$\vec{\mathbf{r}}_2=-\frac{\mu}{m_2}\vec{\mathbf{r}} \tag{25.2.19}$$

When one mass is much smaller than the other, for example $m_1 << m_2$, then the reduced mass is approximately the smaller mass,

$$\mu=\frac{m_1\,m_2}{m_1+m_2}\cong\frac{m_1\,m_2}{m_2}=m_1\,. \tag{25.2.20}$$

Our result has a special interpretation using Newton's Second Law. Let $\mu$ be the mass of a ***single body*** with position vector $\vec{\mathbf{r}}_{1,2}=r_{1,2}\,\hat{\mathbf{r}}_{1,2}$ with respect to an origin $O$, where $\hat{\mathbf{r}}_{1,2}$ is the unit vector pointing from the origin $O$ to the single body. Then the equation of motion, Equation (25.2.17), implies that the single body of mass $\mu$ is under the influence of an attractive gravitational force pointing toward the origin. So, the original two-body gravitational problem has now been reduced to an equivalent one-body problem, involving a single body with mass $\mu$ under the influence of an attractive central force $\vec{\mathbf{F}}_{2,1}$. Note that in this reformulation, there is no body located at the central point (the origin $O$). In what follows we shall drop the indices 1,2 and write Eq. (25.2.17) as

$$-\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}=\mu\frac{d^2\vec{\mathbf{r}}}{dt^2} \tag{25.2.21}$$

The parameter $r$ in the one-body problem is the distance between the reduced mass and the central point, and also the relative distance between bodies 1 and 2. This reduction of the physical two-body problem to a mathematical description of a one body problem generalizes to all central forces.

## 25.3 Energy and Angular Momentum, Constants of the Motion

The equivalent one-body problem has two constants of the motion, energy $E$ and the angular momentum $L$ about the origin $O$. Energy is a constant because in our original two-body problem, the gravitational interaction was an internal conservative force. Angular momentum is constant about the origin because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (the vector from the origin to the single body is anti-parallel to the force vector and $\sin\pi = 0$). Because angular momentum is constant, the orbit of the single body lies in a plane with the angular momentum vector pointing perpendicular to this plane.

In the plane of the orbit, choose polar coordinates $(r,\theta)$ for the single body (see Figure 25.3), where $r$ is the distance of the single body from the central point that is now taken as the origin $O$, and $\theta$ is the angle that the single body makes with respect to a chosen direction, and which increases positively in the counterclockwise direction.

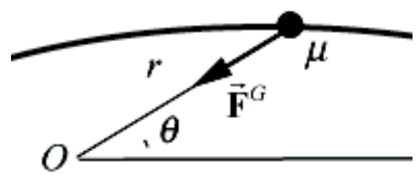



**Figure 25.3** Coordinate system for the orbit of the single body

There are two approaches to describing the motion of the single body. We can try to find both the distance from the origin, $r(t)$ and the angle, $\theta(t)$, as functions of the parameter time, but in most cases explicit functions can't be found analytically. We can also find the distance from the origin, $r(\theta)$, as a function of the angle $\theta$. This second approach offers a spatial description of the motion of the single body (see Appendix 25A).

### 25.3.1 The Orbit Equation for the One-Body Problem

Consider the single body with mass $\mu$ given by Equation (25.2.1), orbiting about a central point under the influence of a radially attractive force given by Equation (25.2.8). Since the force is conservative, the potential energy (from the two-body problem) with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}. \tag{25.3.1}$$

The total energy $E$ is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}. \tag{25.3.2}$$

The kinetic energy term $\mu v^2 / 2$ is written in terms of the mass $\mu$ and the relative speed $v$ of the two bodies. Choose polar coordinates such that

$$\vec{\mathbf{v}} = v_r \,\hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}}, \qquad v = |\vec{\mathbf{v}}| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right|, \tag{25.3.3}$$

where $v_r = dr / dt$ and $v_\theta = r(d\theta / dt)$. Equation (25.3.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G\,m_1\,m_2}{r}. \tag{25.3.4}$$

The angular momentum with respect to the origin $O$ is given by

$$\vec{\mathbf{L}}_O = \vec{\mathbf{r}}_O \times \mu\vec{\mathbf{v}} = r\hat{\mathbf{r}} \times \mu(v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}) = \mu r v_\theta \hat{\mathbf{k}} = \mu r^2 \frac{d\theta}{dt}\hat{\mathbf{k}} \equiv L\hat{\mathbf{k}} \tag{25.3.5}$$

with magnitude

$$L = \mu r v_\theta = \mu r^2 \frac{d\theta}{dt}. \tag{25.3.6}$$

We shall explicitly eliminate the $\theta$ dependence from Equation (25.3.4) by using our expression in Equation (25.3.6),

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}. \tag{25.3.7}$$

The mechanical energy as expressed in Equation (25.3.4) then becomes

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{\mu r^2} - \frac{G\,m_1\,m_2}{r}. \tag{25.3.8}$$

Equation (25.3.8) is a separable differential equation involving the variable $r$ as a function of time $t$ and can be solved for the first derivative $dr / dt$,

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G\,m_1\,m_2}{r}\right)^{\frac{1}{2}}. \tag{25.3.9}$$

Equation (25.3.9) can in principle be integrated directly for $r(t)$. In fact, doing the integrals is complicated and beyond the scope of this book. The function $r(t)$ can then, in principle, be substituted into Equation (25.3.7) and can then be integrated to find $\theta(t)$.

Instead of solving for the position of the single body as a function of time, we shall find a geometric description of the orbit by finding $r(\theta)$. We first divide Equation (25.3.7) by Equation (25.3.9) to obtain

$$\frac{d\theta}{dr} = \frac{\dfrac{d\theta}{dt}}{\dfrac{dr}{dt}} = \frac{L}{\sqrt{2\mu}} \frac{\left(1/r^2\right)}{\left(E - \dfrac{L^2}{2\mu r^2} + \dfrac{G m_1 m_2}{r}\right)^{1/2}} . \qquad (25.3.10)$$

The variables $r$ and $\theta$ are separable;

$$d\theta = \frac{L}{\sqrt{2\mu}} \frac{\left(1/r^2\right)}{\left(E - \dfrac{L^2}{2\mu r^2} + \dfrac{G m_1 m_2}{r}\right)^{1/2}} dr . \qquad (25.3.11)$$

Equation (25.3.11) can be integrated to find the radius as a function of the angle $\theta$; see Appendix 25A for the exact integral solution. The result is called the ***orbit equation*** for the reduced body and is given by

$$r = \frac{r_0}{1 - \varepsilon \cos\theta} \qquad (25.3.12)$$

where

$$r_0 = \frac{L^2}{\mu G m_1 m_2} \qquad (25.3.13)$$

is a constant (known as the ***semilatus rectum***) .

$$\varepsilon = \left(1 + \frac{2 E L^2}{\mu (G m_1 m_2)^2}\right)^{\frac{1}{2}} \qquad (25.3.14)$$

is the ***eccentricity*** of the orbit. The two constants of the motion, $E$ and $L$ determine all the properties of the orbit. Any other pair of properties of the orbit can always be expressed in terms of the constants $E$ and $L$.

The two constants of the motion, angular momentum $L$ and mechanical energy $E$, also determine in terms of $r_0$ and $\varepsilon$, are

$$L = (\mu G m_1 m_2 r_0)^{1/2} \qquad (25.3.15)$$

$$E = \frac{G m_1 m_2 (\varepsilon^2 - 1)}{2 r_0} . \qquad (25.3.16)$$

The orbit equation as given in Equation (25.3.12) is a general ***conic section*** and is perhaps somewhat more familiar in Cartesian coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$, with $r^2 = x^2 + y^2$. The orbit equation can be rewritten as

$$r = r_0 + \varepsilon r\cos\theta . \tag{25.3.17}$$

Using the Cartesian substitutions for $x$ and $y$, rewrite Equation (25.3.17) as

$$(x^2 + y^2)^{1/2} = r_0 + \varepsilon x . \tag{25.3.18}$$

Squaring both sides of Equation (25.3.18),

$$x^2 + y^2 = r_0^{\;2} + 2\varepsilon x r_0 + \varepsilon^2 x^2 . \tag{25.3.19}$$

After rearranging terms, Equation (25.3.19) is the general expression of a conic section with axis on the $x$-axis,

$$x^2(1-\varepsilon^2) - 2\varepsilon x r_0 + y^2 = r_0^{\;2} . \tag{25.3.20}$$

(We now see that the horizontal axis in Figure 25.3 can be taken to be the $x$-axis).

For a given $r_0 > 0$, corresponding to a given nonzero angular momentum as in Equation (25.3.12), there are four cases determined by the value of the eccentricity.

Case 1: when $\varepsilon = 0$, $E = E_{\min} < 0$ and $r = r_0$, Equation (25.3.20) is the equation for a *circle*,

$$x^2 + y^2 = r_0^{\;2} . \tag{25.3.21}$$

Case 2: when $0 < \varepsilon < 1$, $E_{\min} < E < 0$, Equation (25.3.20) describes an *ellipse*,

$$y^2 + Ax^2 - Bx = k . \tag{25.3.22}$$

where $A > 0$ and $k$ is a positive constant. (Appendix 25C shows how this expression may be expressed in the more traditional form involving the coordinates of the center of the ellipse and the semi-major and semi-minor axes.)

Case 3: when $\varepsilon = 1$, $E = 0$, Equation (25.3.20) describes a *parabola*,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2} . \tag{25.3.23}$$

Case 4: when $\varepsilon > 1$, $E > 0$, Equation (25.3.20) describes a *hyperbola*,

$$y^2 - Ax^2 - Bx = k\,, \qquad (25.3.24)$$

where $A > 0$ and $k$ is a positive constant.

