MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Problem Set 4

Readings:
(a) Notes from Lecture 6 and 7.
(b) [Cinlar] Sections I.4, I.5 and II.2
(c) [GS] Chapter 3

Exercise 1. Let N be a random variable that takes nonnegative integer values. Let $X_1, X_2, \ldots,$ be a sequence of i.i.d. discrete random variables that have finite expectation and are independent from N. Use iterated expectations to show that the expected value of $\sum_{i=1}^{N} X_i$ is $\mathbb{E}[N]\mathbb{E}[X_1]$.

Exercise 2. Let X and Y be binomial with parameters (m, p) and (n, q), respectively.

(a) Show that if X is independent from Y, $m = n$, and $p = q$ then $X + Y$ is binomial. *Hint:* Use the interpretation of the binomial, not algebra.

(b) Does the conclusion of part (a) remain valid if $m \neq n$? If X and Y are not independent? If $p \neq q$?

(c) Show that if X and Y are independent, then

$$\mathbb{P}(X + Y = k) = \sum_{i=-\infty}^{\infty} p_X(i) p_Y(k - i).$$

(d) Use the result from part (c) to find the PMF of $X + Y$ where X and Y are independent Poisson random variables with parameters λ and μ, respectively. *Hint:* The "binomial theorem" states that

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Exercise 3. A 4-sided die has its four faces labeled as a, b, c, d. Each time the die is rolled, the result is a, b, c, or d, with probabilities p_a, p_b, p_c, p_d, respectively. Different rolls are statistically independent. The die is rolled n times. Let N_a and N_b be the number of rolls that resulted in a or b, respectively. Find the covariance of N_a and N_b.

Exercise 4. Suppose that X and Y are discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. An elegant way of defining the conditional expectation of Y given X is as a random variable of the form $\phi(X)$ (where ϕ is a measurable function), such that

$$\mathbb{E}[\phi(X)g(X)] = \mathbb{E}[Yg(X)],$$

for all measurable functions g. In this problem, we will prove that this condition defines the conditional expectation uniquely; that is, if we also have

$$\mathbb{E}[\psi(X)g(X)] = \mathbb{E}[Yg(X)],$$

for every measurable function g, then $\phi(X)$ and $\psi(X)$ are almost surely equal, i.e., $\mathbb{P}(\phi(X) = \psi(X)) = 1$.

(a) Prove that the following sets are $\mathcal{F}$-measurable: $\{\phi(X) > \psi(X)\}$ and, for any integer n, $A_n := \{\phi(X) > \psi(X) + 1/n\}$.

(b) Assume the contradiction $\mathbb{P}(\phi(X) = \psi(X)) < 1$ and use $g(x) = \mathbf{1}_{A_n}$ for some appropriate n to show that the conditional expectation is unique.

Exercise 5. A machine is refilled each morning with n portions of vanilla and chocolate ice creams each (a total of $2n$ portions). Customers arrive sequentially, each getting one of the ice creams independently with probability $1/2$. Consider the first moment when a customer receives an "out of order" message. Let X be the number of portions of the other type left at this moment, $0 \leq X \leq n$. Find the distribution of X.

Exercise 6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. (So, μ is a measure, but not necessarily a probability measure.) Let $g : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Let $\{B_i\}$ be a sequence of disjoint measurable sets. Prove that

$$\int_{\cup_i B_i} g\, d\mu = \sum_{i=1}^{\infty} \int_{B_i} g\, d\mu.$$

(Be rigorous!)

Note: As an application, this exercise gives another rich source of probability measures. Namely, take f – a nonnegative measurable function on the real line with $\int_{\mathbb{R}} f(x)dx = 1$ (integral w.r.t. Lebesgue measure), and define a set-function $\mathbb{P}(A) = \int_A f dx$. The exercise shows that $\mathbb{P}(\cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B})$. Function f is called the probability density function (PDF) of $\mathbb{P}$.

Exercise 7. ***[Optional, not to be graded]*** Let μ and ν be two finite measures on $(\mathbb{R}, \mathcal{B})$. Show that if

$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f \, d\nu$$

for all bounded continuous functions f then $\mu = \nu$. (*Hint:* write $\mathbb{1}_{(a,b)}(x)$ as an increasing limit of continuous functions.)

Note: This exercise shows that measure on Borel σ-algebra is uniquely characterized by its values on continuous functions. This is true on $\mathbb{R}$, $\mathbb{R}^n$ and any other topological space. Similar to how it is sufficient to know measures only on intervals $(-\infty, a)$ it is sufficient to consider only a handful of functions (such as all sines and cosines, or all exponents). This will be discussed later.