MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

# Receivers, Antennas, and Signals – 6.661

Solutions -- Problem Set No. 1 December 22, 2003

## Problem 1.1

a) $v(t) = u_{-1}(t)\, e^{-\alpha t}$, [See (1.2.1) in notes; $u_{-1}(t) = \{0$ for $t < 0$; $= 1$ otherwise$\}$]
$\underline{V}(f) = \int_{-\infty}^{\infty} u_{-1}(t)\, e^{-\alpha t}\, e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-(j2\pi f + \alpha)t}\, dt = (-1/(j2\pi f + \alpha))\, e^{-(j2\pi f + \alpha)t}\Big|_{t=0}^{\infty}$
$\boxed{\underline{V}(f) = +1/(j2\pi f + \alpha)}$

b) $R_v(\tau) = \int_{-\infty}^{\infty} [u_{-1}(t)\, e^{-\alpha t}]\, [u_{-1}(t-\tau)\, e^{-\alpha(t-\tau)}]dt$ [See (1.2.5) in notes]
If $\tau > 0$, $R_v(\tau) = \int_{\tau}^{\infty} e^{-\alpha(2t-\tau)}\, dt = e^{\alpha\tau} \int_{\tau}^{\infty} e^{-2\alpha t}\, dt = -(e^{\alpha\tau}/2\alpha)\, e^{-2\alpha t}\Big|_{t=\tau}^{\infty} = e^{-\alpha\tau}/2\alpha$
If $\tau < 0$, $R_v(\tau) = \int_0^{\infty} e^{-\alpha(2t-\tau)}\, dt = e^{\alpha\tau}/2\alpha$
In general, $\boxed{R_v(\tau) = e^{-\alpha|\tau|}/2\alpha}$

c) (i) $S(f) = |\underline{V}(f)|^2 = |-1/(j2\pi f + \alpha)|^2 = \boxed{1/[(2\pi f)^2 + \alpha^2]}$
(ii) $S(f) = \mathcal{F}\{R_v(\tau)\} = \int_{-\infty}^{\infty} (e^{-\alpha|\tau|}/2\alpha)\, e^{-j2\pi f\tau}\, d\tau$ [See (1.2.5) and Solution to 1.1b]
$= \int_{-\infty}^{0} (e^{\alpha\tau}/2\alpha)\, e^{-j2\pi f\tau}\, d\tau + \int_0^{\infty} (e^{-\alpha\tau}/2\alpha)\, e^{-j2\pi f\tau}\, d\tau$
$= (1/2\alpha)\{[1/(\alpha - j2\pi f)] + [1/(\alpha + j2\pi f)]\}$
$= (1/2\alpha)[(\alpha + j2\pi f + \alpha - j2\pi f)/(\alpha^2 + (2\pi f)^2)] = \boxed{1/(\alpha^2 + (2\pi f)^2)}$

## Problem 1.2

a) $y(t) = x(t) * h(t) : \underline{Y}(f) = \underline{X}(f)\underline{H}(f)$
$\underline{S}_y(f) = \underline{Y}(f)\underline{Y}^*(f) = \underline{X}(f)\underline{X}^*(f)\underline{H}(f)\underline{H}^*(f)$
where $\underline{H}(f) = -1/(j2\pi f + \alpha)$ [See solution to 1.1a]. Therefore
$\boxed{\underline{S}_y(f) = S_x(f)/(\alpha^2 + (2\pi f)^2)}$

b) We designate $\sigma_y$= rms deviation of y(t)
$\sigma_y^2 = \int_{-\infty}^{\infty} S_y(f)\, df = \int_{-\infty}^{\infty} |y(t)|^2 dt$,
where one possible $y(t) = u_{-1}(t)\, e^{-\alpha t}$ [See solution to 1.1c(i)]. Thus
$\sigma_y^2 = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_0^{\infty} e^{-2\alpha\tau} dt = 1/2\alpha$, Therefore $\boxed{\sigma_y = (2\alpha)^{-0.5}}$