## 25.4 Energy Diagram, Effective Potential Energy, and Orbits

The energy (Equation (25.3.8)) of the single body moving in two dimensions can be reinterpreted as the energy of a single body moving in one dimension, the radial direction $r$, in an ***effective potential energy*** given by two terms,

$$U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\,. \qquad (25.4.1)$$

The energy is still the same, but our interpretation has changed,

$$E = K_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\,, \qquad (25.4.2)$$

where the ***effective kinetic energy*** $K_{\text{eff}}$ associated with the one-dimensional motion is

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2. \qquad (25.4.3)$$

The graph of $U_{\text{eff}}$ as a function of $u = r / r_0$, where $r_0$ as given in Equation (25.3.13), is shown in Figure 25.4.

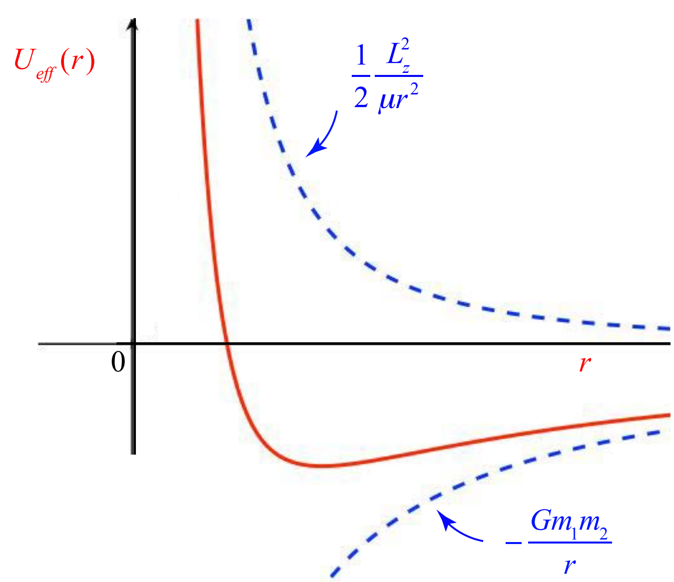


**Figure 25.4** Graph of effective potential energy

The upper red curve is proportional to $L^2/(2\mu r^2) \sim 1/2r^2$. The lower blue curve is proportional to $-Gm_1m_2/r \sim -1/r$. The sum $U_{\text{eff}}$ is represented by the middle green curve. The minimum value of $U_{\text{eff}}$ is at $r = r_0$, as will be shown analytically below. The vertical scale is in units of $-U_{\text{eff}}(r_0)$. Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}} = 0$, the energy is equal to the effective potential energy,

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}. \qquad (25.4.4)$$

Recall that the potential energy is defined to be the negative integral of the work done by the force. For our reduction to a one-body problem, using the effective potential, we will introduce an ***effective force*** such that

$$U_{\text{eff},B} - U_{\text{eff},A} = -\int_A^B \vec{\mathbf{F}}^{\text{eff}} \cdot d\vec{\mathbf{r}} = -\int_A^B F_r^{\text{eff}}\, dr \qquad (25.4.5)$$

The fundamental theorem of calculus (for one variable) then states that the integral of the derivative of the effective potential energy function between two points is the effective potential energy difference between those two points,

$$U_{\text{eff},B} - U_{\text{eff},A} = \int_A^B \frac{dU_{\text{eff}}}{dr} dr \qquad (25.4.6)$$

Comparing Equation (25.4.6) to Equation (25.4.5) shows that the radial component of the effective force is the negative of the derivative of the effective potential energy,

$$F_r^{\text{eff}} = -\frac{dU_{\text{eff}}}{dr} \qquad (25.4.7)$$

The effective potential energy describes the potential energy for a reduced body moving in one dimension. (Note that the effective potential energy is only a function of the variable $r$ and is independent of the variable $\theta$). There are two contributions to the effective potential energy, and the radial component of the force is then

$$F_r^{\text{eff}} = -\frac{d}{dr}U_{\text{eff}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\right) \qquad (25.4.8)$$

Thus there are two "forces" acting on the reduced body,

$$F_r^{\text{eff}} = F_{r,\text{ centifugal}} + F_{r,\text{ gravity}}\,, \tag{25.4.9}$$

with an ***effective centrifugal force*** given by

$$F_{r,\text{centrifugal}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3}\,. \tag{25.4.10}$$

Recall from Eq. (25.3.6) that the angular momentum $L = \mu r v_\theta$. The effective centrifugal force (Eq. (25.4.10) can then be rewritten as

$$F_{r,\text{centrifugal}} = \frac{\mu\, v_\theta^{\;2}}{r}\,. \tag{25.4.11}$$

The centripetal gravitational force is given by

$$F_{r,\text{ gravity}} = -\frac{G\, m_1\, m_2}{r^2}\,. \tag{25.4.12}$$

With this nomenclature, let's review the four cases presented in Section 25.3.

### 25.4.1 Circular Orbit $E = E_0$

The lowest energy state, $E_0$, corresponds to the minimum of the effective potential energy, $E_0 = (U_{\text{eff}})_{\min}$ (Figure 25.5a).

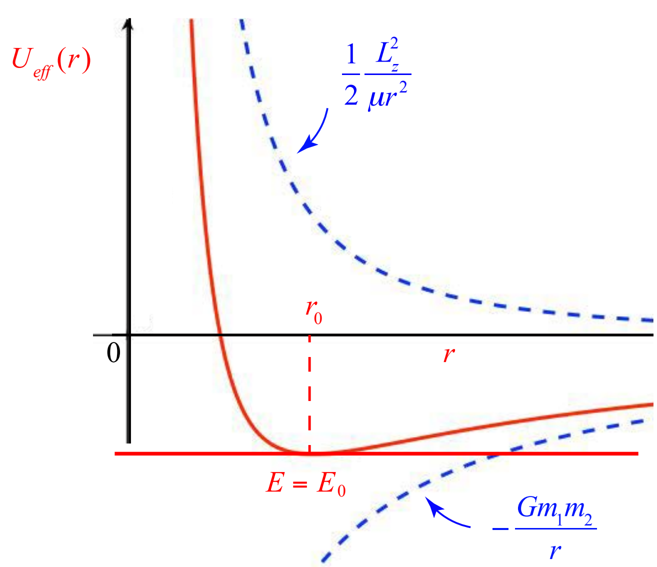



**Figure 25.5**a Plot of $U_{\text{eff}}(r)$ vs. $r$ with energies corresponding to circular orbit

We can minimize the effective potential energy

$$0 = \left.\frac{dU_{\text{eff}}}{dr}\right|_{r=r_0} = -\frac{L^2}{\mu r_0^{\,3}} + \frac{G\,m_1 m_2}{r_0^{\,2}}$$

$$\frac{L^2}{\mu G\, m_1 m_2} \qquad . \qquad (25.4.13)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

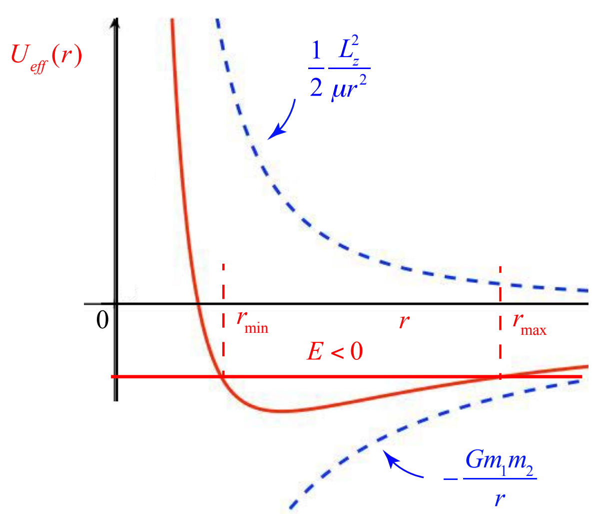
and solve Equation (25.4.13) for $r_0$,

$$r_0 = \frac{L^2}{\mu G\, m_1 m_2}, \qquad (25.4.14)$$

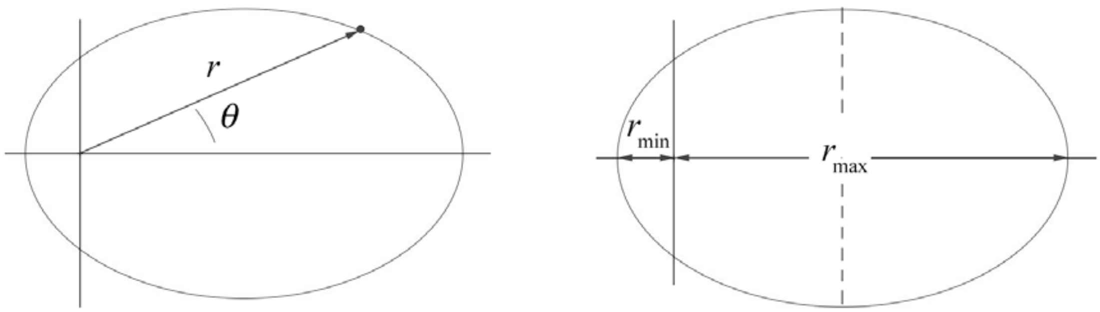
reproducing Equation (25.3.13). For $E = E_0$, $r = r_0$ which corresponds to a circular orbit. For the circular orbit $L = \mu r_0 v_0$, hence Eq. (25.4.14) can be cast into Newton's Second Law for the reduced mass undergoing a circular orbit:

$$\frac{G\,m_1 m_2}{r_0^{\,2}} = \mu \frac{v_0^{\,2}}{r_0}. \qquad (25.4.15)$$

**25.4.2 Elliptic Orbit $E_0 < E < 0$**




For $E_0 < E < 0$, there are two points $r_{\text{min}}$ and $r_{\text{max}}$ such that $E = U_{\text{eff}}(r_{\text{min}}) = U_{\text{eff}}(r_{\text{max}})$. At these points $K_{\text{eff}} = 0$, therefore $dr/dt = 0$ which corresponds to a point of closest or furthest approach (Figure 25.6). This condition corresponds to the minimum and maximum values of $r$ for an elliptic orbit.




