Chapter 16 Two Dimensional Rotational Kinematics

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Most galaxies exhibit rising rotational velocities at the largest measured velocity; only for the very largest galaxies are the rotation curves flat. Thus the smallest SC's (i.e. lowest luminosity) exhibit the same lack of Keplerian velocity decrease at large R as do the high-luminosity spirals. The form for the rotation curves implies that the mass is not centrally condensed, but that significant mass is located at large R. The integral mass is increasing at least as fast as R. The mass is not converging to a limiting mass at the edge of the optical image. The conclusion is inescapable than non-luminous matter exists beyond the optical galaxy.[1]

Vera Rubin

16.1 Introduction

The physical objects that we encounter in the world consist of collections of atoms that are bound together to form systems of particles. When forces are applied, the shape of the body may be stretched or compressed like a spring, or sheared like jello. In some systems the constituent particles are very loosely bound to each other as in fluids and gasses, and the distances between the constituent particles will vary. We shall begin by restricting ourselves to an ideal category of objects, rigid bodies, which do not stretch, compress, or shear.

A body is called a ***rigid body*** if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. Let's start by considering the simplest example of rigid body motion, rotation about a fixed axis.

16.2 Fixed Axis Rotation: Rotational Kinematics

16.2.1 Fixed Axis Rotation

A simple example of rotation about a fixed axis is the motion of a compact disc in a CD player, which is driven by a motor inside the player. In a simplified model of this motion, the motor produces angular acceleration, causing the disc to spin. As the disc is set in motion, resistive forces oppose the motion until the disc no longer has any angular acceleration, and the disc now spins at a constant angular velocity. Throughout this process, the CD rotates about an axis passing through the center of the disc, and is perpendicular to the plane of the disc (see Figure 16.1). This type of motion is called ***fixed-axis rotation***.

[1]V.C. Rubin, W.K. Jr. Ford, N Thonnard, *Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/*, Astrophysical Journal, Part 1, vol. 238, June 1, 1980, p. 471-487.

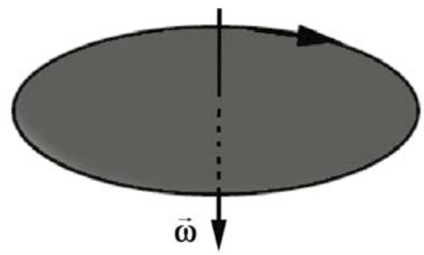

Figure 16.1 Rotation of a compact disc about a fixed axis.

When we ride a bicycle forward, the wheels rotate about an axis passing through the center of each wheel and perpendicular to the plane of the wheel (Figure 16.2). As long as the bicycle does not turn, this axis keeps pointing in the same direction. This motion is more complicated than our spinning CD because the wheel is both moving (translating) with some center of mass velocity, $\vec{\mathbf{v}}_{\text{cm}}$, and rotating with an angular speed ω.

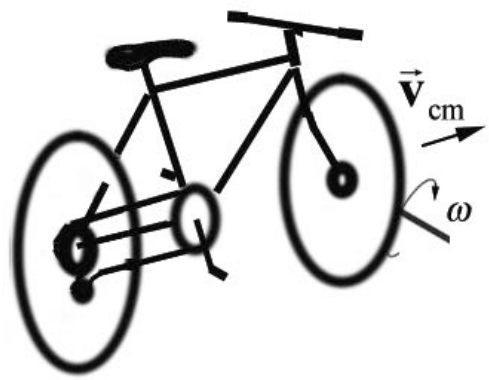

Figure 16.2 Fixed axis rotation and center of mass translation for a bicycle wheel.

When we turn the bicycle's handlebars, we change the bike's trajectory and the axis of rotation of each wheel changes direction. Other examples of non-fixed axis rotation are the motion of a spinning top, or a gyroscope, or even the change in the direction of the earth's rotation axis. This type of motion is much harder to analyze, so we will restrict ourselves in this chapter to considering fixed axis rotation, with or without translation.

16.2.2 Angular Velocity and Angular Acceleration

For a rigid body undergoing fixed-axis rotation, we can divide the body up into small volume elements with mass Δm_i. Each of these volume elements is moving in a circle of radius r_i about the axis of rotation (Figure 16.3).

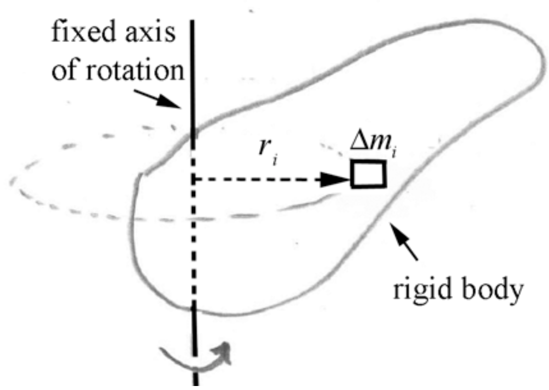

Figure 16.3 Coordinate system for fixed-axis rotation.

