

## Chapter 17 Two Dimensional Rotational Dynamics

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## Chapter 17 Two Dimensional Rotational Dynamics

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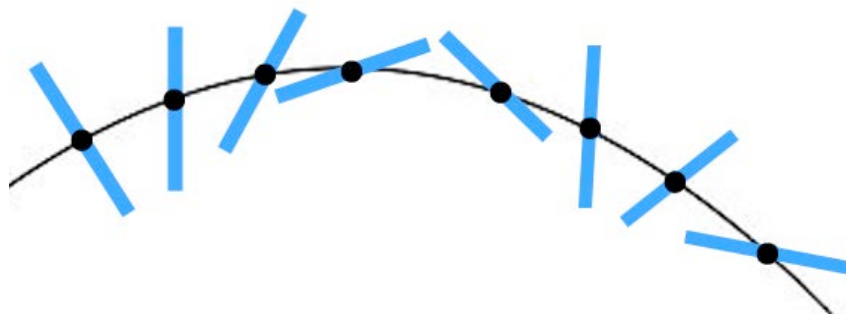
*a. The twisting or rotary force in a piece of mechanism (as a measurable quantity); the moment of a system of forces producing rotation.*

*Oxford English Dictionary*

### 17.1 Introduction

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Forces that are distributed over a body are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we shall demonstrate that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that  $\vec{g}$  may be taken as constant within the body) is identical to the *center of mass* of the body.

Let's consider a rigid rod thrown in the air (Figure 17.1) so that the rod is spinning as its center of mass moves with velocity  $\vec{v}_{\text{cm}}$ . We have explored the physics of translational motion; now, we wish to investigate the properties of rotational motion exhibited in the rod's motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.



**Figure 17.1** The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We can use Newton's Second Law to predict how the center of mass will move. Because the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is the moment of inertia. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space.

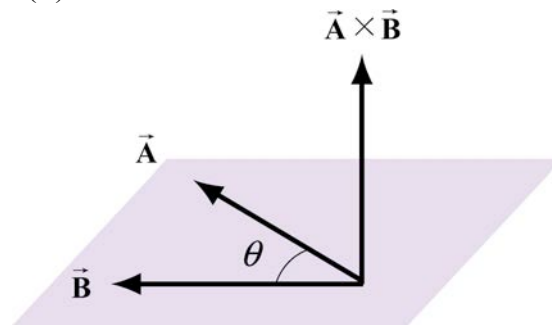
In order to describe the relationship between torque, moment of inertia, and angular acceleration, we will introduce a new vector operation called the **vector product** also known as the “cross product” that takes any two vectors and generates a new vector. The vector product is a type of “multiplication” law that turns our vector space (law for addition of vectors) into a vector algebra (a vector algebra is a vector space with an additional rule for multiplication of vectors).

## 17.2 Vector Product (Cross Product)

Let  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  be two vectors. Because any two non-parallel vectors form a plane, we denote the angle  $\theta$  to be the angle between the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  as shown in Figure 17.2. The **magnitude of the vector product**  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$  of the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  is defined to be product of the magnitude of the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  with the sine of the angle  $\theta$  between the two vectors,

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}}| |\vec{\mathbf{B}}| \sin(\theta). \quad (17.2.1)$$

The angle  $\theta$  between the vectors is limited to the values  $0 \leq \theta \leq \pi$  ensuring that  $\sin(\theta) \geq 0$ .



**Figure 17.2** Vector product geometry.

The direction of the vector product is defined as follows. The vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  form a plane. Consider the direction perpendicular to this plane. There are two possibilities: we shall choose one of these two (the one shown in Figure 17.2) for the direction of the vector product  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$  using a convention that is commonly called the “**right-hand rule**”.

### 17.2.1 Right-hand Rule for the Direction of Vector Product

The first step is to redraw the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  so that the tails are touching. Then draw an arc starting from the vector  $\vec{\mathbf{A}}$  and finishing on the vector  $\vec{\mathbf{B}}$ . Curl your right fingers the same way as the arc. Your right thumb points in the direction of the vector product  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$  (Figure 17.3).

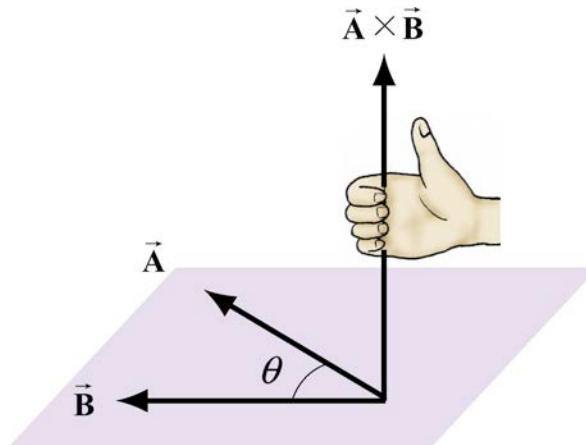


Figure 17.3 Right-Hand Rule.

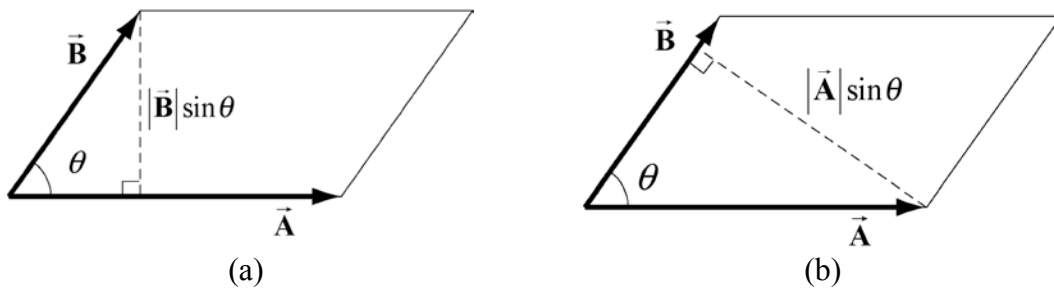
You should remember that the direction of the vector product  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$  is perpendicular to the plane formed by  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$ . We can give a geometric interpretation to the magnitude of the vector product by writing the magnitude as

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}}| (|\vec{\mathbf{B}}| \sin \theta). \quad (17.2.2)$$

The vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  form a parallelogram. The area of the parallelogram is equal to the height times the base, which is the magnitude of the vector product. In Figure 17.4, two different representations of the height and base of a parallelogram are illustrated. As depicted in Figure 17.4a, the term  $|\vec{\mathbf{B}}| \sin \theta$  is the projection of the vector  $\vec{\mathbf{B}}$  in the direction perpendicular to the vector  $\vec{\mathbf{A}}$ . We could also write the magnitude of the vector product as

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = (|\vec{\mathbf{A}}| \sin \theta) |\vec{\mathbf{B}}|. \quad (17.2.3)$$

The term  $|\vec{\mathbf{A}}| \sin \theta$  is the projection of the vector  $\vec{\mathbf{A}}$  in the direction perpendicular to the vector  $\vec{\mathbf{B}}$  as shown in Figure 17.4(b). The vector product of two vectors that are parallel (or anti-parallel) to each other is zero because the angle between the vectors is 0 (or  $\pi$ ) and  $\sin(0) = 0$  (or  $\sin(\pi) = 0$ ). Geometrically, two parallel vectors do not have a unique component perpendicular to their common direction.



**Figure 17.4** Projection of (a)  $\vec{B}$  perpendicular to  $\vec{A}$ , (b) of  $\vec{A}$  perpendicular to  $\vec{B}$

### 17.2.2 Properties of the Vector Product

- (1) The vector product is anti-commutative because changing the order of the vectors changes the direction of the vector product by the right hand rule:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \quad (17.2.4)$$

- (2) The vector product between a vector  $c\vec{A}$  where  $c$  is a scalar and a vector  $\vec{B}$  is

$$c\vec{A} \times \vec{B} = c(\vec{A} \times \vec{B}). \quad (17.2.5)$$

Similarly,

$$\vec{A} \times c\vec{B} = c(\vec{A} \times \vec{B}). \quad (17.2.6)$$

- (3) The vector product between the sum of two vectors  $\vec{A}$  and  $\vec{B}$  with a vector  $\vec{C}$  is

$$(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C} \quad (17.2.7)$$

Similarly,

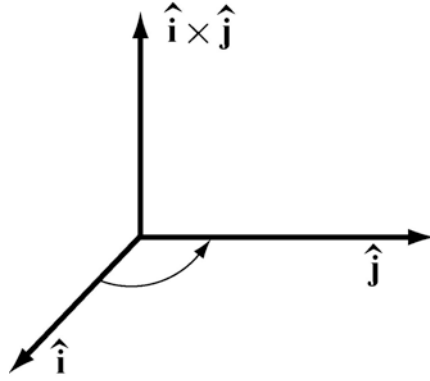
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \quad (17.2.8)$$

### 17.2.3 Vector Decomposition and the Vector Product: Cartesian Coordinates

We first calculate that the magnitude of vector product of the unit vectors  $\hat{i}$  and  $\hat{j}$ :

$$|\hat{i} \times \hat{j}| = |\hat{i}| |\hat{j}| \sin(\pi/2) = 1, \quad (17.2.9)$$

because the unit vectors have magnitude  $|\hat{i}| = |\hat{j}| = 1$  and  $\sin(\pi/2) = 1$ . By the right hand rule, the direction of  $\hat{i} \times \hat{j}$  is in the  $+\hat{k}$  as shown in Figure 17.5. Thus  $\hat{i} \times \hat{j} = \hat{k}$ .



