MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Recitation 5

1 Sum of independent random variables

Lemma 1. *If X and Y are independent random variables, then*

$$\mathbb{P}(X+Y \leq z) = \mathbb{E}[F_X(z-Y)] = \mathbb{E}[F_Y(z-X)].$$

Proof. We have

$$\begin{aligned}
\mathbb{P}(X+Y \leq z) &= \mathbb{E}[1_{\{X+Y\leq z\}}] \\
&= \int_{\mathbb{R}^2} 1_{\{x+y\leq z\}} d(\mathbb{P}_X \times \mathbb{P}_Y)(x,y) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{\{x+y\leq z\}} d\mathbb{P}_X(x) \right) d\mathbb{P}_Y(y) \\
&= \int_{\mathbb{R}} F_X(z-y) d\mathbb{P}_Y(y) \\
&= \mathbb{E}[F_X(z-Y)],
\end{aligned}$$

where in the third inequality we used Fubini's Theorem. □

If X and Y are continuous, $X+Y$ is also continuous, and its density can be derived by differentiating the above expression, and using Exercise 7 of HW 5 to bring the differentiation inside the integral.

2 Gaussian, Gamma, and Exponential distributions

Theorem 1.

(a) If $N_1 \sim N(\mu_1, \sigma_1^2)$ and $N_2 \sim N(\mu_2, \sigma_2^2)$, then $N_1 + N_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(b) If $G_1 \sim Gamma(k_1, \theta)$ and $G_2 \sim Gamma(k_2, \theta)$, then $G_1 + G_2 \sim Gamma(k_1 + k_2, \theta)$.

(c) If $N \sim N(0,1)$, then $N^2 \sim Gamma(1/2, 2)$

(d) If $X, Y \sim N(0,1)$, then $X^2 + Y^2 \sim Exp(2)$.

(e) If $X, Y \sim N(0,1)$, then $\sqrt{X^2+Y^2}$ and $\arcsin(Y/\sqrt{X^2+Y^2})$ are independent. Furthermore, $\arcsin(Y/\sqrt{X^2+Y^2})$ is uniform over $(-\pi/2, \pi/2)$.

Proof. (a) It follows from applying the convolution formula for continuous random variables, and doing lots of algebra. The whole thing is even in Wikipedia: https://en.wikipedia.org/wiki/Sum_of_normally_distributed_random_variables

(b) It also follows from applying the convolution formula, and doing some algebra. For the sake of simplicity, we prove it for the case $\theta = 1$.

$$\begin{aligned} f_{G_1+G_2}(z) &= \int_0^z f_{G_1}(x) f_{G_2}(z-x)\,\mathrm{d}x \\ &= \int_0^z \frac{x^{k_1-1}e^{-x}}{\Gamma(k_1)} \frac{(z-x)^{k_2-1}e^{-(z-x)}}{\Gamma(k_2)}\,\mathrm{d}x \\ &= e^{-z}\int_0^z \frac{x^{k_1-1}(z-x)^{k_2-1}}{\Gamma(k_1)\Gamma(k_2)}\,\mathrm{d}x && \text{variable change:} x{=}zt \\ &= e^{-z}z^{k_1+k_2-1}\int_0^1 \frac{t^{k_1-1}(1-t)^{k_2-1}}{\Gamma(k_1)\Gamma(k_2)}\,\mathrm{d}t && \text{almost the density of a Beta}(k_1,k_2)\text{ r.v.} \\ &= \frac{e^{-z}z^{k_1+k_2-1}}{\Gamma(k_1+k_2)} \end{aligned}$$

(c) We have

$$\mathbb{P}(N^2 \le z) = \mathbb{P}(|N| \le \sqrt{z}) = 2\int_0^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.$$

Then, differentiating with respect to z, we obtain

$$f_{N^2}(z) = \frac{z^{\frac{1}{2}-1}e^{-\frac{z}{2}}}{\sqrt{2\pi}},$$

which is the density of a $Gamma(1/2, 2)$.

(d) From (c), we know that X^2 and Y^2 are $Gamma(1/2, 2)$. Then, applying (b) we get that $X^2 + Y^2$ is $Gamma(1, 2)$, which is the same as $Exp(2)$.

(e) Note that $R = \sqrt{X^2+Y^2}$ and $\Theta = \arcsin(Y/\sqrt{X^2+Y^2})$ correspond to the radius and angle in polar coordinates. As a result, the probability of the event $\{0 \le \Theta \le \theta_0\} \cap \{R \le r_0\}$ can be computed using polar coordinates as follows:

$$\begin{aligned} \mathbb{P}\Big(\{0 \le \Theta \le \theta_0\} \cap \{R \le r_0\}\Big) &= \int\limits_{\{0\le\Theta\le\theta_0\}\cap\{R\le r_0\}} \frac{1}{2\pi} e^{\frac{x^2+y^2}{2}}\,dxdy \\ &= \int_0^{\theta_0}\int_0^{r_0} \frac{1}{2\pi} e^{\frac{r^2}{2}} r\,dr\,d\theta \\ &= \theta_0 \int_0^{r_0} \frac{1}{2\pi} e^{\frac{r^2}{2}} r\,dr \\ &= \mathbb{P}\Big(0 \le \Theta \le \theta_0\Big)\mathbb{P}\Big(R \le r_0\Big). \end{aligned}$$

Thus, they are independent, and Θ is uniform.

$\square$