We will adopt the notation implied in Figure 16.3, and denote the vector from the axis to the point where the mass element is located as $\vec{\mathbf{r}}_i$, with magnitude $r_i = |\vec{\mathbf{r}}_i|$. Suppose the fixed axis of rotation is the z-axis. Introduce a right-handed coordinate system for an angle θ in the plane of rotation and the choice of the positive z-direction perpendicular to that plane of rotation. Recall our definition of the angular velocity vector. The angular velocity vector is directed along the z-axis with z-component equal to the time derivative of the angle θ,

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}} . \tag{16.1.1}$$

The angular velocity vector for the mass element undergoing fixed axis rotation with $\omega_z > 0$ is shown in Figure 16.4. Because the body is rigid, all the mass elements will have the same angular velocity $\vec{\boldsymbol{\omega}}$ and hence the same angular acceleration $\vec{\alpha}$. If the bodies did not have the same angular velocity, the mass elements would "catch up to" or "pass" each other, precluded by the rigid-body assumption.

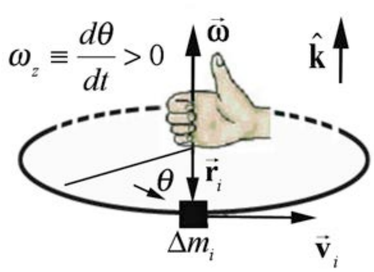

Figure 16.4 Angular velocity vector for a mass element for fixed axis rotation

In a similar fashion, all points in the rigid body have the same angular acceleration,

$$\vec{\boldsymbol{\alpha}} = \frac{d^2\theta}{dt^2}\hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}} . \tag{16.1.2}$$

The angular acceleration vector is shown in Figure 16.5.

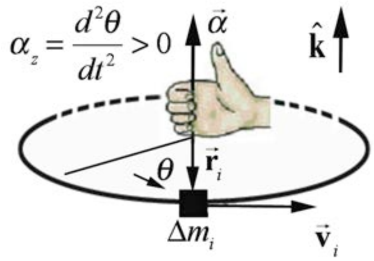

Figure 16.5 Angular acceleration vector for a rigid body rotating about the z-axis

16.2.3 Sign Convention: Angular Velocity and Angular Acceleration

For rotational problems we shall always choose a right-handed cylindrical coordinate system. If the positive z-axis points up, then we choose θ to be increasing in the counterclockwise direction as shown in Figures 16.4 and 16.5. If the rigid body rotates in the counterclockwise direction, then the z-component of the angular velocity is positive, $\omega_z = d\theta / dt > 0$. The angular velocity vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 16.4. If the rigid body rotates in the clockwise direction, then the z-component of the angular velocity angular velocity is negative, $\omega_z = d\theta / dt < 0$. The angular velocity vector then points in the $-\hat{\mathbf{k}}$-direction.

If the rigid body *increases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is positive, $\alpha_z \equiv d^2\theta / dt^2 = d\omega_z / dt > 0$. The angular acceleration vector then points in the $+\hat{\mathbf{k}}$-direction as shown in Figure 16.5. If the rigid body *decreases* its rate of rotation in the counterclockwise (positive) direction then the z-component of the angular acceleration is negative, $\alpha_z = d^2\theta / dt^2 = d\omega_z / dt < 0$. The angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction. To phrase this more generally, if $\vec{\alpha}$ and $\vec{\boldsymbol{\omega}}$ point in the same direction, the body is speeding up, if in opposite directions, the body is slowing down. This general result is independent of the choice of positive direction of rotation. Note that in Figure 16.1, the CD has the angular velocity vector points downward (in the $-\hat{\mathbf{k}}$-direction).

16.2.4 Tangential Velocity and Tangential Acceleration

Because the small element of mass, Δm_i, is moving in a circle of radius r_i with angular velocity $\vec{\boldsymbol{\omega}} = \omega_z \hat{\mathbf{k}}$, the element has a tangential velocity component

$$v_{\theta,\,i} = r_i\,\omega_z\,. \tag{16.1.3}$$

If the magnitude of the tangential velocity is changing, the mass element undergoes a tangential acceleration given by

$$a_{\theta,\, i} = r_i \alpha_z \,. \tag{16.1.4}$$

Recall that the mass element is always accelerating inward with radial component given by

$$a_{r,\, i} = -\frac{v_{\theta,\, i}^2}{r_i} = -r_i \omega_z^2 \,. \tag{16.1.5}$$

Example 16.1 Turntable

A turntable is a uniform disc of mass $1.2\ \text{kg}$ and a radius $1.3 \times 10^1\ \text{cm}$. The turntable is spinning initially in a counterclockwise direction when seen from above at a constant rate of $f_0 = 33\ \text{cycles} \cdot \text{min}^{-1}$ ($33\ \text{rpm}$). The motor is turned off and the turntable slows to a stop in $8.0\ \text{s}$. Assume that the angular acceleration is constant. (a) What is the initial angular velocity of the turntable? (b) What is the angular acceleration of the turntable?