**Figure 17.5** Vector product of  $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$

We note that the same rule applies for the unit vectors in the  $y$  and  $z$  directions,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}. \quad (17.2.10)$$

By the anti-commutativity property (1) of the vector product,

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \quad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \quad (17.2.11)$$

The vector product of the unit vector  $\hat{\mathbf{i}}$  with itself is zero because the two unit vectors are parallel to each other, ( $\sin(0) = 0$ ),

$$|\hat{\mathbf{i}} \times \hat{\mathbf{i}}| = |\hat{\mathbf{i}}| |\hat{\mathbf{i}}| \sin(0) = 0. \quad (17.2.12)$$

The vector product of the unit vector  $\hat{\mathbf{j}}$  with itself and the unit vector  $\hat{\mathbf{k}}$  with itself are also zero for the same reason,

$$|\hat{\mathbf{j}} \times \hat{\mathbf{j}}| = 0, \quad |\hat{\mathbf{k}} \times \hat{\mathbf{k}}| = 0. \quad (17.2.13)$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. Let's choose a Cartesian coordinate system with the vector  $\vec{\mathbf{B}}$  pointing along the positive  $x$ -axis with positive  $x$ -component  $B_x$ . Then the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$  can be written as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (17.2.14)$$

$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}}, \quad (17.2.15)$$

respectively. The vector product in vector components is

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \times B_x \hat{\mathbf{i}}. \quad (17.2.16)$$

This becomes,

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x \hat{\mathbf{i}} \times B_x \hat{\mathbf{i}}) + (A_y \hat{\mathbf{j}} \times B_x \hat{\mathbf{i}}) + (A_z \hat{\mathbf{k}} \times B_x \hat{\mathbf{i}}) \\ &= A_x B_x (\hat{\mathbf{i}} \times \hat{\mathbf{i}}) + A_y B_x (\hat{\mathbf{j}} \times \hat{\mathbf{i}}) + A_z B_x (\hat{\mathbf{k}} \times \hat{\mathbf{i}}) \quad . \quad (17.2.17) \\ &= -A_y B_x \hat{\mathbf{k}} + A_z B_x \hat{\mathbf{j}}\end{aligned}$$

The vector component expression for the vector product easily generalizes for arbitrary vectors

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \quad (17.2.18)$$

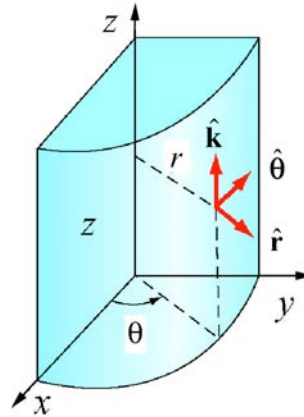
$$\vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}, \quad (17.2.19)$$

to yield

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}}. \quad (17.2.20)$$

### 17.2.4 Vector Decomposition and the Vector Product: Cylindrical Coordinates

Recall the cylindrical coordinate system, which we show in Figure 17.6. We have chosen two directions, radial and tangential in the plane, and a perpendicular direction to the plane.



**Figure 17.6** Cylindrical coordinates

The unit vectors are at right angles to each other and so using the right hand rule, the vector product of the unit vectors are given by the relations

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \quad (17.2.21)$$

$$\hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \hat{\mathbf{r}} \quad (17.2.22)$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}. \quad (17.2.23)$$

Because the vector product satisfies  $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$ , we also have that

$$\hat{\boldsymbol{\theta}} \times \hat{\mathbf{r}} = -\hat{\mathbf{k}} \quad (17.2.24)$$

$$\hat{\mathbf{k}} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} \quad (17.2.25)$$

$$\hat{\mathbf{r}} \times \hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}. \quad (17.2.26)$$

Finally

$$\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \mathbf{0}. \quad (17.2.27)$$

### Example 17.1 Vector Products

Given two vectors,  $\vec{\mathbf{A}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$  and  $\vec{\mathbf{B}} = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ , find  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ .

**Solution:**

$$\begin{aligned} \vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}} \\ &= ((-3)(2) - (7)(1))\hat{\mathbf{i}} + ((7)(5) - (2)(2))\hat{\mathbf{j}} + ((2)(1) - (-3)(5))\hat{\mathbf{k}} \\ &= -13\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 17\hat{\mathbf{k}}. \end{aligned}$$

### Example 17.2 Law of Sines

For the triangle shown in Figure 17.7a, prove the law of sines,  $|\vec{\mathbf{A}}|/\sin\alpha = |\vec{\mathbf{B}}|/\sin\beta = |\vec{\mathbf{C}}|/\sin\gamma$ , using the vector product.

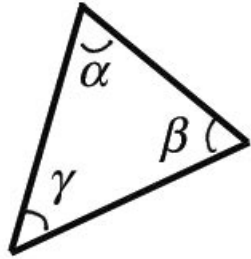


Figure 17.7 (a) Example 17.2

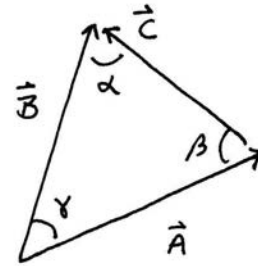


Figure 17.7 (b) Vector analysis

**Solution:** Consider the area of a triangle formed by three vectors  $\vec{\mathbf{A}}$ ,  $\vec{\mathbf{B}}$ , and  $\vec{\mathbf{C}}$ , where  $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = \mathbf{0}$  (Figure 17.7b). Because  $\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = \mathbf{0}$ , we have that  $0 = \vec{\mathbf{A}} \times (\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$ . Thus  $\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{A}} \times \vec{\mathbf{C}}$  or  $|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}} \times \vec{\mathbf{C}}|$ . From Figure 17.7b we see that  $|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\gamma$  and  $|\vec{\mathbf{A}} \times \vec{\mathbf{C}}| = |\vec{\mathbf{A}}||\vec{\mathbf{C}}|\sin\beta$ . Therefore  $|\vec{\mathbf{A}}||\vec{\mathbf{B}}|\sin\gamma = |\vec{\mathbf{A}}||\vec{\mathbf{C}}|\sin\beta$ , and hence  $|\vec{\mathbf{B}}|/\sin\beta = |\vec{\mathbf{C}}|/\sin\gamma$ . A similar argument shows that  $|\vec{\mathbf{B}}|/\sin\beta = |\vec{\mathbf{A}}|/\sin\alpha$  proving the law of sines.

### Example 17.3 Unit Normal

Find a unit vector perpendicular to  $\vec{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$  and  $\vec{\mathbf{B}} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ .



**Solution:** The vector product  $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$  is perpendicular to both  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$ . Therefore the unit vectors  $\hat{\mathbf{n}} = \pm \vec{\mathbf{A}} \times \vec{\mathbf{B}} / |\vec{\mathbf{A}} \times \vec{\mathbf{B}}|$  are perpendicular to both  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$ . We first calculate

$$\begin{aligned}\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_y B_z - A_z B_y) \hat{\mathbf{i}} + (A_z B_x - A_x B_z) \hat{\mathbf{j}} + (A_x B_y - A_y B_x) \hat{\mathbf{k}} \\ &= ((1)(3) - (-1)(-1)) \hat{\mathbf{i}} + ((-1)(2) - (1)(3)) \hat{\mathbf{j}} + ((1)(-1) - (1)(2)) \hat{\mathbf{k}} \\ &= 2 \hat{\mathbf{i}} - 5 \hat{\mathbf{j}} - 3 \hat{\mathbf{k}}.\end{aligned}$$

We now calculate the magnitude

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = (2^2 + 5^2 + 3^2)^{1/2} = (38)^{1/2}.$$

Therefore the perpendicular unit vectors are

$$\hat{\mathbf{n}} = \pm \vec{\mathbf{A}} \times \vec{\mathbf{B}} / |\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = \pm (2 \hat{\mathbf{i}} - 5 \hat{\mathbf{j}} - 3 \hat{\mathbf{k}}) / (38)^{1/2}.$$

### Example 17.4 Volume of Parallelepiped

Show that the volume of a parallelepiped with edges formed by the vectors  $\vec{\mathbf{A}}$ ,  $\vec{\mathbf{B}}$ , and  $\vec{\mathbf{C}}$  is given by  $\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}})$ .