(a) (b)

**Figure 25.6** (a) elliptic orbit, (b) closest and furthest approach

The energy condition at these two points

$$E = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}, \quad r = r_{\text{min}} = r_{\text{max}}, \tag{25.4.16}$$

is a quadratic equation for the distance $r$,

$$r^2 + \frac{G m_1 m_2}{E} r - \frac{L^2}{2\mu E} = 0. \tag{25.4.17}$$

There are two roots

$$r = -\frac{G m_1 m_2}{2E} \pm \left( \left( \frac{G m_1 m_2}{2E} \right)^2 + \frac{L^2}{2\mu E} \right)^{1/2}. \tag{25.4.18}$$

Equation (25.4.18) may be simplified somewhat as

$$r = -\frac{G m_1 m_2}{2E} \left( 1 \pm \left( 1 + \frac{2L^2 E}{\mu (G m_1 m_2)^2} \right)^{1/2} \right) \tag{25.4.19}$$

From Equation (25.3.14), the square root is the eccentricity $\varepsilon$,

$$\varepsilon = \left( 1 + \frac{2EL^2}{\mu (G m_1 m_2)^2} \right)^{\frac{1}{2}}, \tag{25.4.20}$$

and Equation (25.4.19) becomes

$$r = -\frac{G m_1 m_2}{2E}(1 \pm \varepsilon)\,. \tag{25.4.21}$$

A little algebra shows that

$$\begin{aligned}\frac{r_0}{1-\varepsilon^2} &= \frac{L^2 / \mu G m_1 m_2}{1 - \left(1 + \dfrac{2L^2 E}{\mu (G m_1 m_2)^2}\right)} \\ &= \frac{L^2 / \mu G m_1 m_2}{-2L^2 E / \mu (G m_1 m_2)^2} \\ &= -\frac{G m_1 m_2}{2E}.\end{aligned} \tag{25.4.22}$$

Substituting the last expression in (25.4.22) into Equation (25.4.21) gives an expression for the points of closest and furthest approach,

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon) = \frac{r_0}{1 \mp \varepsilon}\,. \tag{25.4.23}$$

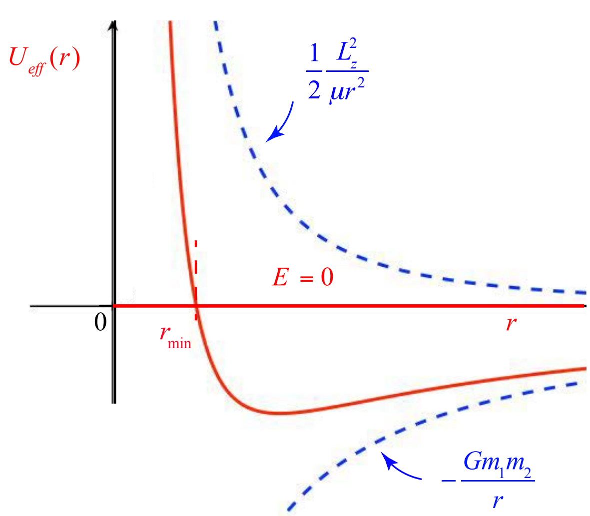
The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\min} = \frac{r_0}{1+\varepsilon} \tag{25.4.24}$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\max} = \frac{r_0}{1-\varepsilon}\,. \tag{25.4.25}$$

### 25.4.3 Parabolic Orbit $E = 0$



The effective potential energy, as given in Equation (25.4.1), approaches zero ($U_{\text{eff}} \to 0$) when the distance $r$ approaches infinity ($r \to \infty$). When $E = 0$, as $r \to \infty$, the kinetic energy also approaches zero, $K_{\text{eff}} \to 0$. This corresponds to a parabolic orbit (see Equation (25.3.23)). Recall that in order for a body to escape from a planet, the body must have an energy $E = 0$ (we set $U_{\text{eff}} = 0$ at infinity). This *escape velocity* condition corresponds to a parabolic orbit. For a parabolic orbit, the body also has a distance of closest approach. This distance $r_{\text{par}}$ can be found from the condition

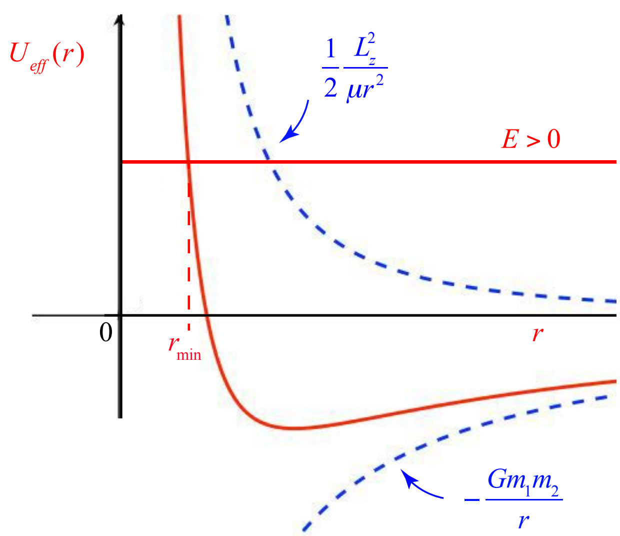
$$E = U_{\text{eff}}(r_{\text{par}}) = \frac{L^2}{2\mu r_{\text{par}}^{\ 2}} - \frac{G m_1 m_2}{r_{\text{par}}} = 0. \qquad (25.4.26)$$

Solving Equation (25.4.26) for $r_{\text{par}}$ yields

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0; \qquad (25.4.27)$$

the fact that the minimum distance to the origin (the *focus* of a parabola) is half the semilatus rectum is a well-known property of a parabola (Figure 25.5).

### 25.4.4 Hyperbolic Orbit $E > 0$




When $E > 0$, in the limit as $r \to \infty$ the kinetic energy is positive, $K_{\text{eff}} > 0$. This corresponds to a hyperbolic orbit (see Equation (25.3.24)). The condition for closest approach is similar to Equation (25.4.16) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (25.4.17), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1+\varepsilon}. \qquad (25.4.28)$$

The constant $r_0$ is independent of the energy and from Equation (25.3.14) as the energy of the single body increases, the eccentricity increases, and hence from Equation (25.4.28), the distance of closest approach gets smaller (Figure 25.5).

## 25.5 Orbits of the Two Bodies

The orbit of the single body can be circular, elliptical, parabolic or hyperbolic, depending on the values of the two constants of the motion, the angular momentum and the energy. Once we have the explicit solution (in this discussion, $r(\theta)$) for the single body, we can find the actual orbits of the two bodies.

Recall that in the center of mass reference frame the position vectors of bodies 1 and 2 are given by Eqs. (25.2.18), $\vec{\mathbf{r}}_1 = (\mu / m_1)\vec{\mathbf{r}}$, and (25.2.19), $\vec{\mathbf{r}}_2 = -(\mu / m_2)\vec{\mathbf{r}}$. Thus each body undergoes a motion about the center of mass in the same manner that the single body moves about the central point given by Equation (25.3.12). The only difference is that the distance from either body to the center of mass is shortened by a factor $\mu / m_i$. When the orbit of the single body is an ellipse, then the orbits of the two bodies are also ellipses, as shown in Figure 25.8.

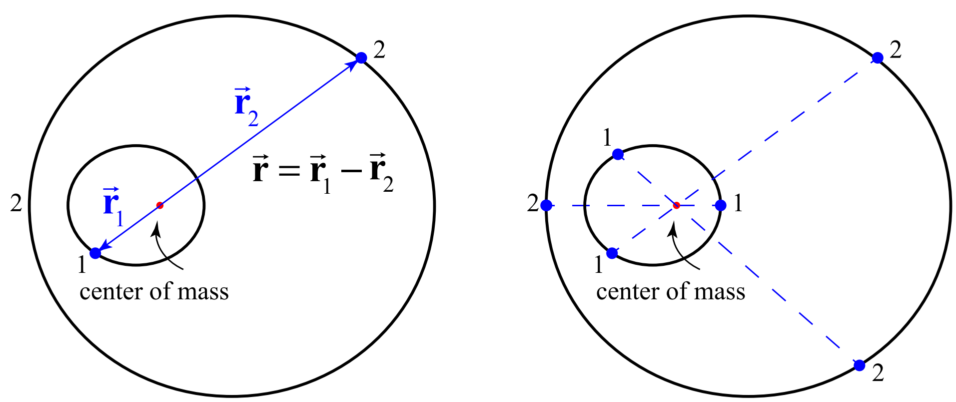



**Figure 25.8** The elliptical motion of bodies interacting gravitationally

When $m_1 << m_2$, then the reduced mass is approximately the smaller mass, $\mu \simeq m_1$. In that case, the center of mass is located approximately at the position of the larger mass, (body 2) and body 1 moves according to

$$\vec{\mathbf{r}}_1' = \frac{\mu}{m_1}\vec{\mathbf{r}} \cong \vec{\mathbf{r}}\,, \tag{25.4.29}$$

while body 2 is approximately stationary,

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}} - \frac{m_1}{m_2}\vec{\mathbf{r}} \cong \vec{\mathbf{0}}\,. \tag{25.4.30}$$

## 25.6 Kepler's Laws

### 25.6.1 Elliptic Orbit Law

*I. Each planet moves in an ellipse with the sun at one focus.*

When the energy is negative, $E < 0$, and according to Equation (25.3.14),

$$\varepsilon = \left( 1 + \frac{2EL^2}{\mu(Gm_1 m_2)^2} \right)^{\frac{1}{2}} \tag{25.5.1}$$

and the eccentricity must fall within the range $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Note the elliptic orbit law is only valid if we assume that there is only one central force acting. We are ignoring the gravitational interactions due to all the other bodies in the universe, a necessary approximation for our analytic solution.

