DEEPTO CHAKRABARTY: Now that we've seen how to describe a rotating vector, we can use that to analyze the motion of our gyroscope. So again, I'll draw a side view of my pivot, my rod. Here's the wheel. I'll call this point S. That's a distance d. And we'll assume that the angular velocity is in that direction, so that the spin angle velocity vector is pointing outwards in the plus r hat direction. That's the k hat direction and theta hat is into the screen. Now, again, the weight is acting downwards at the center of mass of the wheel. There's a normal force acting upwards.

So the spin angular momentum with respect to point S is just equal to the moment of inertia of the disk about its center of mass times the angular speed of the spin, and that's directed in the plus r hat direction. And this is the angular momentum with respect to point S. Now, the torque with respect to point S, again, it's R cross F relative to point S. That's going to be Mgd in the plus theta hat direction.

And for a fast omega, if omega is a large angular speed, and therefore if the angular momentum vector is a large vector, then the torque, which acts perpendicular to the angular momentum, will cause the angular momentum vector to rotate. And so as a rotating vector, we can write that the magnitude of the time derivative of that rotating vector is equal to the angular velocity of the rotation times the length of that vector. And this is Ls. And so the length of that vector is just I times little omega, and then we multiply that by capital omega.

Now, this quantity here, dL dT, is the torque. So that's the magnitude of the torque. But the torque we said is equal to Mgd in the theta hat direction. And so the torque, Mgd, is equal to I little omega times big omega. So I can solve for big omega. And big omega, which is the angular speed of rotation of the angular momentum vector, is Mgd divided by moment of inertia, I, times little omega.

Now recall, little omega is the angular velocity, the angular speed of the spin of the disk or the wheel. Capital omega is the angular speed of the rotation of the angular momentum vector. It basically tells us the speed at which the center of mass of the wheel orbits around the vertical axis through the pivot. We call capital omega the precessional angular velocity. So the precessional angular velocity is capital omega. And notice that the faster little omega is, the bigger little omega is, the slower the precession angular velocity is.

Now, this expression tells us what the magnitude of capital omega is, what the magnitude of

that precessional angular velocity is, but it doesn't tell us which way the system is processing, whether say, viewed from the top, the motion is clockwise or counterclockwise, or equivalently, which way the vector, capital omega, is pointing. We know it must point along the vertical axis, but does it point upwards, in the plus K hat direction or downwards in the minus K hat direction.

To see that, we need to look at which way the angular momentum vector is rotating. And there are two possibilities. So let's again go to a top view of our gyroscope. So suppose here is the pivot point, and here is my gyroscope. And the way I'm drawing things is that r hat is this way, is the top view, so theta hat is that way, and k hat is out of the screen.

So in the example that I did earlier, the angular momentum was pointing in the plus r hat direction. So that's L sub s pointing that way. If the torque is pointing in the theta hat direction, then that's going to act to rotate the angular momentum vector this way. And so the sense of rotation will be like that. And so in that case, looking down on the system, we would see a counterclockwise rotation. And that's equivalent to omega vector pointing in the plus k hat direction.

Alternatively, suppose the wheel were spinning in the other direction. OK, so the sense of rotation of this wheel around the axle, we're in the opposite direction. In that case, even when the wheel was on this side of the pivot --again, this is a top view-- the angular momentum vector would be pointing in the opposite direction. So now this is my angular momentum vector. It's pointing in the opposite direction that the axle is pointing. Still along a line, but in the minus R hat direction.

But the torque would still be in the theta hat direction. And so my new angular momentum vector would rotate this way, which would be equivalent to the axle rotating in this direction. And that's equivalent to a rotation in the opposite sense, clockwise as viewed downward from the top, or in other words, with capital omega hat vector pointing in the minus K hat direction. So the direction of precession depends upon which way the wheel is spinning, which way the spin angular momentum vector is actually pointing.

Now, I'd just like to point out that there's an approximation that we've been making here. I've alluded to it, but I want to make it very specific right now. We have been assuming that the spin angular momentum vector L is large enough that the torque vector, which is perpendicular, provides only a small angular impulse that rotates L without changing it in

length. Now, as L rotates, the direction of r hat and the direction of theta rotate with it, and the torque is always in the theta hat direction. So after any small delta T, when the angular momentum vector rotates, the instantaneous torque at that next instant is still perpendicular to the rotating momentum vector.

And so we're making an approximation that L is large enough that we can consider the instantaneous angular impulse as a small perpendicular perturbation. That's equivalent, it turns out, to saying that, so this is the approximation that we're making, that we've been making so far, in this vector, is that little omega, the spin angular velocity, is much, much larger than capital omega, the precession angular velocity. This is called the gyroscopic approximation. We'll see a more precise statement of it later on. But it's basically equivalent to saying that the spin angular momentum vector is so large that we can consider the angular impulse due to the torque as causing a pure rotation of the vector without any change in its length.

It's important to keep in mind that a vector can include both a rotating component and a constant component. And in that case, it's important to identify what the rotating component is in order to use the analysis that we presented earlier. So for example, let's consider the case of a tilted gyroscope, instead of one that's horizontal and parallel to the ground. So here is my pivot point. That's the vertical. And instead of a horizontal gyroscope, now I'll draw my gyroscope at some angle like this. Here's my wheel. I'll call that angle with the vertical phi. This distance is still d. And I'll use the usual coordinate system, that r hat is pointing out this way, k hat is pointing vertically, and theta hat is pointing into the screen.

Now, in this case, the gyroscope will still process around the vertical axis to the pivot point, and the angle phi will remain constant. Now, the angular momentum vector due to spin points now not in the r hat direction, but again, outward along the axis of rotation. And that angular momentum vector can be decomposed into two components, an r hat component and a k hat component. So in other words, my angular momentum vector can be written as the sum of a vector pointing along Z axis or the k hat direction plus a vector pointing in the radial direction. I can also write that as L sub z times k hat plus L sub r times r hat.

Now, notice as this gyroscope precesses around, the z hat component is constant, but the r hat component rotates around. OK, so the r hat component, the vector along the r hat direction, is a purely rotating vector, whereas the z component is a constant vector. So L sub z is constant and L sub r is rotating. And so now, the magnitude of dL dT, which is equal to the

magnitude of dLz dT plus dLr dT, well dLz dT, since that's a constant vector, is just 0. So the time derivative just involves the r component. And so that's equal to the angular speed of rotation times the length of the rotating vector, which is just the r component, not the z component.

So this is multiplied by L sub r. Now, in this particular case, the r component of the angular momentum vector is L, the angular momentum vector, times the sine of phi. So in this particular case, this would be omega times the full L times the sine of phi. So in the case that we have here, the angular momentum vector consists of a constant part and a rotating part. And the magnitude of its time derivative is equal to the angular velocity rotation times the length of the rotating part of the vector.