MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018

Problem Set 9

Readings:
Notes from Lectures 16-19.
[GS], Section 7.1-7.6
[Cinlar], Chapter III

Exercise 1. We study convergence of algebraic operations:

(a) Show that

$$X_n \stackrel{\text{i.p.}}{\to} X, Y_n \stackrel{\text{i.p.}}{\to} Y \quad \Rightarrow \quad X_n Y_n \stackrel{\text{i.p.}}{\to} XY$$

(*Hint:* reduce to $\stackrel{\text{a.s.}}{\to}$.)

(b) Show, however, that

$$X_n \stackrel{\text{d}}{\to} X, Y_n \stackrel{\text{d}}{\to} Y \quad \not\Rightarrow \quad X_n Y_n \stackrel{\text{d}}{\to} XY$$

(c) Assume $X_n \perp\!\!\!\perp Y_n$ and $X \perp\!\!\!\perp Y$. Show that then

$$X_n \stackrel{\text{d}}{\to} X, Y_n \stackrel{\text{d}}{\to} Y \quad \Rightarrow \quad X_n Y_n \stackrel{\text{d}}{\to} XY$$

(*Hint:* reduce to $\stackrel{\text{a.s.}}{\to}$.)

Exercise 2 (Metrization of convergence in probability)**.** Define a pseudo-metric on the space of random-variables:

$$d(X,Y) \triangleq \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right].$$

Show $X_n \stackrel{\text{i.p.}}{\to} X$ iff $d(X_n, X) \to 0$.

Exercise 3 (20 pts)**.** Prove Cauchy criterions for convergence a.s. and i.P.:

(i) Show that X_n converges almost surely iff

$$\forall \epsilon > 0 \quad \mathbb{P}[\sup_{k\geq 0} |X_{n+k} - X_n| > \epsilon] \to 0 \quad n \to \infty$$

(ii) Show that X_n converges in probability iff

$$\forall \epsilon > 0 \quad \sup_{k \geq 0} \mathbb{P}[|X_{n+k} - X_n| > \epsilon] \to 0 \quad n \to \infty$$

Exercise 4. Let $\{X_n\}$ be a sequence of random variables defined on the same probability space.

(a) Show $\mathbb{E}[|X_n - X|] \to 0$ implies $X_n \overset{\text{i.p.}}{\to} X$.

(b) Suppose that $X_n \overset{\text{i.p.}}{\to} 0$ and that for some constant c, we have $|X_n| \leq c$, for all n, with probability 1. Show that

$$\lim_{n \to \infty} \mathbb{E}[|X_n|] = 0.$$

(c) Suppose that each X_n can only take the values 0 and 1 and, that $\mathbb{P}(X_n = 1) = 1/n$.

(i) Give an example in which we **have** almost sure convergence of X_n to 0.

(ii) Give an example in which we **do not have** almost sure convergence of X_n to 0.

Exercise 5. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter $\lambda = 1$. Let $S_n = X_1 + \cdots + X_n$. Let $a > 1$. What is the Chernoff upper bound for $\mathbb{P}(S_n \geq na)$?

Exercise 6. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, uniformly distributed on the interval $[0, 1]$. For n odd, let M_n be the median of $X_1, X_2, \ldots, X_n$, i.e. the $(\frac{n+1}{2})$ order statistic $X^{(\frac{n+1}{2})}$. Show that M_n converges to 1/2, in probability.

Exercise 7. [Optional, not to be graded] Show that for every $\mathbb{P}_X$ on $(\mathbb{R}, \mathcal{B})$ there exist a sequence $\mathbb{P}_{X_n} \overset{\text{d}}{\to} \mathbb{P}_X$ such that every $\mathbb{P}_{X_n}$ has a continuous, bounded, infinitely-differentiable PDF. Steps:

(i) Show $X_\epsilon = X + \epsilon Z \overset{\text{d}}{\to} X$ as $\epsilon \to 0$.

(ii) Let $X \perp\!\!\!\perp Z$ and $Z \sim \mathcal{N}(0,1)$. Show that CDF of X_ϵ is continuous (*Hint:* BCT) and differentiable (*Hint: Fubini*) with derivative

$$f_{X_\epsilon}(a) = \mathbb{E}\left[f_Z\left(\frac{a-X}{\epsilon}\right)\frac{1}{\epsilon}\right].$$

(iii) Show that $a \mapsto f_{X_\epsilon}(a)$ is continuous.

(iv) [Optional] Conclude the proof (*Hint:* derivatives of f_Z are uniformly bounded on $\mathbb{R}$).