### 25.6.2 Equal Area Law

*II. The radius vector from the sun to a planet sweeps out equal areas in equal time.*

Using analytic geometry in the limit of small $\Delta\theta$, the sum of the areas of the triangles in Figure 25.9 is given by

$$\Delta A = \frac{1}{2}(r\,\Delta\theta)r + \frac{(r\,\Delta\theta)}{2}\Delta r \tag{25.5.2}$$

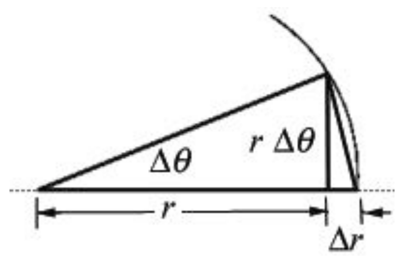



**Figure** 25.**9** Kepler's equal area law.

The average rate of the change of area, $\Delta A$, in time, $\Delta t$, is given by

$$\Delta A = \frac{1}{2}\frac{(r\,\Delta\theta)r}{\Delta t} + \frac{(r\,\Delta\theta)}{2}\frac{\Delta r}{\Delta t}. \tag{25.5.3}$$

In the limit as $\Delta t \to 0$, $\Delta\theta \to 0$, this becomes

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}. \tag{25.5.4}$$

Recall that according to Equation (25.3.7) (reproduced below as Equation (25.5.5)), the angular momentum is related to the angular velocity $d\theta / dt$ by

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \qquad (25.5.5)$$

and Equation (25.5.4) is then

$$\frac{dA}{dt} = \frac{L}{2\mu}. \qquad (25.5.6)$$

Because $L$ and $\mu$ are constants, the rate of change of area with respect to time is a constant. This is often familiarly referred to by the expression: *equal areas are swept out in equal times* (see Kepler's Laws at the beginning of this chapter).

### 25.6.3 Period Law

*III. The period of revolution $T$ of a planet about the sun is related to the semi-major axis $a$ of the ellipse by $T^2 = k\,a^3$ where $k$ is the same for all planets.*

When Kepler stated his period law for planetary orbits based on observation, he only noted the dependence on the larger mass of the sun. Because the mass of the sun is much greater than the mass of the planets, his observation is an excellent approximation.

In order to demonstrate the third law we begin by rewriting Equation (25.5.6) in the form

$$2\mu \frac{dA}{dt} = L. \qquad (25.5.7)$$

Equation (25.5.7) can be integrated as

$$\int_{\text{orbit}} 2\mu\, dA = \int_0^T L\, dt\,, \qquad (25.5.8)$$

where $T$ is the period of the orbit. For an ellipse,

$$\text{Area} = \int_{\text{orbit}} dA = \pi\, ab\,, \qquad (25.5.9)$$

where $a$ is the semi-major axis and $b$ is the semi-minor axis (Figure 25.10).

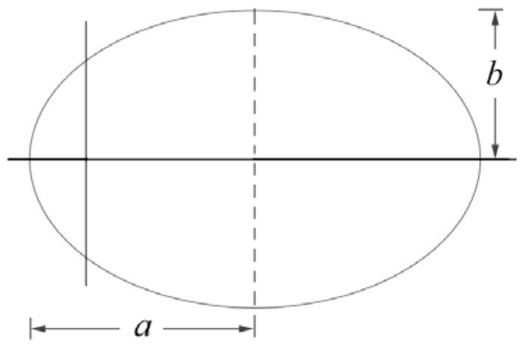



**Figure 25.10** Semi-major and semi-minor axis for an ellipse

Thus we have

$$T = \frac{2\mu \pi ab}{L}. \tag{25.5.10}$$

Squaring Equation (25.5.10) then yields

$$T^2 = \frac{4\pi^2 \mu^2 a^2 b^2}{L^2}. \tag{25.5.11}$$

In Appendix 25B, Equation (25.B.20) gives the angular momentum in terms of the semi-major axis and the eccentricity. Substitution for the angular momentum into Equation (25.5.11) yields

$$T^2 = \frac{4\pi^2 \mu^2 a^2 b^2}{\mu G m_1 m_2 a(1-\varepsilon^2)}. \tag{25.5.12}$$

In Appendix 25B, Equation (25.B.17) gives the semi-minor axis which upon substitution into Equation (25.5.12) yields

$$T^2 = \frac{4\pi^2 \mu^2 a^3}{\mu G m_1 m_2}. \tag{25.5.13}$$

Using Equation (25.2.1) for reduced mass, the square of the period of the orbit is proportional to the semi-major axis cubed,

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}. \tag{25.5.14}$$

## 25.7 Worked Examples

### Example 25.1 Elliptic Orbit

A satellite of mass $m_s$ is in an elliptical orbit around a planet of mass $m_p >> m_s$. The planet is located at one focus of the ellipse. The satellite is at the distance $r_a$ when it is furthest from the planet. The distance of closest approach is $r_p$ (Figure 25.11). What is (i) the speed $v_p$ of the satellite when it is closest to the planet and (ii) the speed $v_a$ of the satellite when it is furthest from the planet?

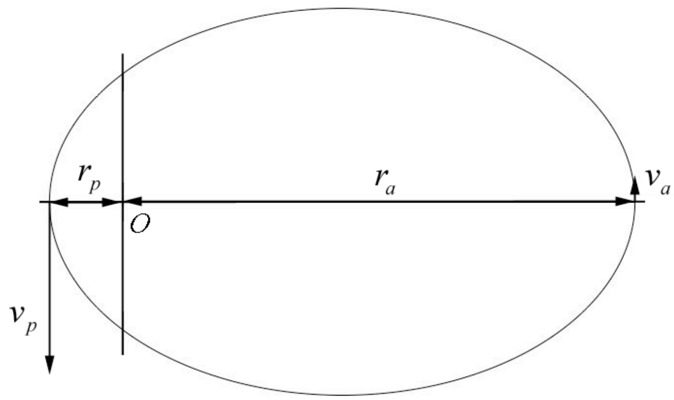



**Figure 25.11** Example 25.1

**Solution:** The angular momentum about the origin is constant and because $\vec{\mathbf{r}}_{O,a} \perp \vec{\mathbf{v}}_a$ and $\vec{\mathbf{r}}_{O,p} \perp \vec{\mathbf{v}}_p$, the magnitude of the angular momentums satisfies

$$L \equiv L_{O,p} = L_{O,a}. \tag{25.6.1}$$

Because $m_s << m_p$, the reduced mass $\mu \cong m_s$ and so the angular momentum condition becomes

$$L = m_s r_p v_p = m_s r_a v_a \tag{25.6.2}$$

We can solve for $v_p$ in terms of the constants $G$, $m_p$, $r_a$ and $r_p$ as follows. Choose zero for the gravitational potential energy, $U(r = \infty) = 0$. When the satellite is at the maximum distance from the planet, the mechanical energy is

$$E_a = K_a + U_a = \frac{1}{2} m_s v_a^{\,2} - \frac{G m_s m_p}{r_a}. \tag{25.6.3}$$

When the satellite is at closest approach the energy is

$$E_p = \frac{1}{2} m_s v_p^{\,2} - \frac{G m_s m_p}{r_p}. \tag{25.6.4}$$

Mechanical energy is constant,

$$E \equiv E_a = E_p, \tag{25.6.5}$$

therefore

$$E = \frac{1}{2} m_s v_p^{\,2} - \frac{G m_s m_p}{r_p} = \frac{1}{2} m_s v_a^{\,2} - \frac{G m_s m_p}{r_a}\,. \qquad (25.6.6)$$

From Eq. (25.6.2) we know that

$$v_a = (r_p / r_a) v_p\,. \qquad (25.6.7)$$

Substitute Eq. (25.6.7) into Eq. (25.6.6) and divide through by $m_s / 2$ yields

$$v_p^{\,2} - \frac{2Gm_p}{r_p} = \frac{r_p^{\,2}}{r_a^{\,2}} v_p^{\,2} - \frac{2Gm_p}{r_a}\,. \qquad (25.6.8)$$

We can solve this Eq. (25.6.8) for $v_p$ :

$$\begin{aligned}
& v_p^{\,2}\left(1 - \frac{r_p^{\,2}}{r_a^{\,2}}\right) = 2Gm_p\left(\frac{1}{r_p} - \frac{1}{r_a}\right) \Rightarrow \\
& v_p^{\,2}\left(\frac{r_a^{\,2} - r_p^{\,2}}{r_a^{\,2}}\right) = 2Gm_p\left(\frac{r_a - r_p}{r_p r_a}\right) \Rightarrow \\
& v_p^{\,2}\left(\frac{(r_a - r_p)(r_a + r_p)}{r_a^{\,2}}\right) = 2Gm_p\left(\frac{r_a - r_p}{r_p r_a}\right) \Rightarrow \\
& v_p = \sqrt{\frac{2Gm_p r_a}{(r_a + r_p) r_p}}.
\end{aligned} \qquad (25.6.9)$$

We now use Eq. (25.6.7) to determine that

$$v_a = (r_p / r_a) v_p = \sqrt{\frac{2Gm_p r_p}{(r_a + r_p) r_a}}\,. \qquad (25.6.10)$$

**Example 25.2 The Motion of the Star SO-2 around the Black Hole at the Galactic Center**

The UCLA Galactic Center Group, headed by Dr. Andrea Ghez, measured the orbits of many stars within $0.8'' \times 0.8''$ of the galactic center. The orbits of six of those stars are shown in Figure 25.12.