**Solution:** The volume of a parallelepiped is given by area of the base times height. If the base is formed by the vectors  $\vec{\mathbf{B}}$  and  $\vec{\mathbf{C}}$ , then the area of the base is given by the magnitude of  $\vec{\mathbf{B}} \times \vec{\mathbf{C}}$ . The vector  $\vec{\mathbf{B}} \times \vec{\mathbf{C}} = |\vec{\mathbf{B}} \times \vec{\mathbf{C}}| \hat{\mathbf{n}}$  where  $\hat{\mathbf{n}}$  is a unit vector perpendicular to the base (Figure 17.8).

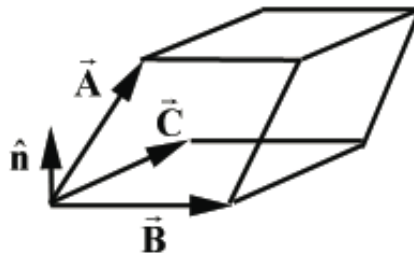


Figure 17.8 Example 17.4

The projection of the vector  $\vec{\mathbf{A}}$  along the direction  $\hat{\mathbf{n}}$  gives the height of the parallelepiped. This projection is given by taking the dot product of  $\vec{\mathbf{A}}$  with a unit vector and is equal to  $\vec{\mathbf{A}} \cdot \hat{\mathbf{n}} = \text{height}$ . Therefore

$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot (|\vec{\mathbf{B}} \times \vec{\mathbf{C}}| \hat{\mathbf{n}}) = (|\vec{\mathbf{B}} \times \vec{\mathbf{C}}|) \vec{\mathbf{A}} \cdot \hat{\mathbf{n}} = (\text{area})(\text{height}) = (\text{volume}).$$

### Example 17.5 Vector Decomposition

Let  $\vec{A}$  be an arbitrary vector and let  $\hat{n}$  be a unit vector in some fixed direction. Show that  $\vec{A} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$ .

**Solution:** Let  $\vec{A} = A_{\parallel}\hat{n} + A_{\perp}\hat{e}$  where  $A_{\parallel}$  is the component  $\vec{A}$  in the direction of  $\hat{n}$ ,  $\hat{e}$  is the direction of the projection of  $\vec{A}$  in a plane perpendicular to  $\hat{n}$ , and  $A_{\perp}$  is the component of  $\vec{A}$  in the direction of  $\hat{e}$ . Because  $\hat{e} \cdot \hat{n} = 0$ , we have that  $\vec{A} \cdot \hat{n} = A_{\parallel}$ . Note that

$$\hat{n} \times \vec{A} = \hat{n} \times (A_{\parallel}\hat{n} + A_{\perp}\hat{e}) = \hat{n} \times A_{\perp}\hat{e} = A_{\perp}(\hat{n} \times \hat{e}).$$

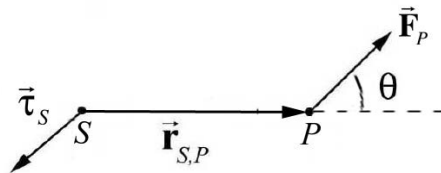
The unit vector  $\hat{n} \times \hat{e}$  lies in the plane perpendicular to  $\hat{n}$  and is also perpendicular to  $\hat{e}$ . Therefore  $(\hat{n} \times \hat{e}) \times \hat{n}$  is also a unit vector that is parallel to  $\hat{e}$  (by the right hand rule). So  $(\hat{n} \times \vec{A}) \times \hat{n} = A_{\perp}\hat{e}$ . Thus

$$\vec{A} = A_{\parallel}\hat{n} + A_{\perp}\hat{e} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}.$$

## 17.3 Torque

### 17.3.1 Definition of Torque about a Point

In order to understand the dynamics of a rotating rigid body we will introduce a new quantity, the torque. Let a force  $\vec{F}_P$  with magnitude  $F = |\vec{F}_P|$  act at a point  $P$ . Let  $\vec{r}_{S,P}$  be the vector from the point  $S$  to a point  $P$ , with magnitude  $r = |\vec{r}_{S,P}|$ . The angle between the vectors  $\vec{r}_{S,P}$  and  $\vec{F}_P$  is  $\theta$  with  $[0 \leq \theta \leq \pi]$  (Figure 17.9).



**Figure 17.9** Torque about a point  $S$  due to a force acting at a point  $P$

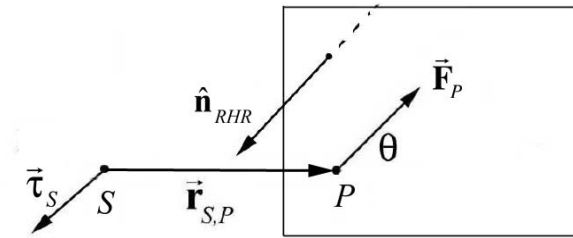
The torque about a point  $S$  due to force  $\vec{F}_P$  acting at  $P$ , is defined by

$$\vec{\tau}_S = \vec{r}_{S,P} \times \vec{F}_P. \quad (17.2.28)$$

The magnitude of the torque about a point  $S$  due to force  $\vec{F}_P$  acting at  $P$ , is given by

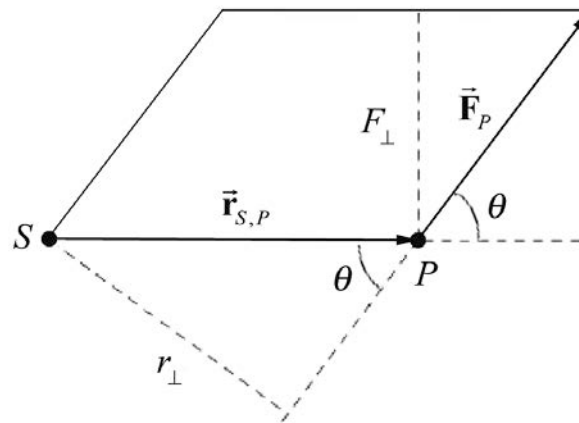
$$\tau_S \equiv |\vec{\tau}_S| = r F \sin \theta. \quad (17.2.29)$$

The SI units for torque are  $[\text{N} \cdot \text{m}]$ . The direction of the torque is perpendicular to the plane formed by the vectors  $\vec{r}_{S,P}$  and  $\vec{F}_P$  (for  $[0 < \theta < \pi]$ ), and by definition points in the direction of the unit normal vector to the plane  $\hat{n}_{RHR}$  as shown in Figure 17.10.



**Figure 17.10** Vector direction for the torque

Figure 17.11 shows the two different ways of defining height and base for a parallelogram defined by the vectors  $\vec{r}_{S,P}$  and  $\vec{F}_P$ .

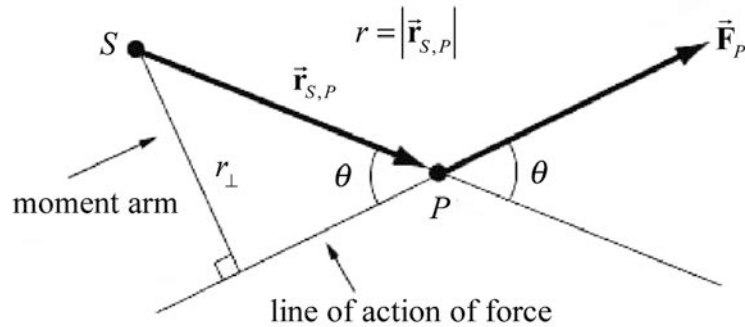


**Figure 17.11** Area of the torque parallelogram.

Let  $r_{\perp} = r \sin \theta$  and let  $F_{\perp} = F \sin \theta$  be the component of the force  $\vec{F}_P$  that is perpendicular to the line passing from the point  $S$  to  $P$ . (Recall the angle  $\theta$  has a range of values  $0 \leq \theta \leq \pi$  so both  $r_{\perp} \geq 0$  and  $F_{\perp} \geq 0$ .) Then the area of the parallelogram defined by  $\vec{r}_{S,P}$  and  $\vec{F}_P$  is given by

$$\text{Area} = \tau_S = r_{\perp} F = r F_{\perp} = r F \sin \theta. \quad (17.2.30)$$

We can interpret the quantity  $r_{\perp}$  as follows.



**Figure 17.12** The moment arm about the point  $S$  and line of action of force passing through the point  $P$

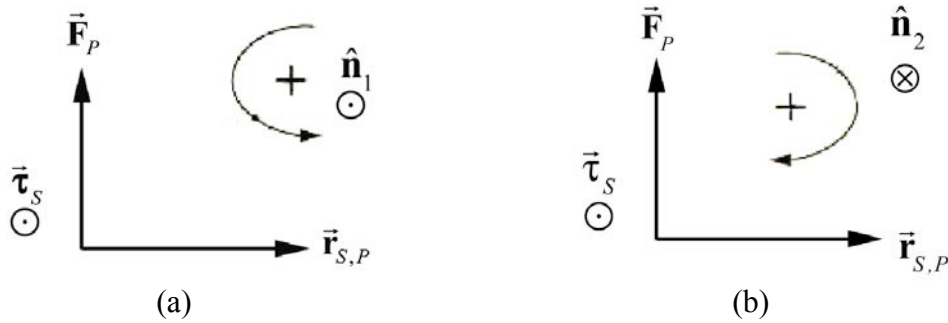
We begin by drawing the **line of action of the force**  $\vec{F}_p$ . This is a straight line passing through  $P$ , parallel to the direction of the force  $\vec{F}_p$ . Draw a perpendicular to this line of action that passes through the point  $S$  (Figure 17.12). The length of this perpendicular,  $r_{\perp} = r \sin \theta$ , is called **the moment arm about the point  $S$  of the force  $\vec{F}_p$** .