Solution: (a) Choose a coordinate system shown in Figure 16.6.

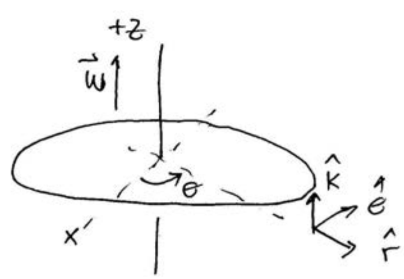

Figure 16.6 Coordinate system for turntable

Initially, the disc is spinning with a frequency

$$f_0 = \left(33\ \frac{\text{cycles}}{\text{min}}\right)\left(\frac{1\,\text{min}}{60\ \text{s}}\right) = 0.55\ \text{cycles} \cdot \text{s}^{-1} = 0.55\ \text{Hz}, \tag{16.1.6}$$

so the initial angular velocity has magnitude

$$\omega_0 = 2\pi\, f_0 = \left(2\pi\ \frac{\text{radian}}{\text{cycle}}\right)\left(0.55\ \frac{\text{cycles}}{\text{s}}\right) = 3.5\ \text{rad} \cdot \text{s}^{-1}\,. \tag{16.1.7}$$

The angular velocity vector points in the $+\hat{\mathbf{k}}$-direction as shown above.

(b) The final angular velocity is zero, so the component of the angular acceleration is

$$\alpha_z = \frac{\Delta\omega_z}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5\ \text{rad}\cdot\text{s}^{-1}}{8.0\ \text{s}} = -4.3\times 10^{-1}\ \text{rad}\cdot\text{s}^{-2}. \tag{16.1.8}$$

The z-component of the angular acceleration is negative, the disc is slowing down and so the angular acceleration vector then points in the $-\hat{\mathbf{k}}$-direction as shown in Figure 16.7.

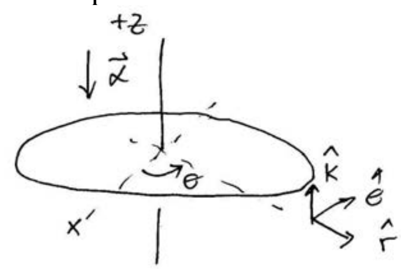

Figure 16.7 Angular acceleration vector for turntable

16.3 Rotational Kinetic Energy and Moment of Inertia

16.3.1 Rotational Kinetic Energy and Moment of Inertia

We have already defined translational kinetic energy for a point object as $K = (1/2)mv^2$; we now define the rotational kinetic energy for a rigid body about its center of mass.

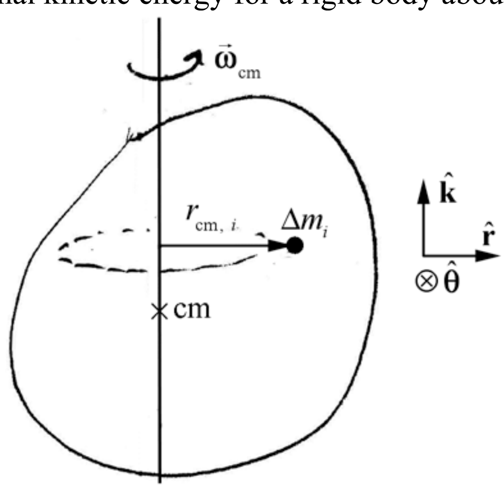

Figure 16.8 Volume element undergoing fixed-axis rotation about the z-axis that passes through the center of mass.

Choose the z-axis to lie along the axis of rotation passing through the center of mass. As in Section 16.2.2, divide the body into volume elements of mass Δm_i (Figure 16.8). Each individual mass element Δm_i undergoes circular motion about the center of mass with z-

component of angular velocity ω_{cm} in a circle of radius $r_{\text{cm},\,i}$. Therefore the velocity of each element is given by $\vec{\mathbf{v}}_{\text{cm},i} = r_{\text{cm},i}\omega_{\text{cm}}\,\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\text{cm},\,i} = \frac{1}{2}\Delta m_i\, v_{\text{cm},\,i}^2 = \frac{1}{2}\Delta m_i r_{\text{cm},\,i}^2 \omega_{\text{cm}}^2 . \tag{16.2.1}$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\text{cm}} &= \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} K_{\text{cm},\,i} = \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} \left(\sum_i \frac{1}{2}\Delta m_i r_{\text{cm},\,i}^2 \right) \omega_{\text{cm}}^2 \\ &= \left(\frac{1}{2} \int_{\text{body}} dm\, r_{\text{dm}}^2 \right) \omega_{\text{cm}}^2 , \end{aligned} \tag{16.2.2}$$

where dm is an infinitesimal mass element undergoing a circular orbit of radius r_{dm} about the axis passing through the center of mass.