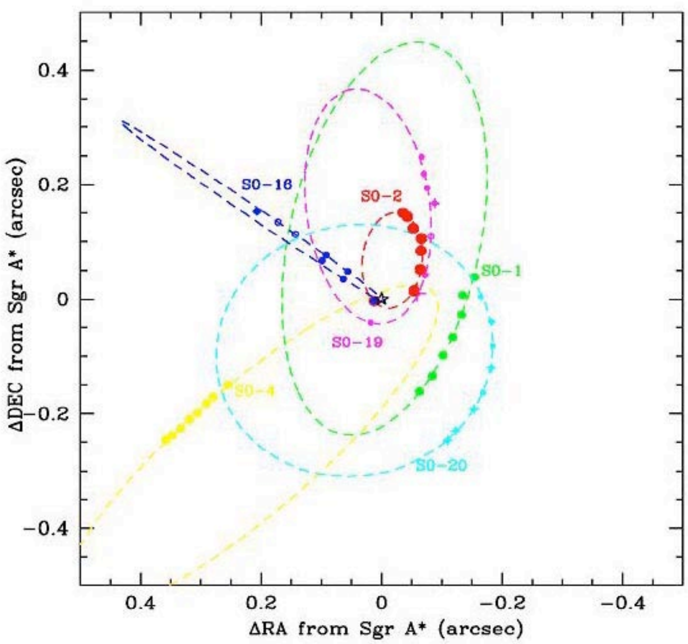



**Figure 25.12** Obits of six stars near black hole at center of Milky Way galaxy.

We shall focus on the orbit of the star S0-2 with the following orbit properties given in Table 25.1[3]. Distances are given in astronomical units, $1\text{au} = 1.50 \times 10^{11}\ \text{m}$, which is the mean distance between the earth and the sun.

**Table 25.1** Orbital Properties of S0-2

| Star | Period (yrs) | Eccentricity | Semi-major axis ($10^{-3}\text{arc sec}$) | Periapse (au) | Apoapse (au) |
|---|---|---|---|---|---|
| S0-2 | 15.2 (0.68/0.76) | 0.8763 (0.0063) | 120.7 (4.5) | 119.5 (3.9) | 1812 (73) |

The period of S0-2 satisfies Kepler's Third Law, given by

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}, \tag{25.6.11}$$

where $m_1$ is the mass of S0-2, $m_2$ is the mass of the black hole, and $a$ is the semi-major axis of the elliptic orbit of S0-2. (a) Determine the mass of the black hole that the star S0-2 is orbiting. What is the ratio of the mass of the black hole to the solar mass? (b) What is the speed of S0-2 at periapse (distance of closest approach to the center of the galaxy) and apoapse (distance of furthest approach to the center of the galaxy)?

[3] A.M.Ghez, et al., Stellar Orbits Around Galactic Center Black Hole, preprint arXiv:astro-ph/0306130v1, 5 June, 2003.

**Solution:** (a) The semi-major axis is given by

$$a = \frac{r_p + r_a}{2} = \frac{119.5\,\text{au} + 1812\,\text{au}}{2} = 965.8\,\text{au}\,. \qquad (25.6.12)$$

In SI units (meters), this is

$$a = 965.8\,\text{au}\,\frac{1.50\times 10^{11}\,\text{m}}{1\,\text{au}} = 1.45\times 10^{14}\,\text{m}\,. \qquad (25.6.13)$$

The mass $m_1$ of the star S0-2 is much less than the mass $m_2$ of the black hole, and Equation (25.6.11) can be simplified to

$$T^2 = \frac{4\pi^2 a^3}{G\,m_2}\,. \qquad (25.6.14)$$

Solving for the mass $m_2$ and inserting the numerical values, yields

$$\begin{aligned} m_2 &= \frac{4\pi^2\,a^3}{G\,T^2} \\ &= \frac{(4\pi^2)(1.45\times 10^{14}\,\text{m})^3}{(6.67\times 10^{-11}\,\text{N}\cdot\text{m}^2\cdot\text{kg}^{-2})((15.2\,\text{yr})(3.16\times 10^7\,\text{s}\cdot\text{yr}^{-1}))^2} \qquad (25.6.15)\\ &= 7.79\times 10^{34}\,\text{kg}. \end{aligned}$$

The ratio of the mass of the black hole to the solar mass is

$$\frac{m_2}{m_{\text{sun}}} = \frac{7.79\times 10^{34}\,\text{kg}}{1.99\times 10^{30}\,\text{kg}} = 3.91\times 10^6\,. \qquad (25.6.16)$$

The mass of black hole corresponds to nearly four million solar masses.

(b) We can use our results from Example 25.1 that

$$v_p = \sqrt{\frac{2Gm_2 r_a}{(r_a + r_p)r_p}} = \sqrt{\frac{Gm_2 r_a}{a r_p}} \qquad (25.6.17)$$

$$v_a = \frac{r_p}{r_a} v_p = \sqrt{\frac{2Gm_2 r_p}{(r_a + r_p)r_a}} = \sqrt{\frac{Gm_2 r_p}{a r_a}}\,, \qquad (25.6.18)$$

where $a = (r_a + r_b)/2$ is the semi-major axis. Inserting numerical values,

$$\begin{aligned} v_p &= \sqrt{\frac{Gm_2}{a}\frac{r_a}{r_p}} \\ &= \sqrt{\frac{(6.67\times 10^{-11}\,\mathrm{N\cdot m^2\cdot kg^{-2}})(7.79\times 10^{34}\,\mathrm{kg})}{(1.45\times 10^{14}\,\mathrm{m})}\left(\frac{1812}{119.5}\right)} \\ &= 7.38\times 10^{6}\,\mathrm{m\cdot s^{-1}}. \end{aligned} \tag{25.6.19}$$

The speed $v_{\mathrm{a}}$ at apoapse is then

$$v_a = \frac{r_p}{r_{\mathrm{a}}} v_p = \left(\frac{1812}{119.5}\right)(7.38\times 10^{6}\,\mathrm{m\cdot s^{-1}}) = 4.87\times 10^{5}\,\mathrm{m\cdot s^{-1}}. \tag{25.6.20}$$

**Example 25.3 Central Force Proportional to Distance Cubed**

A particle of mass $m$ moves in plane about a central point under an attractive central force of magnitude $F = br^3$ . The magnitude of the angular momentum about the central point is equal to $L$. (a) Find the effective potential energy and make sketch of effective potential energy as a function of $r$. (b) Indicate on a sketch of the effective potential the total energy for circular motion. (c) The radius of the particle's orbit varies between $r_0$ and $2r_0$. Find $r_0$.

**Solution:** a) The potential energy, taking the zero of potential energy to be at $r = 0$, is

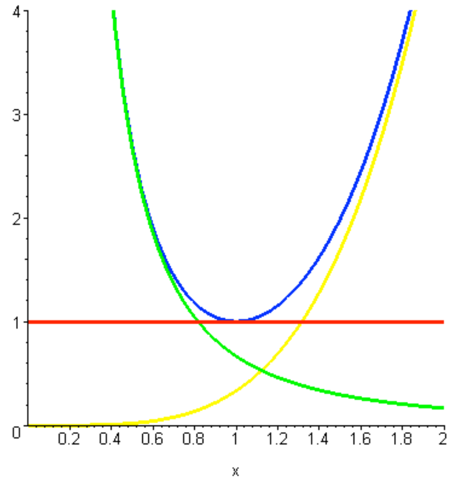
$$U(r) = -\int_0^r (-br'^3)\,dr' = \frac{b}{4}r^4$$

The effective potential energy is

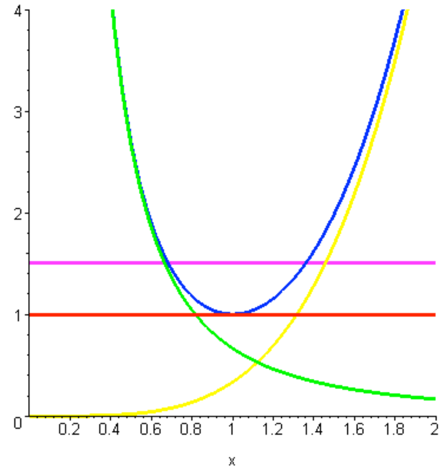
$$U_{\mathrm{eff}}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} + \frac{b}{4}r^4 .$$

A plot is shown in Figure 25.13a, including the potential (yellow, right-most curve), the term $L^2/2m$ (green, left-most curve) and the effective potential (blue, center curve). The horizontal scale is in units of $r_0$ (corresponding to radius of the lowest energy circular orbit) and the vertical scale is in units of the minimum effective potential.

b) The minimum effective potential energy is the horizontal line (red) in Figure 25.13a.



(a)



(b)

**Figure 25.13** (a) Effective potential energy with lowest energy state (red line), (b) higher energy state (magenta line)

c) We are trying to determine the value of $r_0$ such that $U_{\text{eff}}(r_0) = U_{\text{eff}}(2r_0)$. Thus

$$\frac{L^2}{mr_0^2} + \frac{b}{4}r_0^4 = \frac{L^2}{m(2r_0)^2} + \frac{b}{4}(2r_0)^4 .$$

Rearranging and combining terms, we can then solve for $r_0$,

$$\frac{3}{8}\frac{L^2}{m}\frac{1}{r_0^2} = \frac{15}{4}b\,r_0^4$$

$$r_0^6 = \frac{1}{10}\frac{L^2}{mb}.$$

In the plot in Figure 25.13b, if we could move the red line up until it intersects the blue curve at two point whose value of the radius differ by a factor of $2$, those would be the respective values for $r_0$ and $2r_0$. A graph, showing the corresponding energy as the horizontal magenta line, is shown in Figure 25.13b.

