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Lecture 23

MARKOV CHAINS III

Contents

1 PERIODICITY

Previously we showed that when a finite state M.c. has only one recurrent class and π is the corresponding stationary distribution, then $\mathbb{E}[N_i(t)|X_0 = k]/t \to \pi_i$ as $t \to \infty$, irrespective of the starting state k. Since $N_i(t) = \sum_{n=1}^{t} I_{\{X_n=i\}}$ is the number of times state i is visited up till time t, we have shown that $\frac{1}{t}\sum_{n=1}^{t} \mathbb{P}(X_n = i|X_0 = k) \to \pi_i$ for every state k, i.e., $p_{ki}^{(n)}$ converges to π_i in the Cesaro sense. However, $p_{ki}^{(n)}$, which from now on we call *transient* probability distribution of a Markov chain, need not converge, as the following example shows. Consider a 2 state Markov Chain with states $\{1, 2\}$ and $p_{12} = 1 = p_{21}$. Then $p_{12}^{(n)} = 1$ when n is odd and 0 when n is even.

Let x be a recurrent state and consider all the times when x is accessible from itself, i.e., the times in the set $I_x = \{n \geq 1 : p_{xx}^{(n)} > 0\}$ (note that this set is non-empty since x is a recurrent state). One property of I_x we will make use of is that it is closed under addition, i.e., if $m, n \in I_x$, then $m + n \in I_x$. This is easily seen by observing that $p_{xx}^{(m+n)} \geq p_{xx}^{(m)} p_{xx}^{(n)} > 0$. Let d_x be the greatest common divisor of the numbers in I_x. We call d_x the *period* of x. We now show that all states in the same recurrent class has the same period.

Lemma 1. *If x and y are in the same recurrent class, then $d_x = d_y$.*

Proof. Let m and n be such that $p_{xy}^{(m)}, p_{yx}^{(n)} > 0$. Then $p_{yy}^{(m+n)} \geq p_{xy}^{(m)} p_{yx}^{(n)} > 0$. So d_y divides $m+n$. Let l be such that $p_{xx}^{(l)} > 0$, then $p_{yy}^{(m+n+l)} \geq p_{yx}^{(n)} p_{xx}^{(l)} p_{xy}^{(m)} > 0$. Therefore d_y divides $m+n+l$, hence it divides l. This implies that d_y divides d_x. A similar argument shows that d_x divides d_y, so $d_x = d_y$. □

A recurrent class is said to be *periodic* if the period d is greater than 1 and *aperiodic* if $d = 1$. The 2 state Markov Chain in the example above has a period of 2 since $p_{11}(n) > 0$ iff n is even. A recurrent class with period d can be divided into d subsets, so that all transitions from one subset lead to the next subset.

Why is periodicity of interest to us? It is because periodicity is exactly what prevents the convergence of $p_{xy}^{(n)}$ to π_y. Suppose y is a recurrent state with period $d > 1$. Then $p_{yy}^{(n)} = 0$ unless n is a multiple of d, but $\pi_y > 0$. However, if $d = 1$, we have positive probability of returning to y for all time steps n sufficiently large.

Lemma 2. *If* $d_y = 1$, *then there exists some* $N \geq 1$ *such that* $p_{yy}^{(n)} > 0$ *for all* $n \geq N$.

Proof. We first show that $I_y = \{n \geq 1 : p_{yy}^{(n)} > 0\}$ contains two consecutive integers. Let n and $n + k$ be elements of I_y. If $k = 1$, then we are done. If not, then since $d_y = 1$, we can find a $n_1 \in I_y$ such that k is not a divisor of n_1. Let $n_1 = mk + r$ where $0 < r < k$. Consider $(m + 1)(n + k)$ and $(m + 1)n + n_1$, which are both in I_y since I_y is closed under addition. We have

$$(m + 1)(n + k) - ((m + 1)n + n_1) = k - r < k.$$

So by repeating the above argument at most k times, we eventually obtain a pair of consecutive integers $m, m + 1 \in I_y$. If $N = m^2$, then for all $n \geq N$, we have $n - N = km + r$, where $0 \leq r < m$. Then $n = m^2 + km + r = r(1 + m) + (m - r + k)m \in I_y$. □

2 COUPLING TECHNIQUE AND MIXING

We now establish that when a Markov chain has one recurrent class (irreducible) and aperiodic, the transient distribution approaches the (unique) steady state distribution as time goes to infinity. Namely, for every two states x, y we have $p_{xy}^{(n)} \to \pi_y$ as $n \to \infty$. This is commonly called *mixing* property of a Markov chain.