> *The quantity*
>
> $$I_{cm} = \int_{body} dm\, r_{dm}^2 . \tag{16.2.3}$$
>
> *is called the **moment of inertia** of the rigid body about a fixed axis passing through the center of mass, and is a physical property of the body. The SI units for moment of inertia are* $\left[\text{kg}\cdot\text{m}^2\right]$.

Thus

$$K_{\text{cm}} = \left(\frac{1}{2} \int_{\text{body}} dm\, r_{\text{dm}}^2 \right) \omega_{\text{cm}}^2 \equiv \frac{1}{2} I_{cm} \omega_{\text{cm}}^2 . \tag{16.2.4}$$

16.3.2 Moment of Inertia of a Rod of Uniform Mass Density

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 16.9. Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.

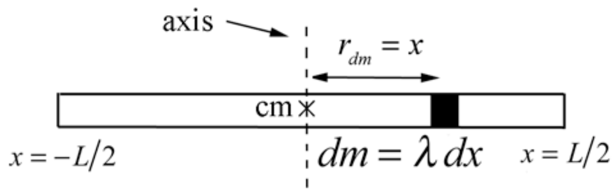

Figure 16.9 Moment of inertia of a uniform rod about center of mass.

Identify an infinitesimal mass element $dm = \lambda\, dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m/L$ is a constant, as we have assumed the rod to be uniform. When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_{dm} = x$. We add together the contributions from each infinitesimal element as we go from $x = -L/2$ to $x = L/2$. The integral is then

$$\begin{aligned} I_{\text{cm}} &= \int_{\text{body}} r_{\text{dm}}^2\, dm = \lambda \int_{-L/2}^{L/2} (x^2)\, dx = \lambda \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} \\ &= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2. \end{aligned} \qquad (16.2.5)$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x- axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the x-y plane if the axis is along the z- axis. The calculation of the moment of inertia in these cases would be more complicated.)

Example 16.2 Moment of Inertia of a Uniform Disc

A thin uniform disc of mass M and radius R is mounted on an axle passing through the center of the disc, perpendicular to the plane of the disc. Calculate the moment of inertia about an axis that passes perpendicular to the disc through the center of mass of the disc

Solution: As a starting point, consider the contribution to the moment of inertia from the mass element dm show in Figure 16.10. Let r denote the distance form the center of mass of the disc to the mass element.

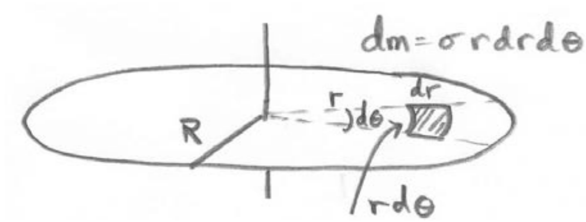

Figure 16.10 Infinitesimal mass element and coordinate system for disc.

Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r\,dr\,d\theta \tag{16.2.6}$$

may be thought of as the product of arc length $r\,d\theta$ and the radial width dr. Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}\,. \tag{16.2.7}$$

Therefore the mass in the infinitesimal area element as given in Equation (16.2.6), a distance r from the axis of rotation, is given by

$$dm = \sigma\, r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta\,. \tag{16.2.8}$$

When the disc rotates, the mass element traces out a circle of radius $r_{dm} = r$; that is, the distance from the center is the perpendicular distance from the axis of rotation. The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus the limits of the integral are

$$I_{\text{cm}} = \int_{\text{body}} r_{dm}^2\, dm = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr\,. \tag{16.2.9}$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr = \frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr \tag{16.2.10}$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2}\frac{r^4}{4}\bigg|_{r=0}^{r=R} = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2\,. \tag{16.2.11}$$

Remark: Instead of taking the area element as a small patch $da = r\,dr\,d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}} = \pi(r+dr)^2 - \pi r^2 = \pi r^2 + 2\pi r\,dr + \pi(dr)^2 - \pi r^2 = 2\pi r\,dr + \pi(dr)^2\,. \tag{16.2.12}$$

In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}} = r\,dr\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) = 2\pi r\,dr\,. \tag{16.2.13}$$

Then the mass element is

$$dm_{\text{ring}} = \sigma da_{\text{ring}} = \frac{M}{\pi R^2} 2\pi r\,dr\,. \tag{16.2.14}$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}} = \int_{\text{body}} (r_{\perp})^2\,dm = \frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3 dr = \frac{1}{2} MR^2\,. \tag{16.2.15}$$

16.3.3 Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm}. The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 16.11).

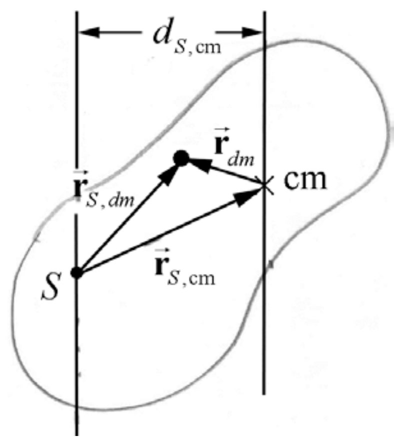

Figure 16.11 Geometry of the parallel axis theorem.

Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\text{cm}} + m\,d_{S,\text{cm}}^2\,. \tag{16.2.16}$$

16.3.4 Parallel Axis Theorem Applied to a Uniform Rod

Let point S be the left end of the rod of Figure 16.9. Then the distance from the center of mass to the end of the rod is $d_{S,\text{cm}} = L/2$. The moment of inertia $I_S = I_{\text{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\text{cm}} = (1/12)\, m L^2$, according to Equation (16.2.16),

$$I_S = \frac{1}{12} m L^2 + \frac{1}{4} m L^2 = \frac{1}{3} m L^2 . \qquad (16.2.17)$$

In this case it's easy and useful to check by direct calculation. Use Equation (16.2.5) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$,

$$\begin{aligned} I_{\text{end}} &= \int_{\text{body}} {r_\perp}^2\, dm = \lambda \int_0^L x'^2\, dx' \\ &= \lambda \left. \frac{x'^3}{3} \right|_0^L = \frac{m}{L} \frac{(L)^3}{3} - \frac{m}{L} \frac{(0)^3}{3} = \frac{1}{3} m L^2 . \end{aligned} \qquad (16.2.18)$$

Example 16.3 Rotational Kinetic Energy of Disk

A disk with mass M and radius R is spinning with angular speed ω about an axis that passes through the rim of the disk perpendicular to its plane. The moment of inertia about cm is $I_{cm} = (1/2)mR^2$. What is the kinetic energy of the disk?

Solution: The parallel axis theorem states the moment of inertia about an axis passing perpendicular to the plane of the disc and passing through a point on the edge of the disc is equal to

$$I_{edge} = I_{cm} + mR^2 . \qquad (16.2.19)$$

The moment of inertia about an axis passing perpendicular to the plane of the disc and passing through the center of mass of the disc is equal to $I_{cm} = (1/2)mR^2$. Therefore

$$I_{edge} = (3/2)mR^2 . \qquad (16.2.20)$$

The kinetic energy is then

$$K = (1/2) I_{edge} \omega^2 = (3/4) mR^2 \omega^2 . \qquad (16.2.21)$$

16.4 Conservation of Energy for Fixed Axis Rotation

Consider a closed system ($\Delta E_{system} = 0$) under action of only conservative internal forces. Then the change in the mechanical energy of the system is zero

$$\Delta E_m = \Delta U + \Delta K = (U_f + K_f) - (U_i + K_i) = 0 . \qquad (16.3.1)$$

For fixed axis rotation with a component of angular velocity ω about the fixed axis, the change in kinetic energy is given by

$$\Delta K \equiv K_f - K_i = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_i^2 , \qquad (16.3.2)$$

where S is a point that lies on the fixed axis. Then conservation of energy implies that

$$U_f + \frac{1}{2} I_S \omega_f^2 = U_i + \frac{1}{2} I_S \omega_i^2 \qquad (16.3.3)$$

Example 16.4 Energy and Pulley System

A wheel in the shape of a uniform disk of radius R and mass m_{p} is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass $I_{\mathrm{cm}} = (1/2) m_{\mathrm{p}} R^2$. A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass m_2 that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass m_1 that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle θ (Figure 16.12). Once the objects are released from rest, the cord moves without slipping around the disk. Calculate the speed of block 2 as a function of distance that it moves down the inclined plane using energy techniques. Assume there are no energy losses due to friction and that the rope does not slip around the pulley

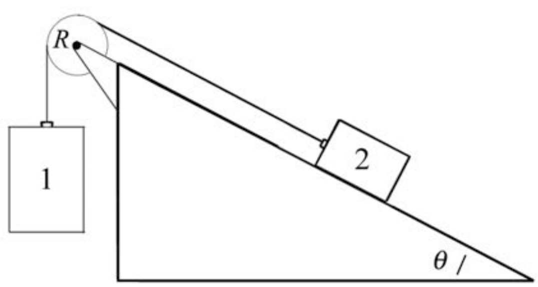

Figure 16.12 Pulley and blocks

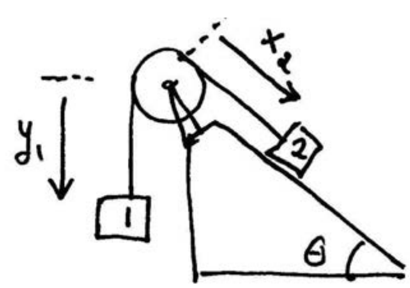

Figure 16.13 Coordinate system for pulley and blocks

Solution: Define a coordinate system as shown in Figure 16.13. Choose the zero for the gravitational potential energy at a height equal to the center of the pulley. In Figure 16.14 illustrates the energy diagrams for the initial state and a dynamic state at an arbitrary time when the blocks are sliding.