**Example 25.4 Transfer Orbit**

A space vehicle is in a circular orbit about the earth. The mass of the vehicle is $m_{\text{s}} = 3.00\times10^3\ \text{kg}$ and the radius of the orbit is $2R_{\text{e}} = 1.28\times10^4\ \text{km}$. It is desired to transfer the vehicle to a circular orbit of radius $4R_{\text{e}}$ (Figure 24.14). The mass of the earth is $M_{\text{e}} = 5.97\times10^{24}\ \text{kg}$. (a) What is the minimum energy expenditure required for the transfer? (b) An efficient way to accomplish the transfer is to use an elliptical orbit from point $A$ on the inner circular orbit to a point $B$ on the outer circular orbit (known as a

Hohmann transfer orbit). What changes in speed are required at the points of intersection, $A$ and $B$?

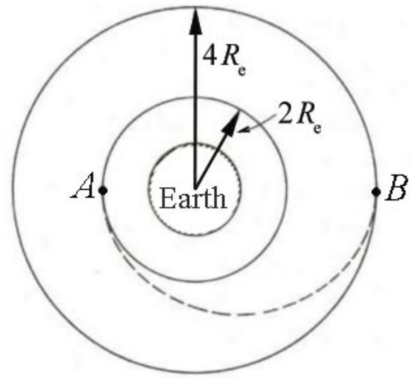



**Figure 24.12** Example 25.5

**Solution:** (a) The mechanical energy is the sum of the kinetic and potential energies,

$$\begin{aligned} E &= K + U \\ &= \frac{1}{2} m_s v^2 - G\frac{m_s M_e}{R_e}. \end{aligned} \tag{25.6.21}$$

For a circular orbit, the orbital speed and orbital radius must be related by Newton's Second Law,

$$\begin{aligned} & F_r = m a_r \\ & -G\frac{m_s M_e}{R_e^{\,2}} = -m_s \frac{v^2}{R_e} \Rightarrow \\ & \frac{1}{2} m_s v^2 = \frac{1}{2} G\frac{m_s M_e}{R_e}. \end{aligned} \tag{25.6.22}$$

Substituting the last result in (25.6.22) into Equation (25.6.21) yields

$$E = \frac{1}{2} G\frac{m_s M_e}{R_e} - G\frac{m_s M_e}{R_e} = -\frac{1}{2} G\frac{m_s M_e}{R_e} = \frac{1}{2} U(R_e). \tag{25.6.23}$$

Equation (25.6.23) is one example of what is known as the ***Virial Theorem***, in which the energy is equal to (1/2) the potential energy for the circular orbit. In moving from a circular orbit of radius $2R_e$ to a circular orbit of radius $4R_e$, the total energy increases, (as the energy becomes less negative). The change in energy is

$$\begin{aligned}\Delta E &= E(r=4R_e) - E(r=2R_e) \\ &= -\frac{1}{2}G\frac{m_s M_e}{4R_e} - \left(-\frac{1}{2}G\frac{m_s M_e}{2R_e}\right) \\ &= \frac{Gm_s M_e}{8R_e}.\end{aligned} \tag{25.6.24}$$

Inserting the numerical values,

$$\begin{aligned}\Delta E &= \frac{1}{8}G\frac{m_s M_e}{R_e} = \frac{1}{4}G\frac{m_s M_e}{2R_e} \\ &= \frac{1}{4}(6.67\times10^{-11}\,\text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2})\frac{(3.00\times10^3\text{ kg})(5.97\times10^{24}\text{ kg})}{(1.28\times10^4\text{ km})} \\ &= 2.3\times10^{10}\text{ J}.\end{aligned} \tag{25.6.25}$$

b) The satellite must increase its speed at point $A$ in order to move to the larger orbit radius and increase its speed again at point $B$ to stay in the new circular orbit. Denote the satellite speed at point $A$ while in the circular orbit as $v_{A,i}$ and after the speed increase (a "rocket burn") as $v_{A,f}$. Similarly, denote the satellite's speed when it first reaches point $B$ as $v_{B,i}$. Once the satellite reaches point $B$, it then needs to increase its speed in order to continue in a circular orbit. Denote the speed of the satellite in the circular orbit at point $B$ by $v_{B,f}$. The speeds $v_{A,i}$ and $v_{B,f}$ are given by Equation (25.6.22). While the satellite is moving from point $A$ to point $B$ in the elliptic orbit (that is, during the transfer, after the first burn and before the second), both mechanical energy and angular momentum are conserved. Conservation of energy relates the speeds and radii by

$$\frac{1}{2}m_s(v_{A,f})^2 - G\frac{m_s m_e}{2R_e} = \frac{1}{2}m_s(v_{B,i})^2 - G\frac{m_s m_e}{4R_e}. \tag{25.6.26}$$

Conservation of angular momentum relates the speeds and radii by

$$m_s v_{A,f}(2R_e) = m_s v_{B,i}(4R_e) \;\Rightarrow\; v_{A,f} = 2v_{B,i}. \tag{25.6.27}$$

Substitution of Equation (25.6.27) into Equation (25.6.26) yields, after minor algebra,

$$v_{A,f} = \sqrt{\frac{2}{3}\frac{GM_e}{R_e}}, \quad v_{B,i} = \sqrt{\frac{1}{6}\frac{GM_e}{R_e}}\,. \tag{25.6.28}$$

We can now use Equation (25.6.22) to determine that

$$v_{A,i} = \sqrt{\frac{1}{2}\frac{G\,M_e}{R_e}}, \quad v_{B,f} = \sqrt{\frac{1}{4}\frac{G\,M_e}{R_e}}\,. \tag{25.6.29}$$

Thus the change in speeds at the respective points is given by

$$\begin{aligned} \Delta v_A &= v_{A,f} - v_{A,i} = \left(\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{2}}\right)\sqrt{\frac{G\,M_e}{R_e}} \\ \Delta v_B &= v_{B,f} - v_{B,i} = \left(\sqrt{\frac{1}{4}} - \sqrt{\frac{1}{6}}\right)\sqrt{\frac{G\,M_e}{R_e}}. \end{aligned} \tag{25.6.30}$$

Substitution of numerical values gives

$$\Delta v_A = 8.6 \times 10^2\ \mathrm{m \cdot s^{-2}}, \quad \Delta v_B = 7.2 \times 10^2\ \mathrm{m \cdot s^{-2}}\,. \tag{25.6.31}$$

## Appendix 25A Derivation of the Orbit Equation

### 25A.1 Derivation of the Orbit Equation: Method 1

Start from Equation (25.3.11) in the form

$$d\theta = \frac{L}{\sqrt{2\mu}} \frac{(1/r^2)}{\left(E - \dfrac{L^2}{2\mu r^2} + \dfrac{G m_1 m_2}{r}\right)^{1/2}} dr\,. \qquad (25.A.1)$$

What follows involves a good deal of hindsight, allowing selection of convenient substitutions in the math in order to get a clean result. First, note the many factors of the reciprocal of $r$. So, we'll try the substitution $u = 1/r$, $du = -(1/r^2)\,dr$, with the result

$$d\theta = -\frac{L}{\sqrt{2\mu}} \frac{du}{\left(E - \dfrac{L^2}{2\mu} u^2 + G m_1 m_2 u\right)^{1/2}}\,. \qquad (25.A.2)$$

Experience in evaluating integrals suggests that we make the absolute value of the factor multiplying $u^2$ inside the square root equal to unity. That is, multiplying numerator and denominator by $\sqrt{2\mu}/L$,

$$d\theta = -\frac{du}{\left(2\mu E/L^2 - u^2 + 2(\mu G m_1 m_2/L^2)u\right)^{1/2}}\,. \qquad (25.A.3)$$

As both a check and a motivation for the next steps, note that the left side $d\theta$ of Equation (25.A.3) is dimensionless, and so the right side must be. This means that the factor of $\mu G m_1 m_2 / L^2$ in the square root must have the same dimensions as $u$, or $\text{length}^{-1}$; so, define $r_0 \equiv L^2/\mu G m_1 m_2$. This is of course the *semilatus rectum* as defined in Equation (25.3.12), and it's no coincidence; this is part of the "hindsight" mentioned above. The differential equation then becomes

$$d\theta = -\frac{du}{(2\mu E/L^2 - u^2 + 2u/r_0)^{1/2}}\,. \qquad (25.A.4)$$

We now rewrite the denominator in order to express it terms of the eccentricity.

$$d\theta = -\frac{du}{\left(2\mu E / L^2 + 1/r_0^{\ 2} - u^2 + 2u/r_0 - 1/r_0^{\ 2}\right)^{1/2}}$$

$$= -\frac{du}{\left(2\mu E / L^2 + 1/r_0^{\ 2} - (u - 1/r_0)^2\right)^{1/2}} \qquad (25.A.5)$$

$$= -\frac{r_0\, du}{\left(2\mu E r_0^{\ 2} / L^2 + 1 - (r_0\, u - 1)^2\right)^{1/2}}.$$

We note that the combination of terms $2\mu E r_0^{\ 2} / L^2 + 1$ is dimensionless, and is in fact equal to the square of the eccentricity $\varepsilon$ as defined in Equation (25.3.13); more hindsight. The last expression in (25.A.5) is then

$$d\theta = -\frac{r_0\, du}{\left(\varepsilon^2 - (r_0\, u - 1)^2\right)^{1/2}}. \qquad (25.A.6)$$

From here, we'll combine a few calculus steps, going immediately to the substitution $r_0\, u - 1 = \varepsilon \cos\alpha$ , $r_0\, du = -\varepsilon \sin\alpha\, d\alpha$, with the final result that

$$d\theta = -\frac{-\varepsilon \sin\alpha\, d\alpha}{\left(\varepsilon^2 - \varepsilon^2 \cos^2\alpha\right)^{1/2}} = d\alpha, \qquad (25.A.7)$$

We now integrate Eq. (25.A.7) with the very simple result that

$$\theta = \alpha + \text{constant}. \qquad (25.A.8)$$

We have a choice in selecting the constant, and if we pick $\theta = \alpha - \pi$ , $\alpha = \theta + \pi$ , $\cos\alpha = -\cos\theta$ , the result is

$$r = \frac{1}{u} = \frac{r_0}{1 - \varepsilon \cos\theta}, \qquad (25.A.9)$$

which is our desired result, Equation (25.3.11). Note that if we chose the constant of integration to be zero, the result would be

$$r = \frac{1}{u} = \frac{r_0}{1 + \varepsilon \cos\theta} \qquad (25.A.10)$$

which is the same trajectory reflected about the "vertical" axis in Figure 25.3, indeed the same as rotating by $\pi$ .