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors  $\vec{r}_{S,P}$  and  $\vec{F}_p$  are parallel ( $\theta = 0$ ) or anti-parallel ( $\theta = \pi$ ).
2. Torque is a vector whose direction and magnitude depend on the choice of a point  $S$  about which the torque is calculated.
3. The direction of torque is perpendicular to the plane formed by the two vectors,  $\vec{F}_p$  and  $r = |\vec{r}_{S,P}|$  (the vector from the point  $S$  to a point  $P$ ).

### 17.3.2 Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces  $\vec{F}_i$  and position vectors  $\vec{r}_{i,p}$  are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 17.13a circles in counterclockwise direction. (Figures 17.13a and 17.13b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question,  $\vec{F}_p$  and  $\vec{r}_{S,P}$  are perpendicular. The point  $S$  about which torques are calculated is not shown.)



**Figure 17.13** (a) Positive torque out of plane, (b) positive torque into plane

We can associate with this counterclockwise orientation a unit normal vector  $\hat{\mathbf{n}}$  according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the  $\hat{\mathbf{n}}_1$  direction (Figure 17.13a). The arc in Figure 17.13b circles in the clockwise direction, and we associate this orientation with the unit normal  $\hat{\mathbf{n}}_2$ .

It's important to note that the terms “clockwise” and “counterclockwise” might be different for different observers. For instance, if the plane containing  $\vec{\mathbf{F}}_P$  and  $\vec{\mathbf{r}}_{S,P}$  is horizontal, an observer above the plane and an observer below the plane would disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign for the component of the torque when the torque is in the same direction as the unit normal  $\hat{\mathbf{n}}_1$ , i.e.  $\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = +|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_1$ , (Figure 17.13a).
2. Suppose we choose clockwise as positive. Then we assign a negative sign for the component of the torque in Figure 17.13b because the torque is directed opposite to the unit normal  $\hat{\mathbf{n}}_2$ , i.e.  $\vec{\tau}_S = \vec{\mathbf{r}}_{S,P} \times \vec{\mathbf{F}}_P = -|\vec{\mathbf{r}}_{S,P}||\vec{\mathbf{F}}_P|\hat{\mathbf{n}}_2$ .

### Example 17.6 Torque and Vector Product

Consider two vectors  $\vec{\mathbf{r}}_{P,F} = x\hat{\mathbf{i}}$  with  $x > 0$  and  $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$  with  $F_x > 0$  and  $F_z > 0$ . Calculate the torque  $\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}}$ .

**Solution:** We calculate the vector product noting that in a right handed choice of unit vectors,  $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$  and  $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$ ,

$$\vec{\mathbf{r}}_{P,F} \times \vec{\mathbf{F}} = x\hat{\mathbf{i}} \times (F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}) = (x\hat{\mathbf{i}} \times F_x\hat{\mathbf{i}}) + (x\hat{\mathbf{i}} \times F_z\hat{\mathbf{k}}) = -xF_z\hat{\mathbf{j}}.$$

Because  $x > 0$  and  $F_z > 0$ , the direction of the vector product is in the negative  $y$ -direction.

### Example 17.7 Calculating Torque

In Figure 17.14, a force of magnitude  $F$  is applied to one end of a lever of length  $L$ . What is the magnitude and direction of the torque about the point  $S$ ?

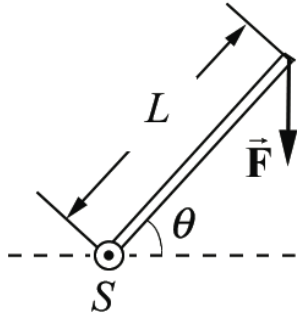


Figure 17.14 Example 17.7

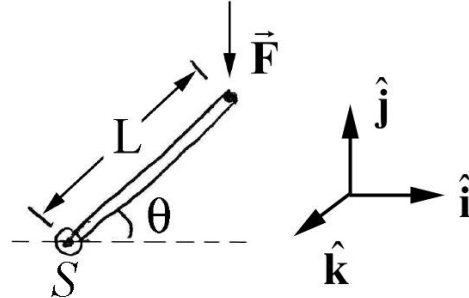


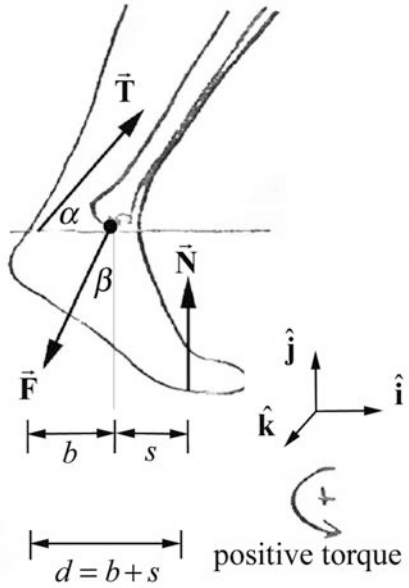
Figure 17.15 Coordinate system

**Solution:** Choose units vectors such that  $\hat{i} \times \hat{j} = \hat{k}$ , with  $\hat{i}$  pointing to the right and  $\hat{j}$  pointing up (Figure 17.15). The torque about the point  $S$  is given by  $\vec{\tau}_S = \vec{r}_{S,F} \times \vec{F}$ , where  $\vec{r}_{S,F} = L \cos \theta \hat{i} + L \sin \theta \hat{j}$  and  $\vec{F} = -F \hat{j}$  then

$$\vec{\tau}_S = (L \cos \theta \hat{i} + L \sin \theta \hat{j}) \times -F \hat{j} = -FL \cos \theta \hat{k}.$$

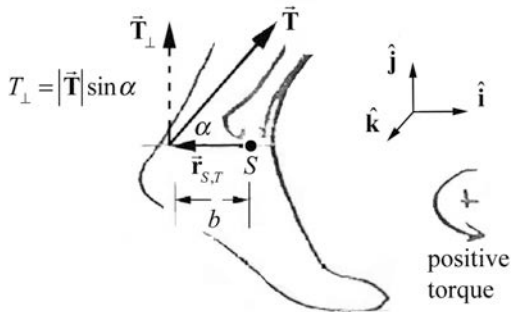
### Example 17.8 Torque and the Ankle

A person of mass  $m$  is crouching with their weight evenly distributed on both tiptoes. The free-body force diagram on the skeletal part of the foot is shown in Figure 17.16. The normal force  $\vec{N}$  acts at the contact point between the foot and the ground. In this position, the tibia acts on the foot at the point  $S$  with a force  $\vec{F}$  of an unknown magnitude  $F = |\vec{F}|$  and makes an unknown angle  $\beta$  with the vertical. This force acts on the ankle a horizontal distance  $s$  from the point where the foot contacts the floor. The Achilles tendon also acts on the foot and is under considerable tension with magnitude  $T \equiv |\vec{T}|$  and acts at an angle  $\alpha$  with the horizontal as shown in the figure. The tendon acts on the ankle a horizontal distance  $b$  from the point  $S$  where the tibia acts on the foot. You may ignore the weight of the foot. Let  $g$  be the gravitational constant. Compute the torque about the point  $S$  due to (a) the tendon force on the foot; (b) the force of the tibia on the foot; (c) the normal force of the floor on the foot.

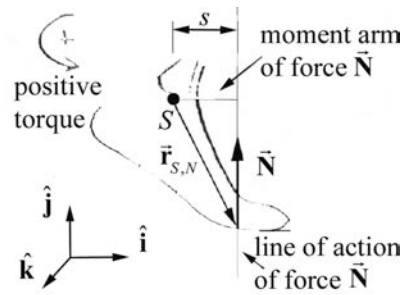


**Figure 17.16** Force diagram and coordinate system for ankle

**Solution:** (a) We shall first calculate the torque due to the force of the Achilles tendon on the ankle. The tendon force has the vector decomposition  $\vec{T} = T \cos \alpha \hat{i} + T \sin \alpha \hat{j}$ .