Theorem 1. *Consider an irreducible, aperiodic Markov chain. Then for all states* x, y, $\lim_{n\to\infty} p_{xy}^{(n)} = \pi_y$.

For the case of periodic chains, there is a similar statement regarding convergence of $p_{xy}^{(n)}$, but now the convergence holds only for certain subsequences of the time index n. See [**?**] for further details.

There are at least two generic ways to prove this theorem. One is based on the Perron-Frobenius Theorem which characterizes eigenvalues and eigenvectors of non-negative matrices. Specifically the largest eigenvalue of P is equal to unity and all other eigenvalues are strictly smaller than unity in absolute value. The P-F Theorem is especially useful in the special case of so-called *reversible* M.c.. These are irreducible M.c. for which the unique stationary distribution satisfies $\pi_x p_{xy} = \pi_y p_{yx}$ for all states x, y. The subject of reversible M.c. is a rich subject on its own and is outside of the scope of this lecture. In the special case of reversible M.c. the following important refinement of Theorem 1 is known.

Theorem 2. *Consider an irreducible aperiodic Markov chain which is reversible. Then there exists a constant* C *such that for all states* x, y, $|p_{xy}^{(n)} - \pi_y| \le C|\lambda_2|^n$, *where* λ_2 *is the second largest (in absolute value) eigenvalue of* P.

Since by P-F Theorem $|\lambda_2| < 1$, this theorem is indeed a refinement of Theorem 1 as it gives a concrete rate of convergence to the steady-state.

We adopt a different approach which does not rely on the reversibility assumption. The main technique underlying our approach is the method of coupling, which we now discuss. The method of coupling allows combining two Markov chains into one by building them on the same probability space. Intuitively, two Markov chains X_n and Y_n are coupled if we construct out of them a single Markov chain $Z_n = (X_n, Y_n)$ such that each "marginal" Markov chain X_n and Y_n behaves as an individual Markov chain before coupling, but the evolution of X_n and Y_n is in general dependent. Formally

Definition 1. Given two Markov chains X_n and Y_n on state spaces $\{1, \ldots, N\}$ and $\{1, \ldots, M\}$, respectively, and transition probability matrices $P = (p_{xy}, 1 \le x, y \le N)$ and $Q = (q_{xy}, 1 \le x, y \le M)$, a coupling of X_n and Y_n is any Markov chain Z_n with a state space $\{1, \ldots, N\} \times \{1, \ldots, M\}$ and a transition matrix

$$R = \left(r_{(x_1,x_2),(y_1,y_2)}, 1 \le x_1, y_1 \le N, 1 \le x_2, y_2 \le M\right),$$

which satisfies the following properties: for every $1 \le x_1, y_1 \le N$ and $1 \le x_2 \le M$,

$$\sum_{y_2=1}^{M} r_{(x_1,x_2),(y_1,y_2)} = p_{x_1,y_1},$$

and for every $1 \le x_1 \le N$ and $1 \le x_2, y_2 \le M$,

$$\sum_{y_1=1}^{N} r_{(x_1,x_2),(y_1,y_2)} = q_{x_2,y_2},$$

In words, coupling $Z_n = (X_n, Y_n)$ of two Markov chains X_n and Y_n means that the Markov chain X_n transitions from state x_1 to state y_1 with probability p_{x_1,y_1}, *regardless* of the state of the Y_n, and vice verse.

How do we know that the definition of the coupling is not vacuous and at least one coupling exists? This is easy: simply consider running chains X_n and Y_n independently and set $Z_n = (X_n, Y_n)$. Formally, set $r_{(x_1,x_2),(y_1,y_2)} = p_{x_1,y_1} q_{x_2,y_2}$ for all x_1, x_2, y_1, y_2. It is easy to check that this leads to a valid coupling. However, this is also the least useful coupling. We now consider a different coupling idea applied in the special case when $M = N$ and $Q = P$. Namely, we will couple a Markov chain X_n with itself. For convenience, we again use notation X_n and Y_n, though now Y_n has the same state space $\{1, \ldots, N\}$ and transition matrix $Q = P$ as X_n. We now defined the coupled chain $Z_n = (X_n, Y_n)$ according to the following rules

$$r_{(x_1,x_2),(y_1,y_2)} = \begin{cases} p_{x_1,y_1} p_{x_2,y_2}, & \text{when } x_1 \ne x_2; \\ p_{x_1,y_1}, & \text{when } x_1 = x_2, y_1 = y_2; \\ 0, & \text{when } x_1 = x_2, y_1 \ne y_2; \end{cases}$$