**25A.2 Derivation of the Orbit Equation: Method 2**

The derivation of Equation (25.A.9) in the form

$$u = \frac{1}{r_0}(1 - \varepsilon\cos\theta) \quad (25.A.11)$$

suggests that the equation of motion for the one-body problem might be manipulated to obtain a simple differential equation. That is, start from

$$\vec{\mathbf{F}} = \mu\vec{\mathbf{a}}$$
$$-G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} = \mu\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{\mathbf{r}}. \quad (25.A.12)$$

Setting the components equal, using the constant of motion $L = \mu r^2 (d\theta / dt)$ and rearranging, Eq. (25.A.12) becomes

$$\mu\frac{d^2 r}{dt^2} = \frac{L^2}{\mu r^3} - \frac{G m_1 m_2}{r^2}. \quad (25.A.13)$$

We now use the same substitution $u = 1/r$ and change the independent variable from $t$ to $r$, using the chain rule twice, since Equation (25.A.13) is a second-order equation. That is, the first time derivative is

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}. \quad (25.A.14)$$

From $r = 1/u$ we have $dr/du = -1/u^2$. Combining with $d\theta/dt$ in terms of $L$ and $u$, $d\theta/dt = Lu^2/\mu$, Equation (25.A.14) becomes

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{\mu} = -\frac{du}{d\theta}\frac{L}{\mu}, \quad (25.A.15)$$

a very tidy result, with the variable $u$ appearing linearly. Taking the second derivative with respect to $t$,

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt}. \quad (25.A.16)$$

Now substitute Eq. (25.A.15) into Eq. (25.A.16) with the result that

$$\frac{d^2 r}{dt^2} = -\frac{d^2 u}{d\theta^2}\left(u^2\frac{L^2}{\mu^2}\right). \quad (25.A.17)$$

Substituting into Equation (25.A.13), with $r = 1/u$ yields

$$-\frac{d^2u}{d\theta^2}u^2\frac{L^2}{\mu}=\frac{L^2}{\mu}u^3-Gm_1m_2\,u^2. \tag{25.A.18}$$

Canceling the common factor of $u^2$ and rearranging, we arrive at

$$-\frac{d^2u}{d\theta^2}=u-\frac{\mu Gm_1m_2}{L^2}\,. \tag{25.A.19}$$

Equation (25.A.19) is mathematically equivalent to the simple harmonic oscillator equation with an additional constant term. The solution consists of two parts: the angle-independent solution

$$u_0=\frac{\mu Gm_1m_2}{L^2} \tag{25.A.20}$$

and a sinusoidally varying term of the form

$$u_{\mathrm{H}}=A\cos(\theta-\theta_0)\,, \tag{25.A.21}$$

where $A$ and $\theta_0$ are constants determined by the form of the orbit. The expression in Equation (25.A.20) is the ***inhomogeneous solution*** and represents a circular orbit. The expression in Equation (25.A.21) is the ***homogeneous solution*** (as hinted by the subscript) and must have two independent constants. We can readily identify $1/u_0$ as the *semilatus rectum* $r_0$, with the result that

$$\begin{aligned}&u=u_0+u_{\mathrm{H}}=\frac{1}{r_0}\big(1+r_0A(\theta-\theta_0)\big)\Rightarrow\\&r=\frac{1}{u}=\frac{r_0}{1+r_0A(\theta-\theta_0)}.\end{aligned} \tag{25.A.22}$$

Choosing the product $r_0A$ to be the eccentricity $\varepsilon$ and $\theta_0=\pi$ (much as was done leading to Equation (25.A.9) above), Equation (25.A.9) is reproduced.

# Appendix 25B Properties of an Elliptical Orbit

## 25B.1 Coordinate System for the Elliptic Orbit

We now consider the special case of an elliptical orbit. Choose coordinates with the central point located at one focal point and coordinates $(r,\theta)$ for the position of the single body (Figure 25B.1a). In Figure 25B.1b, let $a$ denote the semi-major axis, $b$ denote the semi-minor axis and $x_0$ denote the distance from the center of the ellipse to the origin of our coordinate system.

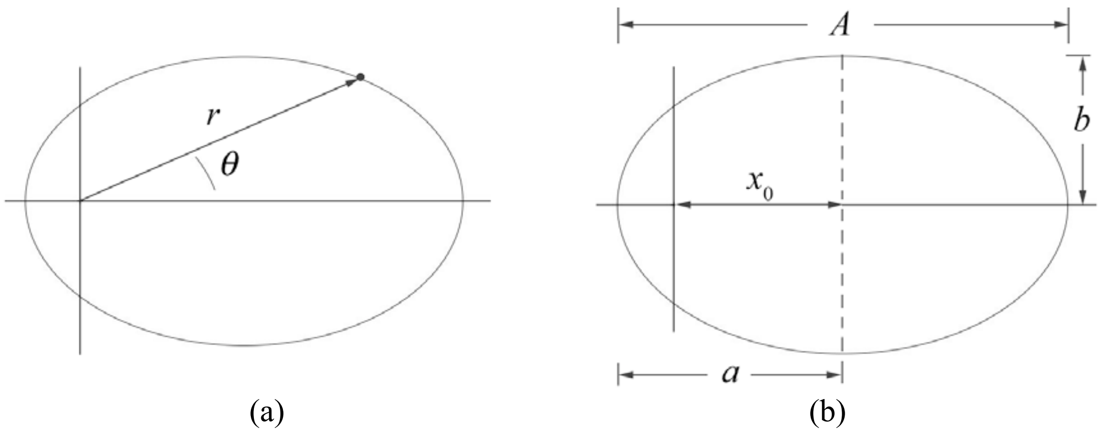



**Figure 25B.1** (a) Coordinate system for elliptic orbit, (b) semi-major axis

## 25B.2 The Semi-major Axis

Recall the orbit equation, Eq, (25.A.9), describes $r(\theta)$,

$$r(\theta) = \frac{r_0}{1-\varepsilon\cos\theta}. \tag{25.B.1}$$

The major axis $A = 2a$ is given by

$$A = 2a = r_a + r_p. \tag{25.B.2}$$

where the distance of furthest approach $r_a$ occurs when $\theta = 0$, hence

$$r_a = r(\theta = 0) = \frac{r_0}{1-\varepsilon}, \tag{25.B.3}$$

and the distance of nearest approach $r_p$ occurs when $\theta = \pi$, hence

$$r_p = r(\theta = \pi) = \frac{r_0}{1+\varepsilon}. \tag{25.B.4}$$

Figure 25B.2 shows the distances of nearest and furthest approach.

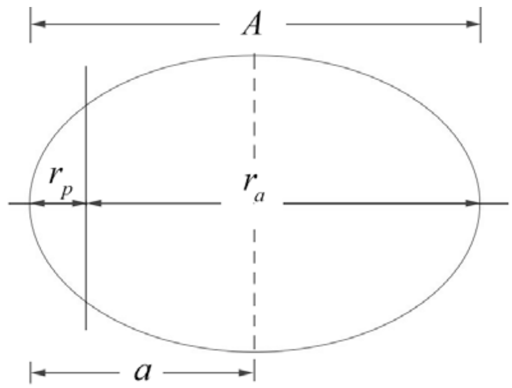



**Figure 25B.2** Furthest and nearest approach

We can now determine the semi-major axis

$$a = \frac{1}{2}\left( \frac{r_0}{1-\varepsilon} + \frac{r_0}{1+\varepsilon} \right) = \frac{r_0}{1-\varepsilon^2} . \tag{25.B.5}$$

The *semilatus rectum* $r_0$ can be re-expressed in terms of the semi-major axis and the eccentricity,

$$r_0 = a(1-\varepsilon^2) . \tag{25.B.6}$$

We can now express the distance of nearest approach, Equation (25.B.4), in terms of the semi-major axis and the eccentricity,

$$r_p = \frac{r_0}{1+\varepsilon} = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon) . \tag{25.B.7}$$

In a similar fashion the distance of furthest approach is

$$r_a = \frac{r_0}{1-\varepsilon} = \frac{a(1-\varepsilon^2)}{1-\varepsilon} = a(1+\varepsilon) . \tag{25.B.8}$$