**Figure 17.17** Torque diagram for tendon force on ankle



**Figure 17.18** Torque diagram for normal force on ankle  $\vec{r}_{S,N}$

The vector from the point  $S$  to the point of action of the force is given by  $\vec{r}_{S,T} = -b\hat{i}$  (Figure 17.17). Therefore the torque due to the force of the tendon  $\vec{T}$  on the ankle about the point  $S$  is then

$$\vec{\tau}_{S,T} = \vec{r}_{S,T} \times \vec{T} = -b\hat{i} \times (T \cos \alpha \hat{i} + T \sin \alpha \hat{j}) = -bT \sin \alpha \hat{k}.$$

(b) The torque diagram for the normal force is shown in Figure 17.18. The vector from the point  $S$  to the point where the normal force acts on the foot is given by

$\vec{r}_{S,N} = (s\hat{i} - h\hat{j})$ . Because the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or  $N = (1/2)mg$ . The normal force is therefore given by  $\vec{N} = N\hat{j} = (1/2)mg\hat{j}$ . Therefore the torque of the normal force about the point  $S$  is

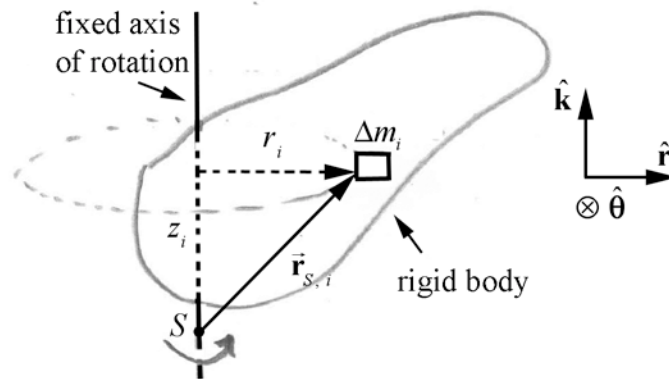
$$\vec{\tau}_{S,N} = \vec{r}_{S,N} \times N\hat{j} = (s\hat{i} - h\hat{j}) \times N\hat{j} = sN\hat{k} = (1/2)smg\hat{k}.$$

(c) The force  $\vec{F}$  that the tibia exerts on the ankle will make no contribution to the torque about this point  $S$  since the tibia force acts at the point  $S$  and therefore the vector  $\vec{r}_{S,F} = \vec{0}$ .

## 17.4 Torque, Angular Acceleration, and Moment of Inertia

### 17.4.1 Torque Equation for Fixed Axis Rotation

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration. Consider the forces that act on the rotating body. Generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass  $\Delta m_i$  by  $\vec{F}_i$ . Choose the  $z$ -axis to lie along the axis of rotation. Divide the body into volume elements of mass  $\Delta m_i$ . Let the point  $S$  denote a specific point along the axis of rotation (Figure 17.19). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius  $r_i = |\vec{r}_i|$  about the fixed axis.



**Figure 17.19:** Volume element undergoing fixed-axis rotation about the  $z$ -axis.

The vector from the point  $S$  to the volume element is given by

$$\vec{r}_{S,i} = z_i\hat{k} + \vec{r}_i = z_i\hat{k} + r_i\hat{r} \quad (17.3.1)$$



where  $z_i$  is the distance along the axis of rotation between the point  $S$  and the volume element. The torque about  $S$  due to the force  $\vec{F}_i$  acting on the volume element is given by

$$\vec{\tau}_{S,i} = \vec{r}_{S,i} \times \vec{F}_i. \quad (17.3.2)$$

Substituting Eq. (17.3.1) into Eq. (17.3.2) gives

$$\vec{\tau}_{S,i} = (z_i \hat{k} + r_i \hat{r}) \times \vec{F}_i. \quad (17.3.3)$$

For fixed-axis rotation, we are interested in the  $z$ -component of the torque, which must be the term

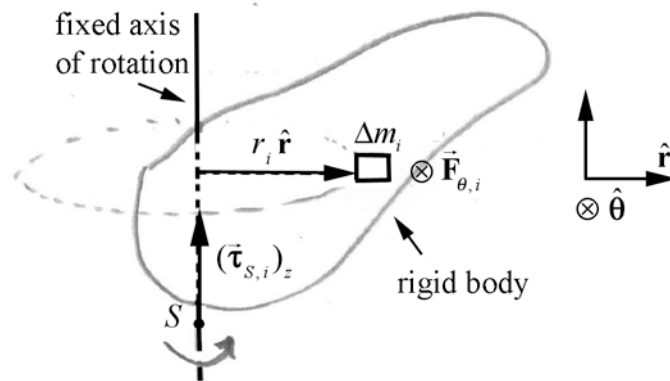
$$(\vec{\tau}_{S,i})_z = (r_i \hat{r} \times \vec{F}_i)_z \quad (17.3.4)$$

because the vector product  $z_i \hat{k} \times \vec{F}_i$  must be directed perpendicular to the plane formed by the vectors  $\hat{k}$  and  $\vec{F}_i$ , hence perpendicular to the  $z$ -axis. The force acting on the volume element has components

$$\vec{F}_i = F_{r,i} \hat{r} + F_{\theta,i} \hat{\theta} + F_{z,i} \hat{k}. \quad (17.3.5)$$

The  $z$ -component  $F_{z,i}$  of the force cannot contribute a torque in the  $z$ -direction, and so substituting Eq. (17.3.5) into Eq. (17.3.4) yields

$$(\vec{\tau}_{S,i})_z = (r_i \hat{r} \times (F_{r,i} \hat{r} + F_{\theta,i} \hat{\theta}))_z. \quad (17.3.6)$$



**Figure 17.20** Tangential force acting on a volume element.

The radial force does not contribute to the torque about the  $z$ -axis, since

$$r_i \hat{r} \times F_{r,i} \hat{r} = \vec{0}. \quad (17.3.7)$$

So, we are interested in the contribution due to torque about the  $z$ -axis due to the tangential component of the force on the volume element (Figure 17.20). The component of the torque about the  $z$ -axis is given by

$$(\vec{\tau}_{S,i})_z = (r_i \hat{r} \times F_{\theta,i} \hat{\theta})_z = r_i F_{\theta,i}. \quad (17.3.8)$$

The  $z$ -component of the torque is directed upwards in Figure 17.20, where  $F_{\theta,i}$  is positive (the tangential force is directed counterclockwise, as in the figure). Applying Newton's Second Law in the tangential direction,

$$F_{\theta,i} = \Delta m_i a_{\theta,i}. \quad (17.3.9)$$

Using our kinematics result that the tangential acceleration is  $a_{\theta,i} = r_i \alpha_z$ , where  $\alpha_z$  is the  $z$ -component of angular acceleration, we have that

$$F_{\theta,i} = \Delta m_i r_i \alpha_z. \quad (17.3.10)$$

From Eq. (17.3.8), the component of the torque about the  $z$ -axis is then given by

$$(\vec{\tau}_{S,i})_z = r_i F_{\theta,i} = \Delta m_i r_i^2 \alpha_z. \quad (17.3.11)$$

The component of the torque about the  $z$ -axis is the summation of the torques on all the volume elements,

$$\begin{aligned} (\vec{\tau}_S)_z &= \sum_{i=1}^{i=N} (\vec{\tau}_{S,i})_z = \sum_{i=1}^{i=N} r_{\perp,i} F_{\theta,i} \\ &= \sum_{i=1}^{i=N} \Delta m_i r_i^2 \alpha_z. \end{aligned} \quad (17.3.12)$$

Because each element has the same  $z$ -component of angular acceleration,  $\alpha_z$ , the summation becomes

$$(\vec{\tau}_S)_z = \left( \sum_{i=1}^{i=N} \Delta m_i r_i^2 \right) \alpha_z. \quad (17.3.13)$$

Recalling our definition of the moment of inertia, (Chapter 16.3) the  $z$ -component of the torque is proportional to the  $z$ -component of angular acceleration,

$$\tau_{S,z} = I_S \alpha_z, \quad (17.3.14)$$

and the moment of inertia,  $I_S$ , is the constant of proportionality. The torque about the point  $S$  is the sum of the external torques and the internal torques

$$\vec{\tau}_S = \vec{\tau}_S^{\text{ext}} + \vec{\tau}_S^{\text{int}}. \quad (17.3.15)$$

The external torque about the point  $S$  is the sum of the torques due to the net external force acting on each element

$$\vec{\tau}_S^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\tau}_{S,i}^{\text{ext}} = \sum_{i=1}^{i=N} \vec{r}_{S,i} \times \vec{F}_i^{\text{ext}}. \quad (17.3.16)$$

The internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\tau}_S^{\text{int}} = \sum_{i=1}^N \vec{\tau}_{S,i}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{\tau}_{S,j,i}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{r}_{S,i} \times \vec{F}_{j,i}. \quad (17.3.17)$$

We know by Newton's Third Law that the internal forces cancel in pairs,  $\vec{F}_{j,i} = -\vec{F}_{i,j}$ , and hence the sum of the internal forces is zero

$$\vec{0} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j \neq i}}^{j=N} \vec{F}_{j,i}. \quad (17.3.18)$$

Does the same statement hold about pairs of internal torques? Consider the sum of internal torques arising from the interaction between the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles

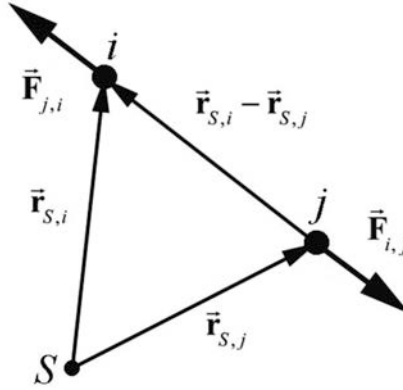
$$\vec{\tau}_{S,j,i}^{\text{int}} + \vec{\tau}_{S,i,j}^{\text{int}} = \vec{r}_{S,i} \times \vec{F}_{j,i} + \vec{r}_{S,j} \times \vec{F}_{i,j}. \quad (17.3.19)$$

By the Newton's Third Law this sum becomes

$$\vec{\tau}_{S,j,i}^{\text{int}} + \vec{\tau}_{S,i,j}^{\text{int}} = (\vec{r}_{S,i} - \vec{r}_{S,j}) \times \vec{F}_{j,i}. \quad (17.3.20)$$

In the Figure 17.21, the vector  $\vec{r}_{S,i} - \vec{r}_{S,j}$  points from the  $j^{\text{th}}$  element to the  $i^{\text{th}}$  element. If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\tau}_{S,j,i}^{\text{int}} + \vec{\tau}_{S,i,j}^{\text{int}} = (\vec{r}_{S,i} - \vec{r}_{S,j}) \times \vec{F}_{j,i} = \vec{0}. \quad (17.3.21)$$



**Figure 17.21** The internal force is directed along the line connecting the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles

This is a stronger version of Newton's Third Law than we have so far since we have added the additional requirement regarding the direction of all the internal forces between pairs of particles. With this assumption, the torque is just due to the external forces

$$\vec{\tau}_S = \vec{\tau}_S^{\text{ext}}. \quad (17.3.22)$$

Thus Eq. (17.3.14) becomes

$$(\tau_S^{\text{ext}})_z = I_S \alpha_z, \quad (17.3.23)$$

This is very similar to Newton's Second Law: the total force is proportional to the acceleration,

$$\vec{F} = m\vec{a}. \quad (17.3.24)$$

where the mass,  $m$ , is the constant of proportionality.

### 17.4.2 Torque Acts at the Center of Gravity

Suppose a rigid body in static equilibrium consists of  $N$  particles labeled by the index  $i = 1, 2, 3, \dots, N$ . Choose a coordinate system with a choice of origin  $O$  such that mass  $m_i$  has position  $\vec{r}_i$ . Each point particle experiences a gravitational force  $\vec{F}_{\text{gravity},i} = m_i \vec{g}$ . The total torque about the origin is then zero (static equilibrium condition),

$$\vec{\tau}_O = \sum_{i=1}^{i=N} \vec{\tau}_{O,i} = \sum_{i=1}^{i=N} \vec{r}_i \times \vec{F}_{\text{gravity},i} = \sum_{i=1}^{i=N} \vec{r}_i \times m_i \vec{g} = \vec{0}. \quad (17.3.25)$$

If the gravitational acceleration  $\vec{g}$  is assumed constant, we can rearrange the summation in Eq. (17.3.25) by pulling the constant vector  $\vec{g}$  out of the summation ( $\vec{g}$  appears in each term in the summation),

$$\vec{\tau}_O = \sum_{i=1}^{i=N} \vec{r}_i \times m_i \vec{g} = \left( \sum_{i=1}^{i=N} m_i \vec{r}_i \right) \times \vec{g} = \vec{0}. \quad (17.3.26)$$

We now use our definition of the center of the center of mass, Eq. (10.5.3), to rewrite Eq. (17.3.26) as

$$\vec{\tau}_O = \sum_{i=1}^{i=N} \vec{r}_i \times m_i \vec{g} = M_T \vec{R}_{\text{cm}} \times \vec{g} = \vec{R}_{\text{cm}} \times M_T \vec{g} = \vec{0}. \quad (17.3.27)$$

Thus the torque due to the gravitational force acting on each point-like particle is equivalent to the torque due to the gravitational force acting on a point-like particle of mass  $M_T$  located at a point in the body called the **center of gravity**, which is equal to the center of mass of the body in the typical case in which the gravitational acceleration  $\vec{g}$  is constant throughout the body.

### Example 17.9 Turntable

The turntable in Example 16.1, of mass 1.2 kg and radius  $1.3 \times 10^1$  cm, has a moment of inertia  $I_S = 1.01 \times 10^{-2}$  kg·m<sup>2</sup> about an axis through the center of the turntable and perpendicular to the turntable. The turntable is spinning at an initial constant frequency  $f_i = 33$  cycles·min<sup>-1</sup>. The motor is turned off and the turntable slows to a stop in 8.0 s due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the turntable?

**Solution:** We have already calculated the angular acceleration of the turntable in Example 16.1, where we found that

$$\alpha_z = \frac{\Delta\omega_z}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{-3.5 \text{ rad} \cdot \text{s}^{-1}}{8.0 \text{ s}} = -4.3 \times 10^{-1} \text{ rad} \cdot \text{s}^{-2} \quad (17.3.28)$$

and so the magnitude of the frictional torque is

$$\begin{aligned} |\tau_z^{\text{fric}}| &= I_S |\alpha_z| = (1.01 \times 10^{-2} \text{ kg} \cdot \text{m}^2)(4.3 \times 10^{-1} \text{ rad} \cdot \text{s}^{-2}) \\ &= 4.3 \times 10^{-3} \text{ N} \cdot \text{m}. \end{aligned} \quad (17.3.29)$$

### Example 17.10 Pulley and blocks

A pulley of mass  $m_p$ , radius  $R$ , and moment of inertia about its center of mass  $I_{\text{cm}}$ , is attached to the edge of a table. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to block 1 that hangs over the edge of the table (Figure 17.22). The other end of the string is attached to block 2 that slides along a table.

The coefficient of sliding friction between the table and the block 2 is  $\mu_k$ . Block 1 has mass  $m_1$  and block 2 has mass  $m_2$ , with  $m_1 > \mu_k m_2$ . At time  $t = 0$ , the blocks are released from rest and the string does not slip around the pulley. At time  $t = t_1$ , block 1 hits the ground. Let  $g$  denote the gravitational constant. (a) Find the magnitude of the acceleration of each block. (b) How far did the block 1 fall before hitting the ground?

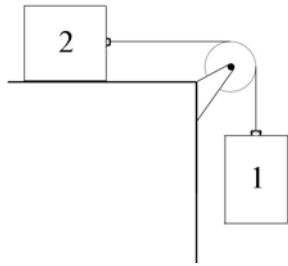


Figure 17.22 Example 17.10

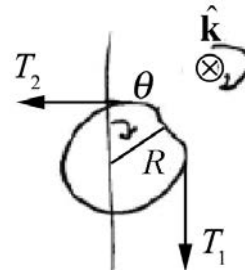


Figure 17.23 Torque diagram for pulley

**Solution:** The torque diagram for the pulley is shown in the figure below where we choose  $\hat{\mathbf{k}}$  pointing into the page. Note that the tensions in the string on either side of the pulley are not equal. The reason is that the pulley is massive. To understand why, remember that the difference in the magnitudes of the torques due to the tension on either side of the pulley is equal to the moment of inertia times the magnitude of the angular acceleration, which is non-zero for a massive pulley. So the tensions cannot be equal. From our torque diagram, the torque about the point  $O$  at the center of the pulley is given by

$$\vec{\tau}_O = \vec{r}_{O,1} \times \vec{T}_1 + \vec{r}_{O,2} \times \vec{T}_2 = R(T_1 - T_2)\hat{\mathbf{k}}. \quad (17.3.30)$$

Therefore the torque equation (17.3.23) becomes

$$R(T_1 - T_2) = I_z \alpha_z. \quad (17.3.31)$$

The free body force diagrams on the two blocks are shown in Figure 17.23.