## Problem 1.3

**Boldface** indicates vectors here; underbars indicate complex quantities

a) Power $= |\mathbf{E}|^2/2\eta_o = 1$, so $|\mathbf{E}| = (2\eta_o)^{0.5} = 27.5$ [where $\eta_o = 377$ ohms] Thus
$\boxed{\mathbf{E}(t,x,y,z) = 27.5\mathbf{y}\cos(\omega t - kz)}$, where $\boxed{\omega = 2\pi\, 10^9 \text{ and } k = 2\pi f/c = 20.9 \text{ throughout}}$
$\boxed{\underline{\mathbf{E}}(x,y,z) = 27.5\mathbf{y}\, e^{-jkz}}$ where **y** is a unit vector in the y direction

b) (i) The boundary conditions at z = 0 dictate that E = 0 there, which is satisfied if

$\mathbf{E}(t,x,y,z) = 27.5\mathbf{y}\,[\cos(\omega t - kz) - \cos(\omega t + kz)] = \boxed{55\mathbf{y}\,(\sin\omega t)(\sin kz)\ \{v/m\}}$

(ii) $\underline{\mathbf{E}}(x,y,z) = 27.5\mathbf{y}\,(e^{-jkz} - e^{+jkz}) = \boxed{-j\,55\mathbf{y}\sin kz\ \{v/m\}}$

c) (i) $\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t)$; $\mathbf{H}(t) = -\mathbf{x}(2/\eta_o)^{0.5}\cos(\omega t\text{-}kz)$

$\boxed{\mathbf{S}(t) = 2\mathbf{z}\cos^2(\omega t\text{-}kz)\ \{W/m^2\}}$

(ii) $\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}^*$; $\underline{\mathbf{H}} = -\mathbf{x}\,(2/\eta_o)^{0.5} e^{-jkz}$; $\boxed{\overline{\underline{\mathbf{S}}} = 2\mathbf{z}\ \{W/m^2\}}$

(iii) $\mathbf{H}(t,x,y,z) = -\mathbf{x}\,(2/\eta_o)^{0.5}\,[\cos(\omega t - kz) + \cos(\omega t + kz)]$

$= -\mathbf{x}\,(8/\eta_o)^{0.5}\cos(\omega t)\cos(kz)$

$\mathbf{E}(t,x,y,z) = \mathbf{y}\,(8\eta_o)^{0.5}\sin(\omega t)\sin(kz)$

$\mathbf{S}(t) = 8\mathbf{z}\sin\omega t\cos\omega t\cos kz\sin kz = \boxed{2\mathbf{z}\sin 2\omega t\sin 2kz\ \{W/m^2\}}$

(iv) $\underline{\mathbf{H}}(x,y,z) = -(2/\eta_o)^{0.5}\mathbf{x}\,(e^{-jkz} + e^{+jkz}) = -(8/\eta_o)^{0.5}\mathbf{x}\cos kz$

$\underline{\mathbf{E}}(x,y,z) = -j\mathbf{y}\,(8\eta_o)^{0.5}\sin(kz)$

$\underline{\mathbf{S}}(x,y,z) = 8\mathbf{z}\sin kz\cos kz = \boxed{4j\mathbf{z}\sin 2kz\ \{W/m^2\}}$

d) $\mathbf{E}(t,x,y,z) = (2\eta_o)^{0.5}\,\mathbf{y}\cos(\omega t - kz)$; $W_e = \varepsilon_o|\mathbf{E}(t)|^2/2 = \boxed{(1/c)\cos^2(\omega t+\omega/c)\ [J/m^3]}$

$\mathbf{H}(t) = -\mathbf{x}(2/\eta_o)^{0.5}\cos(\omega t\text{-}kz)$; $W_m = \mu_o|\mathbf{H}(t)|^2/2 = \boxed{(1/c)\cos^2(\omega t+\omega/c)\ [J/m^3]}$

$\langle W_m\rangle = \langle W_e\rangle = 1/2c$

## Problem 1.4

a) Within a 100-MHz band kTB = 270k $10^8$ = $\boxed{3.7 \times 10^{-13}\ \text{Watts}}$ would flow to a matched load.

b) $v_{rms}^2/Z_o$ = power, so $v_{rms} = (270kZ_o\,10^8)^{0.5}$ volts = $\boxed{4.3\ \text{microvolts}}$

c) Raleigh-Jeans applies when hf<<kT, or f << kT/h = $\boxed{\sim 6 \times 10^{12}\ \text{Hz}}$