MITOCW | MIT8_01F16_L14v03_360p

Now we're going to analyze a more complicated example of drag forces, where we have an object falling in a gravitational field with gravity.

We have a resistive force.

And this is an object in air.

And so our model will be for the resistive force that it's going to be proportional to the velocity squared.

Now to get its direction right, opposing the motion, j hat.

So this will turn out to be a more complicated analysis.

But first again, let's think about the units of our coefficient beta.

It has the units of force divided by velocity squared.

So we write the units of force, kilogram meters per second squared, and the units of velocity squared, meter squared per second squared.

And so we see that we have units of kilogram per meter for our coefficient beta.

Now, we'll apply as usual Newton's second law, F equals m a, to get our equation of motion.

We're looking at the j direction.

And our forces are gravity minus the velocity squared resistive drag force.

And that's equal to the derivative of the velocity dv/dt.

In this example, it's a one dimensional motion.

So I'm dropping any mention of y direction for simplicity.

Now we can rewrite this equation as dv/dt.

Let's divide through by m is g minus beta over m v square.

And this is a linear-- it's a first order differential equation, dv/dt.

It's a non-linear equation because the velocity term appears here as a square and there's a constant term.

But we can still apply our technique of separation of variables.

And so when we write this equation as dv/dt we'll separate out dv's and t's, so we have d d times g minus beta over and m v squared is equal to dt.

Now, I'm going to do two things just to clean this up for algebra a little bit later.

I'm going to multiply both sides by a minus sign.

And I'm going to pull the g out.

So I have 1 minus beta over m g v squared.

And on the other side, I have minus dt.

And now I can get this equation in the form that I'd like to integrate, which is minus dv times 1 minus beta over m g v squared equals minus gdt.

Now, the trouble here is this integral is a little bit complicated.

So I'd like to make a change of variable.

And my change of variable will be u equals the square root of beta m g times v. And that implies that du is beta m g dv.

And the limits, if we start our object at rest, so if u0 equals 0, then because v0 equal 0, that's our first limit.

Now we have to be a little bit careful because if we drop this object at rest, initially it will be moving very slowly.

And so our resistive model doesn't actually apply.

However, we're going to neglect that effect even though if we were to do a more complicated analysis, we would have to change our model as the object is following, so it would be a multi-stage motion.

First, at the beginning with our only velocity dependent resistant.

And then as it gets some initial speed and it's going faster, we change our model.

That's why the actual problem can be quite complicated.

But we're just trying to keep things simple here.

And then u of t is square root of beta mg d of t.

And then with this change of variable, my integration, remember I have a dv, so I have to multiply the left side by mg over beta.

And I'm integrating with a minus sign du times 1 minus u squared from 0 to this final value u of t And over here I'm just integrating minus g dt.

Now again, for simplicity, I'm going to bring this term, the beta over mg over to the other side.

So I'll use the magic of our light board by just erasing that and bringing it to the other side, which makes my life a little easier.

And now, this interval can be done by the method of integration by parts.

It's a nice problem in calculus.

And you can verify for yourself that the result is one half natural log of 1 plus u over 1 minus u evaluated at our limits.

And over here I have minus g square root of beta over mg.

Now, once again, for a little bit of simplicity, I'm going to bring the 2 over to the other side.

And now, I evaluate my limits.

Now recall that when you have a minus log, we're flipping, because log of b over a equals minus log over ab, so when I put out my limits in, I have natural log of-- now remember, what are our limits?

We have 1-- I'm flipping-- minus u is beta mg times v of t.

And I have the 1 plus beta mg v of t.

And that's equal to minus as 2g square root of beta mg.

Now, again, we'll use the fact that e to log of x is x.

And so if I exponentiate both sides, I end up with 1 minus the square root of beta mg d of t over 1 plus square root of beta mg v of t is equal to exponential minus 2g beta mg times t.

And we'll just move that.

OK, now this is a little bit of algebra.

I want to solve for v of t.

If I bring this side over to there, I'll just do that to make the first step a little simpler to see.

So we have 1 minus square root of beta mg v of t equals 1 plus square root of beta over mg v of t times e to the minus this factor 2g square root of beta mg t.

Now this is a lot of stuff to carry around.

I'd like to introduce a constant here, tao, which I'm going to find to be square root of mg beta 1 over 2g.

And so this whole term is going to just be e to the minus tao.

It's a nice example for you to work out that the units of tao are the units of seconds.

And that's a little exercise to work out.

Now, I just have to collect my terms.

And what I'll do is I'll collect the T terms on the right and the terms that don't have vt on the left.

So I have 1 minus e to the minus t over tao on the left is equal to 1 plus e to the minus t over tao on the right times beta mg v of t.

And so I get my solution, v of t equals the square root of mg over beta times 1 minus e to the minus t over tau over 1 plus e to the minus t over tao.

Well, it's not a simple solution at all.

But let's examine when you have a case like this-- again, it would be a nice exercise to graph this out.

But right now we're going to consider the limit as t goes to infinity.

And remember that e minus t over t goes to 0 when t goes to infinity.

So we just have 1 over 1.

And what we get as t goes to infinity is the quantity mg over beta.

And this is what we call the terminal velocity.

Now what does terminal velocity mean?

Well, when object is falling and there's a resistive force, as the object falls faster and faster, the resistive force gets greater and greater until if we go back to Newton's second law and look at it, as v gets faster and faster, eventually these two terms are equal.

And when these two terms are equal, that's the statement that the right-hand side has to be zero.

So what we mean by terminal velocity is it's the velocity is no longer changing in time.

And then we can immediately check our work by going to Newton's second law and see what that case is if we set this quantity equal to zero.

In other words, when we set mg minus beta v squared terminal equal to 0 we can solve for v terminal, and we get a square root of mg divided by beta.

And that agrees with our lengthy calculation.

So we think we're on the right track.