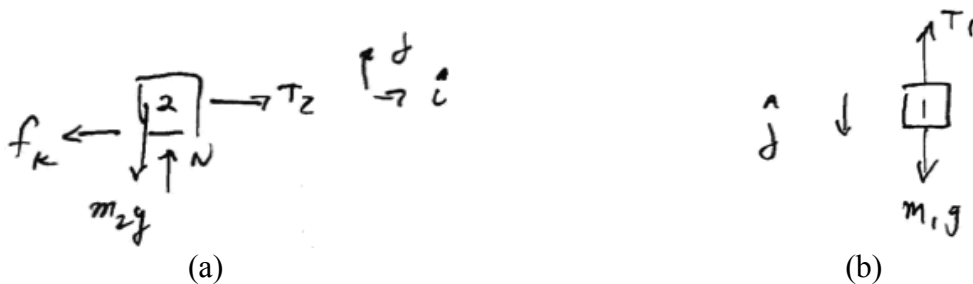


Figure 17.23 Free-body force diagrams on (a) block 2, (b) block 1

Newton's Second Law on block 1 yields

$$m_1 g - T_1 = m_1 a_{y1}. \quad (17.3.32)$$

Newton's Second Law on block 2 in the  $\hat{\mathbf{j}}$  direction yields

$$N - m_2 g = 0. \quad (17.3.33)$$

Newton's Second Law on block 2 in the  $\hat{\mathbf{i}}$  direction yields

$$T_2 - f_k = m_2 a_{x2}. \quad (17.3.34)$$

The kinetic friction force is given by

$$f_k = \mu_k N = \mu_k m_2 g \quad (17.3.35)$$

Therefore Eq. (17.3.34) becomes

$$T_2 - \mu_k m_2 g = m_2 a_{x2}. \quad (17.3.36)$$

Block 1 and block 2 are constrained to have the same acceleration so

$$a \equiv a_{x1} = a_{x2}. \quad (17.3.37)$$

We can solve Eqs. (17.3.32) and (17.3.36) for the two tensions yielding

$$T_1 = m_1 g - m_1 a, \quad (17.3.38)$$

$$T_2 = \mu_k m_2 g + m_2 a. \quad (17.3.39)$$

At point on the rim of the pulley has a tangential acceleration that is equal to the acceleration of the blocks so

$$a = a_\theta = R\alpha_z. \quad (17.3.40)$$

The torque equation (Eq. (17.3.31)) then becomes

$$T_1 - T_2 = \frac{I_z}{R^2} a. \quad (17.3.41)$$

Substituting Eqs. (17.3.38) and (17.3.39) into Eq. (17.3.41) yields

$$m_1 g - m_1 a - (\mu_k m_2 g + m_2 a) = \frac{I_z}{R^2} a, \quad (17.3.42)$$

which we can now solve for the accelerations of the blocks

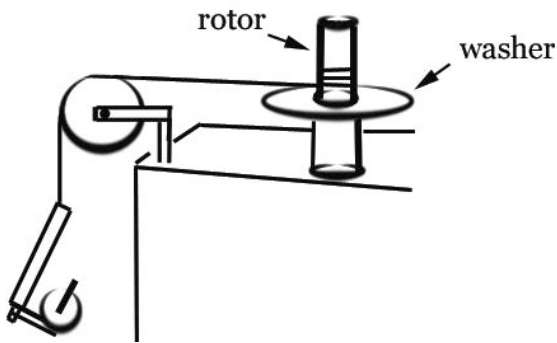
$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}. \quad (17.3.43)$$

Block 1 hits the ground at time  $t_1$ , therefore it traveled a distance

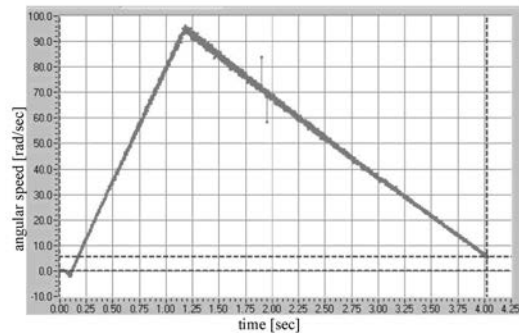
$$y_1 = \frac{1}{2} \left( \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2} \right) t_1^2. \quad (17.3.44)$$

### Example 17.11 Experimental Method for Determining Moment of Inertia

A steel washer is mounted on a cylindrical rotor of radius  $r = 12.7 \text{ mm}$ . A massless string, with an object of mass  $m = 0.055 \text{ kg}$  attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley (Figure 17.24). Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the object falls, the rotor undergoes an angular acceleration of magnitude  $\alpha_1$ . After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude  $\alpha_2$ . Let  $g = 9.8 \text{ m} \cdot \text{s}^{-2}$  denote the gravitational constant. Based on the data in the Figure 17.25, what is the moment of inertia  $I_R$  of the rotor assembly (including the washer) about the rotation axis?



**Figure 17.24** Steel washer, rotor, pulley, and hanging object

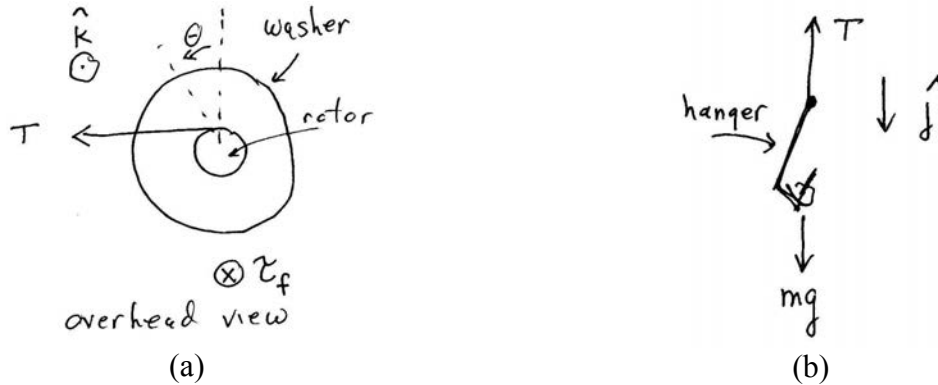


**Figure 17.26** Graph of angular speed vs. time for falling object

**Solution:** We begin by drawing a force-torque diagram (Figure 17.26a) for the rotor and a free-body diagram for hanger (Figure 17.26b). (The choice of positive directions are indicated on the figures.) The frictional torque on the rotor is then given by  $\vec{\tau}_f = -\tau_f \hat{\mathbf{k}}$  where we use  $\tau_f$  as the magnitude of the frictional torque. The torque about the center of the rotor due to the tension in the string is given by  $\vec{\tau}_T = rT \hat{\mathbf{k}}$  where  $r$  is the radius of



the rotor. The angular acceleration of the rotor is given by  $\vec{\alpha}_1 = \alpha_1 \hat{k}$  and we expect that  $\alpha_1 > 0$  because the rotor is speeding up.



**Figure 17.26** (a) Force-torque diagram on rotor and (b) free-body force diagram on hanging object

While the hanger is falling, the rotor-washer combination has a net torque due to the tension in the string and the frictional torque, and using the rotational equation of motion,

$$Tr - \tau_f = I_R \alpha_1. \quad (17.4.1)$$

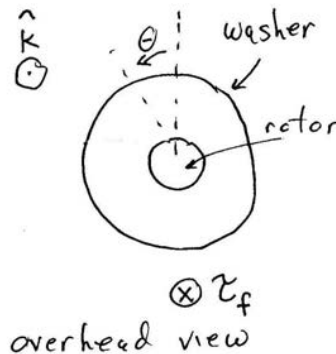
We apply Newton's Second Law to the hanger and find that

$$mg - T = ma_1 = m\alpha_1 r, \quad (17.4.2)$$

where  $a_1 = r\alpha_1$  has been used to express the linear acceleration of the falling hanger to the angular acceleration of the rotor; that is, the string does not stretch. Before proceeding, it might be illustrative to multiply Eq. (17.4.2) by  $r$  and add to Eq. (17.4.1) to obtain

$$mgr - \tau_f = (I_R + mr^2)\alpha_1. \quad (17.4.3)$$

Eq. (17.4.3) contains the unknown frictional torque, and this torque is determined by considering the slowing of the rotor/washer after the string has detached.



**Figure 17.27** Torque diagram on rotor when string has detached

The torque on the system is just this frictional torque (Figure 17.27), and so

$$-\tau_f = I_R \alpha_2 \quad (17.4.4)$$

Note that in Eq. (17.4.4),  $\tau_f > 0$  and  $\alpha_2 < 0$ . Subtracting Eq. (17.4.4) from Eq. (17.4.3) eliminates  $\tau_f$ ,

$$mgr = mr^2 \alpha_1 + I_R (\alpha_1 - \alpha_2). \quad (17.4.5)$$

We can now solve for  $I_R$  yielding

$$I_R = \frac{mr(g - r\alpha_1)}{\alpha_1 - \alpha_2}. \quad (17.4.6)$$

For a numerical result, we use the data collected during a trial run resulting in the graph of angular speed vs. time for the falling object shown in Figure 17.25. The values for  $\alpha_1$  and  $\alpha_2$  can be determined by calculating the slope of the two straight lines in Figure 17.28 yielding

$$\begin{aligned} \alpha_1 &= (96 \text{ rad} \cdot \text{s}^{-1}) / (1.15 \text{ s}) = 83 \text{ rad} \cdot \text{s}^{-2}, \\ \alpha_2 &= -(89 \text{ rad} \cdot \text{s}^{-1}) / (2.85 \text{ s}) = -31 \text{ rad} \cdot \text{s}^{-2}. \end{aligned}$$