In words, the Markov chains X_n and Y_n run independently until they collide for the first time in the same state $X_n = Y_n = x$. Once this happen, they transition to a new state y which is the *same* for X_n and Y_n with probability p_{xy} - the transition probability of the original Markov chain. Again, it is easy to check that $R = (r_{(x_1,x_2),(y_1,y_2)})$ defines a valid coupling of the Markov chain X_n with itself. We define T to be the first (random) time when two copies of M.c. collide for the first time. Namely, $T = \min\{n \geq 0 : X_n = Y_n\}$. Then, $X_n = Y_n$ for all $n \geq T$. We define T to be infinite if the states never collide. We now ready to prove the "mixing" theorem.

Proof of Theorem 1. Fix an arbitrary two states x_0 and y_0. We need to show that $\lim_{n\to\infty} p^{(n)}_{x_0,y_0} = \pi_{y_0}$. We assume for simplicity that all transition probabilities are positive: $p_{xy} > 0,\ \forall x, y \in \{1, \ldots, N\}$. The general case is the subject of an exercise. Fix any $\delta > 0$ such that $p_{xy} \geq \delta$ for all states x, y. Consider a coupling $Z_n = (X_n, Y_n)$ of X_n with itself described above. To completely describe the probabilistic evolution of Z_n we need to specify the initial distribution Z_0. We will be judicious about this. Specifically, let $X_n = x_0$ with probability one and let Y_n be distributed according to π. Formally, $\mathbb{P}(Z_0 = (x_0, x)) = \pi_x$, and $\mathbb{P}(Z_0 = (x', x)) = 0$ for all $x' \neq x$. This in particular, means that $\mathbb{P}(Y_n = x) = \pi_x$ for all n and $\mathbb{P}(X_n = x) = p^{(n)}_{x_0,x}$, though notice that we explicitly write down the joint probability of X_n and Y_n in terms of P and π, since X_n and Y_n run dependently.

Let $T \geq 0$ be defined as above - the first time when $X_n = Y_n$. Observe that, if the Markov chains X_n and Y_n did not collide by time n, they will collide at time $n + 1$ with probability at least δ, since every state is reachable with probability δ. Therefore, $\mathbb{P}(T \geq t) \leq \delta^t$. In particular, by continuity of probabilities, $\mathbb{P}(T = \infty) = \lim_t \mathbb{P}(T \geq t) = 0$. Now we have

$$\begin{aligned}
\mathbb{P}(X_n = y_0) &= \mathbb{P}(X_n = y_0, T \leq n) + \mathbb{P}(X_n = y_0, T > n) \\
&\leq \mathbb{P}(X_n = y_0, T \leq n) + \mathbb{P}(T > n) \\
&= \mathbb{P}(Y_n = y_0, T \leq n) + \mathbb{P}(T > n) \\
&\leq \mathbb{P}(Y_n = y_0) + \mathbb{P}(T > n) \\
&= \pi_{y_0} + \mathbb{P}(T > n).
\end{aligned}$$

Here the second equality is valid since on the event $T \leq n$ we have $X_n = Y_n$ (the collision took place at time n or earlier). Taking, the limit of both sides we obtain

$$\limsup_n \mathbb{P}(X_n = y_0) \leq \pi_{y_0}.$$

But recall that $\mathbb{P}(X_n = y_0) = p^{(n)}_{x_0,y_0}$. Similarly, we have

$$\begin{aligned}
\mathbb{P}(X_n = y_0) &\geq \mathbb{P}(X_n = y_0, T \leq n) \\
&= \mathbb{P}(Y_n = y_0, T \leq n) \\
&= \mathbb{P}(Y_n = y_0) - \mathbb{P}(Y_n = y_0, T > n) \\
&\geq \mathbb{P}(Y_n = y_0) - \mathbb{P}(T > n) \\
&\geq \pi_{y_0} - \mathbb{P}(T > n).
\end{aligned}$$

Again, by taking limits, we obtain $\liminf_n \mathbb{P}(X_n = y_0) \geq \pi_{y_0}$. Combining, we obtain $\lim_n \mathbb{P}(X_n = y_0) = \lim_n p^{(n)}_{x_0,y_0} = \pi_{y_0}$ and the proof is complete.