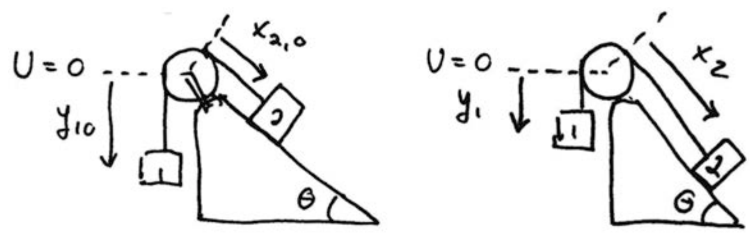

Figure 16.14 Energy diagrams for initial state and dynamic state at arbitrary time

Then the initial mechanical energy is

$$E_i = U_i = -m_1 g y_{1,i} - m_2 g x_{2,i} \sin\theta \,. \tag{16.3.4}$$

The mechanical energy, when block 2 has moved a distance

$$d = x_2 - x_{2,i} \tag{16.3.5}$$

is given by

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2} m_1 v_1^{\,2} + \frac{1}{2} m_2 v_2^{\,2} + \frac{1}{2} I_P \omega^2 \,. \tag{16.3.6}$$

The rope connects the two blocks, and so the blocks move at the same speed

$$v \equiv v_1 = v_2 \,. \tag{16.3.7}$$

The rope does not slip on the pulley; therefore as the rope moves around the pulley the tangential speed of the rope is equal to the speed of the blocks

$$v_{\tan} = R\omega = v \,. \tag{16.3.8}$$

Eq. (16.3.6) can now be simplified

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2 \,. \tag{16.3.9}$$

Because we have assumed that there is no loss of mechanical energy, we can set $E_i = E$ and find that

$$-m_1 g y_{1,i} - m_2 g x_{2,i} \sin\theta = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2, \quad (16.3.10)$$

which simplifies to

$$-m_1 g(y_{1,0} - y_1) + m_2 g(x_2 - x_{2,0}) \sin\theta = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2. \quad (16.3.11)$$

We finally note that the movement of block 1 and block 2 are constrained by the relationship

$$d = x_2 - x_{2,i} = y_{1,i} - y_1. \quad (16.3.12)$$

Then Eq. (16.3.11) becomes

$$gd(-m_1 + m_2 \sin\theta) = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2. \quad (16.3.13)$$

We can now solve for the speed as a function of distance $d = x_2 - x_{2,i}$ that block 2 has traveled down the incline plane

$$v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{\left(m_1 + m_2 + (I_P / R^2) \right)}}. \quad (16.3.14)$$

If we assume that the moment of inertial of the pulley is $I_{\text{cm}} = (1/2) m_{\text{p}} R^2$, then the speed becomes

$$v = \sqrt{\frac{2gd(-m_1 + m_2 \sin\theta)}{\left(m_1 + m_2 + (1/2) m_P \right)}}. \quad (16.3.15)$$

Example 16.5 Physical Pendulum

A physical pendulum consists of a uniform rod of mass m_1 pivoted at one end about the point S. The rod has length l_1 and moment of inertia I_1 about the pivot point. A disc of mass m_2 and radius r_2 with moment of inertia I_{cm} about its center of mass is rigidly attached a distance l_2 from the pivot point. The pendulum is initially displaced to an angle θ_i and then released from rest. (a) What is the moment of inertia of the physical pendulum about the pivot point S? (b) How far from the pivot point is the center of mass of the system? (c) What is the angular speed of the pendulum when the pendulum is at the bottom of its swing?

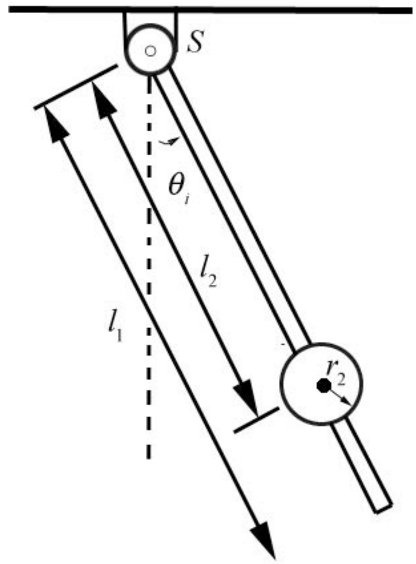

Figure 16.15 Rod and with fixed disc pivoted about the point S

Solution: a) The moment of inertia about the pivot point is the sum of the moment of inertia of the rod, given as I_1, and the moment of inertia of the disc about the pivot point. The moment of inertia of the disc about the pivot point is found from the parallel axis theorem,

$$I_{\text{disc}} = I_{\text{cm}} + m_2 l_2^2. \tag{16.3.16}$$

The moment of inertia of the system consisting of the rod and disc about the pivot point S is then

$$I_S = I_1 + I_{\text{disc}} = I_1 + I_{\text{cm}} + m_2 l_2^2. \tag{16.3.17}$$

The center of mass of the system is located a distance from the pivot point

$$l_{\text{cm}} = \frac{m_1(l_1/2) + m_2 l_2}{m_1 + m_2}. \tag{16.3.18}$$

b) We can use conservation of mechanical energy, to find the angular speed of the pendulum at the bottom of its swing. Take the zero point of gravitational potential energy to be the point where the bottom of the rod is at its lowest point, that is, $\theta = 0$. The initial state energy diagram for the rod is shown in Figure 16.16a and the initial state energy diagram for the disc is shown in Figure 16.16b.