Inserting these values into Eq. (17.4.6) yields

$$I_R = 5.3 \times 10^{-5} \text{ kg} \cdot \text{m}^2. \quad (17.4.7)$$

## 17.5 Torque and Rotational Work

When a constant torque  $\tau_{S,z}$  is applied to an object, and the object rotates through an angle  $\Delta\theta$  about a fixed  $z$ -axis through the center of mass, then the torque does an amount of work  $\Delta W = \tau_{S,z} \Delta\theta$  on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_f^2 - \frac{1}{2} I_{\text{cm}} \omega_i^2 = K_{\text{rot},f} - K_{\text{rot},i}. \quad (17.4.8)$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_{S,z} \frac{d\theta}{dt} = \tau_{S,z} \omega_z. \quad (17.4.9)$$

### 17.5.1 Rotational Work

Consider a rigid body rotating about an axis. Each small element of mass  $\Delta m_i$  in the rigid body is moving in a circle of radius  $(r_{S,i})_{\perp}$  about the axis of rotation passing through the point  $S$ . Each mass element undergoes a small angular displacement  $\Delta\theta$  under the action of a tangential force,  $\vec{\mathbf{F}}_{\theta,i} = F_{\theta,i} \hat{\boldsymbol{\theta}}$ , where  $\hat{\boldsymbol{\theta}}$  is the unit vector pointing in the tangential direction (Figure 17.20). The element will then have an associated displacement vector for this motion,  $\Delta\vec{\mathbf{r}}_{S,i} = r_i \Delta\theta \hat{\boldsymbol{\theta}}$  and the work done by the tangential force is

$$\Delta W_{\text{rot},i} = \vec{\mathbf{F}}_{\theta,i} \cdot \Delta\vec{\mathbf{r}}_{S,i} = (F_{\theta,i} \hat{\boldsymbol{\theta}}) \cdot (r_i \Delta\theta \hat{\boldsymbol{\theta}}) = r_i F_{\theta,i} \Delta\theta. \quad (17.4.10)$$

Recall the result of Eq. (17.3.8) that the component of the torque (in the direction along the axis of rotation) about  $S$  due to the tangential force,  $\vec{\mathbf{F}}_{\theta,i}$ , acting on the mass element  $\Delta m_i$  is

$$(\tau_{S,i})_z = r_i F_{\theta,i}, \quad (17.4.11)$$

and so Eq. (17.4.10) becomes

$$\Delta W_{\text{rot},i} = (\tau_{S,i})_z \Delta\theta. \quad (17.4.12)$$

Summing over all the elements yields

$$W_{\text{rot}} = \sum_i \Delta W_{\text{rot},i} = \left( (\tau_{S,i})_z \right) \Delta\theta = \tau_{S,z} \Delta\theta, \quad (17.4.13)$$

the rotational work is the product of the torque and the angular displacement. In the limit of small angles,  $\Delta\theta \rightarrow d\theta$ ,  $\Delta W_{\text{rot}} \rightarrow dW_{\text{rot}}$  and the differential rotational work is

$$dW_{\text{rot}} = \tau_{S,z} d\theta. \quad (17.4.14)$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value  $\theta = \theta_i$  to some final value  $\theta = \theta_f$ ,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\theta_i}^{\theta_f} \tau_{S,z} d\theta. \quad (17.4.15)$$

### 17.5.2 Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Eq. (17.3.14) into Eq. (17.4.14) for the infinitesimal rotational work,

$$dW_{\text{rot}} = I_S \alpha_z d\theta. \quad (17.4.16)$$

Recall that the rate of change of angular velocity is equal to the angular acceleration,  $\alpha_z \equiv d\omega_z/dt$  and that the angular velocity is  $\omega_z \equiv d\theta/dt$ . Note that in the limit of small displacements,

$$\frac{d\omega_z}{dt} d\theta = d\omega_z \frac{d\theta}{dt} = d\omega_z \omega_z. \quad (17.4.17)$$

Therefore the infinitesimal rotational work is

$$dW_{\text{rot}} = I_S \alpha_z d\theta = I_S \frac{d\omega_z}{dt} d\theta = I_S d\omega_z \frac{d\theta}{dt} = I_S d\omega_z \omega_z. \quad (17.4.18)$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value  $\omega_z = \omega_{z,i}$  to some final value  $\omega_z = \omega_{z,f}$ ,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_{z,i}}^{\omega_{z,f}} I_S d\omega_z \omega_z = \frac{1}{2} I_S \omega_{z,f}^2 - \frac{1}{2} I_S \omega_{z,i}^2. \quad (17.4.19)$$

When a rigid body is rotating about a fixed axis passing through a point  $S$  in the body, there is both rotation and translation about the center of mass unless  $S$  is the center of mass. If we choose the point  $S$  in the above equation for the rotational work to be the center of mass, then

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},i}^2 = K_{\text{rot},f} - K_{\text{rot},i} \equiv \Delta K_{\text{rot}}. \quad (17.4.20)$$

Note that because the  $z$ -component of the angular velocity of the center of mass appears as a square, we can just use its magnitude in Eq. (17.4.20).

### 17.5.3 Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt}. \quad (17.4.21)$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \tau_{S,z} \frac{d\theta}{dt} = \tau_{S,z} \omega_z. \quad (17.4.22)$$

### Example 17.12 Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is  $I_0$ . The washer is set into motion. When it reaches an initial angular velocity  $\omega_0$ , at  $t = 0$ , the power to the motor is shut off, and the washer slows down during a time interval  $\Delta t_1 = t_a$  until it reaches an angular velocity of  $\omega_a$  at time  $t_a$ . At that instant, a second steel washer with a moment of inertia  $I_w$  is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time  $\Delta t_{\text{int}} = t_b - t_a$  after which the two washers and rotor rotate with angular speed  $\omega_b$ . Assume the frictional torque on the axle (magnitude  $\tau_f$ ) is independent of speed, and remains the same when the second washer is dropped. (a) What angle does the rotor rotate through during the collision? (b) What is the work done by the friction torque from the bearings during the collision? (c) Write down an equation for conservation of energy. Can you solve this equation for  $\omega_b$ ? (d) What is the average rate that work is being done by the friction torque during the collision?

**Solution:** We begin by solving for the frictional torque during the first stage of motion when the rotor is slowing down. We choose a coordinate system shown in Figure 17.29.

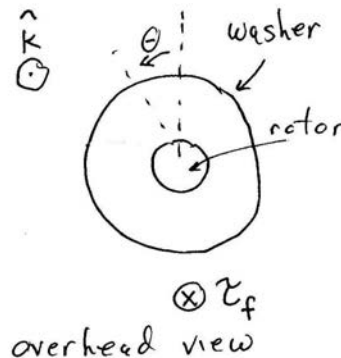


Figure 17.29 Coordinate system for Example 17.12

The component of average angular acceleration is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{t_a} < 0.$$

We can use the rotational equation of motion, and find that the frictional torque satisfies

$$-\tau_f = I_0 \left( \frac{\omega_a - \omega_0}{\Delta t_1} \right).$$

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_2 = \frac{\omega_b - \omega_a}{(\Delta t_{\text{int}})} < 0.$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2} \alpha_2 \Delta t_{\text{int}}^2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2} \left( \frac{\omega_b - \omega_a}{\Delta t_{\text{int}}} \right) \Delta t_{\text{int}}^2 = \frac{1}{2} (\omega_b + \omega_a) \Delta t_{\text{int}} > 0.$$

The non-conservative work done by the bearing friction during the collision is

$$W_{f,b} = -\tau_f \Delta\theta_{\text{rotor}} = -\tau_f \frac{1}{2} (\omega_a + \omega_b) \Delta t_{\text{int}}.$$

Using our result for the frictional torque, the work done by the bearing friction during the collision is

$$W_{f,b} = \frac{1}{2} I_0 \left( \frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) \Delta t_{\text{int}} < 0.$$

The negative work is consistent with the fact that the kinetic energy of the rotor is decreasing as the rotor is slowing down. Using the work energy theorem during the collision the kinetic energy of the rotor has decreased by

$$W_{f,b} = \frac{1}{2} (I_0 + I_w) \omega_b^2 - \frac{1}{2} I_0 \omega_a^2.$$

Using our result for the work, we have that

$$\frac{1}{2} I_0 \left( \frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) \Delta t_{\text{int}} = \frac{1}{2} (I_0 + I_w) \omega_b^2 - \frac{1}{2} I_0 \omega_a^2.$$

This is a quadratic equation for the angular speed  $\omega_b$  of the rotor and washer immediately after the collision that we can in principle solve. However remember that we assumed that the frictional torque is independent of the speed of the rotor. Hence the best practice would be to measure  $\omega_0$ ,  $\omega_a$ ,  $\omega_b$ ,  $\Delta t_1$ ,  $\Delta t_{\text{int}}$ ,  $I_0$ , and  $I_w$  and then determine how closely our model agrees with conservation of energy. The rate of work done by the frictional torque is given by

$$P_f = \frac{W_{f,b}}{\Delta t_{\text{int}}} = \frac{1}{2} I_0 \left( \frac{\omega_a - \omega_0}{\Delta t_1} \right) (\omega_a + \omega_b) < 0.$$

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