### 25B.2.3 The Location $x_0$ of the Center of the Ellipse

From Figure 25B.3a, the distance from a focus point to the center of the ellipse is

$$x_0 = a - r_p . \tag{25.B.9}$$

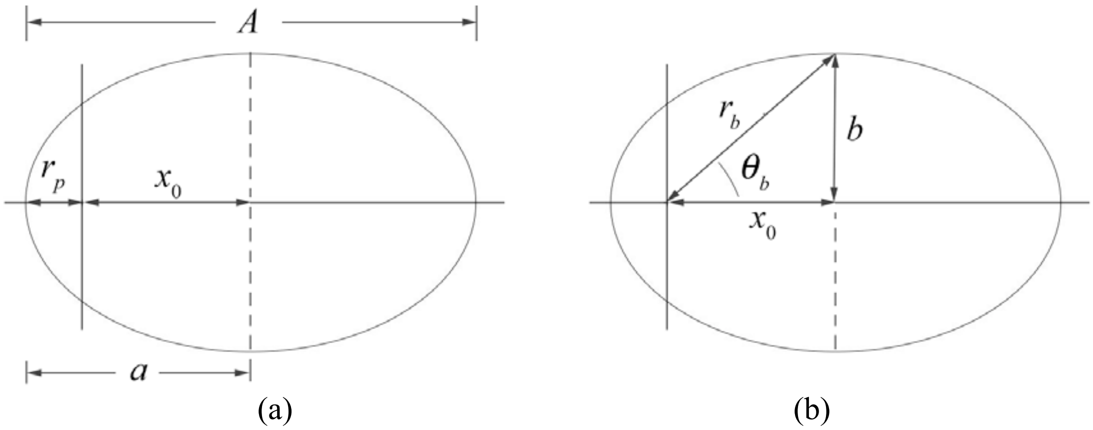



**Figure 25B.3** Location of the center of the ellipse and semi-minor axis.

Using Equation (25.B.7) for $r_p$, we have that

$$x_0 = a - a(1-\varepsilon) = \varepsilon\, a. \tag{25.B.10}$$

### 25B.2.4 The Semi-minor Axis

From Figure 25B.3b, the semi-minor axis can be expressed as

$$b = \sqrt{(r_b^{\;2} - x_0^{\;2})}\,, \tag{25.B.11}$$

where

$$r_b = \frac{r_0}{1-\varepsilon\cos\theta_b}\,. \tag{25.B.12}$$

We can rewrite Eq. (25.B.12) as

$$r_b - r_b\,\varepsilon\cos\theta_b = r_0\,. \tag{25.B.13}$$

The horizontal projection of $r_b$ is given by (Figure 25B.2b),

$$x_0 = r_b\cos\theta_b\,, \tag{25.B.14}$$

which upon substitution into Eq. (25.B.13) yields

$$r_b = r_0 + \varepsilon\, x_0. \tag{25.B.15}$$

Substituting Equation (25.B.10) for $x_0$ and Equation (25.B.6) for $r_0$ into Equation (25.B.15) yields

$$r_b = a(1-\varepsilon^2) + a\varepsilon^2 = a\,. \tag{25.B.16}$$

The fact that $r_b = a$ is a well-known property of an ellipse reflected in the geometric construction, that the sum of the distances from the two foci to any point on the ellipse is a constant. We can now determine the semi-minor axis $b$ by substituting Eq. (25.B.16) into Eq. (25.B.11) yielding

$$b = \sqrt{\left(r_b^{\ 2} - x_0^{\ 2}\right)} = \sqrt{a^2 - \varepsilon^2 a^2} = a\sqrt{1-\varepsilon^2}\,. \qquad (25.B.17)$$

### 25B.2.5 Constants of the Motion for Elliptic Motion

We shall now express the parameters $a$, $b$ and $x_0$ in terms of the constants of the motion $L$, $E$, $\mu$, $m_1$ and $m_2$. Using our results for $r_0$ and $\varepsilon$ from Equations (25.3.13) and (25.3.14) we have for the semi-major axis

$$\begin{aligned} a &= \frac{L^2}{\mu G m_1 m_2} \frac{1}{(1-(1+2EL^2/\mu(Gm_1 m_2)^2))} \\ &= -\frac{G m_1 m_2}{2E} \end{aligned}. \qquad (25.B.18)$$

The energy is then determined by the semi-major axis,

$$E = -\frac{G m_1 m_2}{2a}. \qquad (25.B.19)$$

The angular momentum is related to the *semilatus rectum* $r_0$ by Equation (25.3.13). Using Equation (25.B.6) for $r_0$, we can express the angular momentum (25.B.4) in terms of the semi-major axis and the eccentricity,

$$L = \sqrt{\mu G m_1 m_2 r_0} = \sqrt{\mu G m_1 m_2 a(1-\varepsilon^2)}\,. \qquad (25.B.20)$$

Note that

$$\sqrt{(1-\varepsilon^2)} = \frac{L}{\sqrt{\mu G m_1 m_2 a}}\,, \qquad (25.B.21)$$

Thus, from Equations (25.3.14), (25.B.10), and (25.B.18), the distance from the center of the ellipse to the focal point is

$$x_0 = \varepsilon a = -\frac{G m_1 m_2}{2E}\sqrt{\left(1+2EL^2/\mu\left(G m_1 m_2\right)^2\right)}, \qquad (25.B.22)$$

a result we will return to later. We can substitute Eq. (25.B.21) for $\sqrt{1-\varepsilon^2}$ into Eq. (25.B.17), and determine that the semi-minor axis is

$$b=\sqrt{aL^2/\mu Gm_1m_2}\,. \qquad (25.B.23)$$

We can now substitute Eq. (25.B.18) for $a$ into Eq. (25.B.23), yielding

$$b=\sqrt{aL^2/\mu Gm_1m_2}=L\sqrt{-\frac{Gm_1m_2}{2E}/\mu Gm_1m_2}=L\sqrt{-\frac{1}{2\mu E}}\,. \qquad (25.B.24)$$

### 25B.2.6 Speeds at Nearest and Furthest Approaches

At nearest approach, the velocity vector is tangent to the orbit (Figure 25B.4), so the magnitude of the angular momentum is

$$L=\mu r_p v_p\,, \qquad (25.B.25)$$

and the speed at nearest approach is

$$v_p=L/\mu r_p\,. \qquad (25.B.26)$$

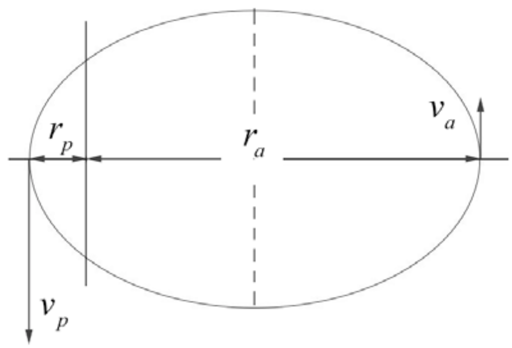



**Figure 25B.4** Speeds at nearest and furthest approach

Using Equation (25.B.20) for the angular momentum and Equation (25.B.7) for $r_p$, Equation (25.B.26) becomes

$$v_p=\frac{L}{\mu r_p}=\frac{\sqrt{\mu Gm_1m_2(1-\varepsilon^2)}}{\mu a(1-\varepsilon)}=\sqrt{\frac{Gm_1m_2(1-\varepsilon^2)}{\mu a(1-\varepsilon)^2}}=\sqrt{\frac{Gm_1m_2(1+\varepsilon)}{\mu a(1-\varepsilon)}}\,. \qquad (25.B.27)$$

A similar calculation show that the speed $v_a$ at furthest approach,

$$v_a=\frac{L}{\mu r_a}=\frac{\sqrt{\mu Gm_1m_2(1-\varepsilon^2)}}{\mu a(1+\varepsilon)}=\sqrt{\frac{Gm_1m_2 1-\varepsilon^2}{\mu a(1+\varepsilon)^2}}=\sqrt{\frac{Gm_1m_2\left(1-\varepsilon\right)}{\mu a(1+\varepsilon)}}\,. \qquad (25.B.28)$$

## Appendix 25C Analytic Geometric Properties of Ellipses

Consider Equation (25.3.20), and for now take $\varepsilon < 1$, so that the equation is that of an ellipse. We shall now show that we can write it as

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1\,, \qquad (25.C.1)$$

where the ellipse has axes parallel to the $x$- and $y$-coordinate axes, center at $(x_0, 0)$, semi-major axis $a$ and semi-minor axis $b$. We begin by rewriting Equation (25.3.20) as

$$x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2}\,. \qquad (25.C.2)$$

We next complete the square,

$$\begin{gathered}
x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{1-\varepsilon^2} + \frac{\varepsilon^2 r_0^2}{(1-\varepsilon^2)^2} \Rightarrow \\
\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{r_0^2}{(1-\varepsilon^2)^2} \Rightarrow \\
\frac{\left(x - \dfrac{\varepsilon r_0}{1-\varepsilon^2}\right)^2}{\left(r_0/(1-\varepsilon^2)\right)^2} + \frac{y^2}{(r_0/\sqrt{1-\varepsilon^2}\,)^2} = 1.
\end{gathered} \qquad (25.C.3)$$

The last expression in (25.C.3) is the equation of an ellipse with semi-major axis

$$a = \frac{r_0}{1-\varepsilon^2}\,, \qquad (25.C.4)$$

semi-minor axis

$$b = \frac{r_0}{\sqrt{1-\varepsilon^2}} = a\sqrt{1-\varepsilon^2}\,, \qquad (25.C.5)$$

and center at

$$x_0 = \frac{\varepsilon r_0}{(1-\varepsilon^2)} = \varepsilon a\,, \qquad (25.C.6)$$

as found in Equation (25.B.10).