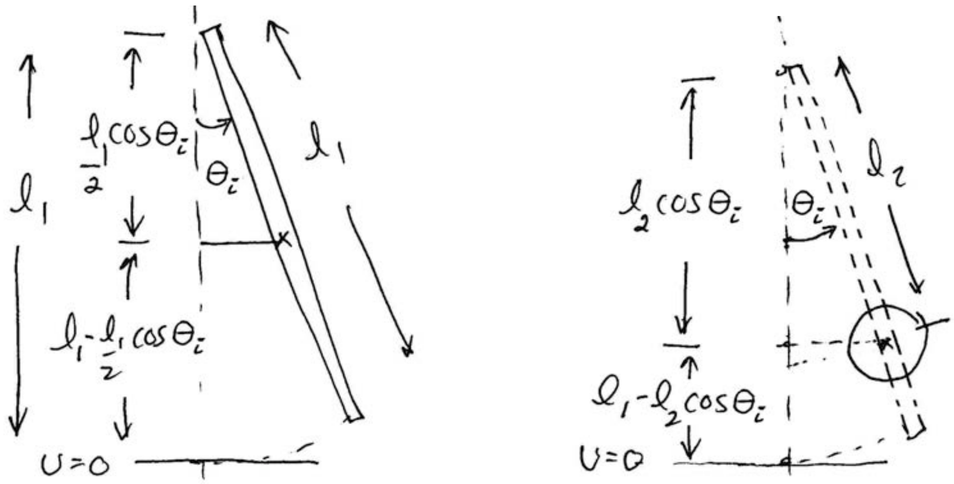

Figure 16.16 (a) Initial state energy diagram for rod (b) Initial state energy diagram for disc

The initial mechanical energy is then

$$E_i = U_i = m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2\cos\theta_i)\,, \tag{16.3.19}$$

At the bottom of the swing, $\theta_f = 0$, and the system has angular velocity ω_f. The mechanical energy at the bottom of the swing is

$$E_f = U_f + K_f = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.20}$$

with I_S as found in Equation (16.3.17). There are no non-conservative forces acting, so the mechanical energy is constant therefore equating the expressions in (16.3.19) and (16.3.20) we get that

$$m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2\cos\theta_i) = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.21}$$

This simplifies to

$$\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g(1 - \cos\theta_i) = \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.22}$$

We now solve for ω_f (taking the positive square root to insure that we are calculating angular speed)

$$\omega_f = \sqrt{\frac{2\left(\dfrac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_S}}\,, \qquad (16.3.23)$$

Finally we substitute in Eq.(16.3.17) in to Eq. (16.3.23) and find

$$\omega_f = \sqrt{\frac{2\left(\dfrac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_1 + I_{\text{cm}} + m_2 l_2^2}}\,. \qquad (16.3.24)$$

Note that we can rewrite Eq. (16.3.22), using Eq. (16.3.18) for the distance between the center of mass and the pivot point, to get

$$(m_1 + m_2) l_{cm} g(1-\cos\theta_i) = \frac{1}{2} I_S \omega_f^2\,, \qquad (16.3.25)$$

We can interpret this equation as follows. Treat the system as a point particle of mass $m_1 + m_2$ located at the center of mass l_{cm}. Take the zero point of gravitational potential energy to be the point where the center of mass is at its lowest point, that is, $\theta = 0$. Then

$$E_i = (m_1 + m_2) l_{cm} g(1-\cos\theta_i)\,, \qquad (16.3.26)$$

$$E_f = \frac{1}{2} I_S \omega_f^2\,. \qquad (16.3.27)$$

Thus conservation of energy reproduces Eq. (16.3.25).

Appendix 16A: Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass dm. The vector from the point S to the mass element is $\vec{\mathbf{r}}_{S,dm}$, the vector from the center of mass to the mass element is $\vec{\mathbf{r}}_{dm}$, and the vector from the point S to the center of mass is $\vec{\mathbf{r}}_{S,\text{cm}}$.

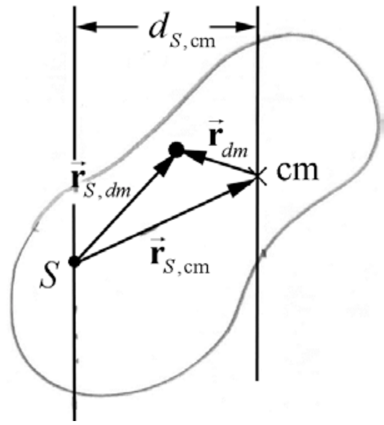

Figure 16A.1 Geometry of the parallel axis theorem.