□

3 ABSORBTION PROBABILITY AND EXPECTED TIME TILL ABSORBTION

We have considered the long-term behavior of Markov chains. Now, we study the short-term behavior. In such considerations, we are concerned with the behavior of the chain starting in a transient state, till it enters one of the recurrent state. For simplicity, we can therefore assume that every recurrent state i is *absorbing*, i.e., $p_{ii} = 1$. The Markov chain that we will work with in this section has only transient and absorbing states.

If there is only one absorbing state i, then $\pi_i = 1$, and i is reached with probability 1. If there are multiple absorbing states, the state that is entered is random, and we are interested in the absorbing probability

$$a_{ki} = \mathbb{P}(X_n \text{ eventually equals } i \mid X_0 = k),$$

i.e., the probability that state i is eventually reached, starting from state k. Note that $a_{ii} = 1$ and $a_{ji} = 0$ for all absorbing $j \neq i$. When k is a transient state, we have

$$\begin{aligned}
a_{ki} &= \mathbb{P}(\exists n : X_n = i \mid X_0 = k) \\
&= \sum_{j=1}^{N} \mathbb{P}(\exists n : X_n = i \mid X_1 = j) p_{kj} \\
&= \sum_{j=1}^{N} a_{ji} p_{kj}.
\end{aligned}$$

So we can find the absorption probabilities by solving the above system of linear equations.

Example: Gambler's Ruin A gambler wins 1 dollar at each round, with probability p, and loses a dollar with probability $1 - p$. Different rounds are independent. The gambler plays continuously until he either accumulates a target amount m or loses all his money. What is the probability of losing his fortune?

We construct a Markov chain with state space $\{0, 1, \ldots, m\}$, where the state i is the amount of money the gambler has. So state $i = 0$ corresponds to losing his entire fortune, and state m corresponds to accumulating the target amount. The states 0 and m are absorbing states. We have the transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = 1-p$ for $i = 1, 2, \ldots, m-1$, and $p_{00} = p_{mm} = 1$. To find the absorbing probabilities for the state 0, we have

$$\begin{aligned}
&a_{00} = 1, \\
&a_{m0} = 0, \\
&a_{i0} = (1-p)a_{i-1,0} + pa_{i+1,0}, \quad \text{for } i = 1, \ldots, m-1.
\end{aligned}$$

Let $b_i = a_{i0} - a_{i+1,0}$, $\rho = (1-p)/p$, then the above equation gives us

$$(1-p)(a_{i-1,0} - a_{i0}) = p(a_{i0} - a_{i+1,0}).$$

Namely, $b_i = \rho b_{i-1}$. So we obtain $b_i = \rho^i b_0$. Note that $b_0 + b_1 + \cdots + b_{m-1} = a_{00} - a_{m0} = 1$, hence $(1 + \rho + \ldots + \rho^{m-1})b_0 = 1$, which gives us

$$b_i = \begin{cases} \frac{\rho^i(1-\rho)}{1-\rho^m}, & \text{if } \rho \neq 1, \\ \frac{1}{m}, & \text{if } \rho = 1. \end{cases}$$

Finally, $a_{i,0}$ can be calculated. For $\rho \neq 1$, we have for $i = 1, \ldots, m-1$,

$$\begin{aligned}
a_{i0} &= a_{00} - b_{i-1} - \ldots - b_0 \\
&= 1 - (\rho^{i-1} + \ldots + \rho + 1)b_0 \\
&= 1 - \frac{1-\rho^i}{1-\rho}\frac{1-\rho}{1-\rho^m} \\
&= \frac{\rho^i - \rho^m}{1-\rho^m}
\end{aligned}$$

and for $\rho = 1$,

$$a_{i0} = \frac{m-i}{m}.$$

This shows that for any fixed i, if $\rho > 1$, i.e., $p < 1/2$, the probability of losing goes to 1 as $m \to \infty$. Hence, it suggests that if the gambler aims for a large target while under unfavorable odds, financial ruin is almost certain.

The expected time of absorption μ_k when starting in a transient state k can be defined as $\mu_k = \mathbb{E}[\min\{n \geq 1 : X_n \text{ is recurrent}\} \mid X_0 = k]$. A similar analysis by conditioning on the first step of the Markov chain shows that the expected time to absorption can be found by solving

$$\mu_k = 0 \quad \text{for all recurrent states } k,$$
$$\mu_k = 1 + \sum_{j=1}^{N} p_{kj}\mu_j \quad \text{for all transient states } k.$$