From Figure 16A.1, we see that

$$\vec{\mathbf{r}}_{S,dm} = \vec{\mathbf{r}}_{S,\text{cm}} + \vec{\mathbf{r}}_{dm} . \tag{16.A.1}$$

The notation gets complicated at this point. The vector $\vec{\mathbf{r}}_{dm}$ has a component vector $\vec{\mathbf{r}}_{\parallel,dm}$ parallel to the axis through the center of mass and a component vector $\vec{\mathbf{r}}_{\perp,dm}$ perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{\text{cm},\perp,dm}\right| = r_{\perp,dm} . \tag{16.A.2}$$

The vector $\vec{\mathbf{r}}_{S,dm}$ has a component vector $\vec{\mathbf{r}}_{S,\parallel,dm}$ parallel to the axis through the point S and a component vector $\vec{\mathbf{r}}_{S,\perp,dm}$ perpendicular to the axis through the point S. The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,dm}\right| = r_{S,\perp,dm} . \tag{16.A.3}$$

The vector $\vec{\mathbf{r}}_{S,\text{cm}}$ has a component vector $\vec{\mathbf{r}}_{S,\parallel,\text{cm}}$ parallel to *both* axes and a perpendicular component vector $\vec{\mathbf{r}}_{S,\perp,\text{cm}}$ that is perpendicular to *both* axes (the axes are parallel, of course). The magnitude of the perpendicular component vector is

$$\left|\vec{\mathbf{r}}_{S,\perp,\text{cm}}\right| = d_{S,\text{cm}} . \tag{16.A.4}$$

Equation (16.A.1) is now expressed as two equations,

$$\begin{aligned}\vec{\mathbf{r}}_{S,\perp,dm} &= \vec{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \vec{\mathbf{r}}_{\perp,dm} \\ \vec{\mathbf{r}}_{S,\|,dm} &= \vec{\mathbf{r}}_{S,\|,\mathrm{cm}} + \vec{\mathbf{r}}_{\|,dm}.\end{aligned} \tag{16.A.5}$$

At this point, note that if we had simply decided that the two parallel axes are parallel to the z-direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that *precesses* (wobbles). Such cases will be considered in later on, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point S is

$$I_S = \int_{\text{body}} dm (r_{S,\perp,dm})^2 . \tag{16.A.6}$$

From (16.A.5) we have

$$\begin{aligned}(r_{S,\perp,dm})^2 &= \vec{\mathbf{r}}_{S,\perp,dm} \cdot \vec{\mathbf{r}}_{S,\perp,dm} \\ &= (\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \vec{\mathbf{r}}_{\perp,dm}) \cdot (\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}} + \vec{\mathbf{r}}_{\perp,dm}) \\ &= d_{S,\mathrm{cm}}^2 + (r_{\perp,dm})^2 + 2\,\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \vec{\mathbf{r}}_{\perp,dm}.\end{aligned} \tag{16.A.7}$$

Thus we have for the moment of inertia about S,

$$I_S = \int_{\text{body}} dm\, d_{S,\mathrm{cm}}^2 + \int_{\text{body}} dm (r_{\perp,dm})^2 + 2 \int_{\text{body}} dm (\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}} \cdot \vec{\mathbf{r}}_{\perp,dm}) . \tag{16.A.8}$$

In the first integral in Equation (16.A.8), $r_{S,\perp,\mathrm{cm}} = d_{S,\mathrm{cm}}$ is the distance between the parallel axes and is a constant. Therefore we can rewrite the integral as

$$d_{S,\mathrm{cm}}^2 \int_{\text{body}} dm = m\, d_{S,\mathrm{cm}}^2 . \tag{16.A.9}$$

The second term in Equation (16.A.8) is the moment of inertia about the axis through the center of mass,

$$I_{\mathrm{cm}} = \int_{\text{body}} dm\, (r_{\perp,dm})^2 . \tag{16.A.10}$$

The third integral in Equation (16.A.8) is zero. To see this, note that the term $\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}$ is a constant and may be taken out of the integral,

$$2\int_{\text{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot\vec{\mathbf{r}}_{\perp,dm}) = \vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot 2\int_{\text{body}} dm\,\vec{\mathbf{r}}_{\perp,dm} \tag{16.A.11}$$

The integral $\int_{\text{body}} dm\;\vec{\mathbf{r}}_{\perp,dm}$ is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence $\vec{\mathbf{0}}$, with the result that

$$2\int_{\text{body}} dm\,(\vec{\mathbf{r}}_{S,\perp,\mathrm{cm}}\cdot\vec{\mathbf{r}}_{\perp,dm}) = 0\,. \tag{16.A.12}$$

Thus, the moment of inertia about S is just the sum of the first two integrals in Equation (16.A.8)

$$I_S = I_{\mathrm{cm}} + m\,d_{S,\mathrm{cm}}^2\,, \tag{16.A.13}$$

proving the parallel axis theorem.