Solutions to Homework 1

6.262 Discrete Stochastics Process

MIT, Spring 2011

Solution to Exercise 1.3:

a) Since $A_1, A_2, \ldots,$ are assumed to be disjoint, the third axiom of probability says that

$$\Pr \bigcup_{m=1}^{\infty} A_m = \sum_{m=1}^{\infty} \Pr A_m$$

Since $\Omega = \bigcup_{m=1}^{\infty} A_m$, the term on the left above is 1. Since $\Pr A_m = 2^{-m-1}$, the term on the right is $2^{-2} + 2^{-3} + \cdots = 1/2$. Thus the assumptions about the probabilities in the problem are inconsistent with the axioms of probability.

b) If the third axiom of probability is replaced with the finite additivity condition in (1.3) of the text, then all we can say from the modified axiom is that for all $n \geq 1$,

$$\Pr \bigcup_{m=1}^{n} A_m = \sum_{m=1}^{n} \Pr A_m$$

The sum on the right is simply a number that is increasing in n but bounded by 1, so we go to the limit $n \to \infty$,

$$\lim_{n\to\infty} \Pr \bigcup_{m=1}^{n} A_m = \lim_{n\to\infty} \sum_{m=1}^{n} \Pr A_m = 2^{-2} + 2^{-3} + \cdots = 1/2$$

Unfortunately, we have no reason to assume that the limit on the left can be interchanged with the probability, and in fact the essence of the third axiom of probability is that this interchange can be made. Thus this can not be used to show that the assumptions about the probabilities are inconsistent with these modified axioms.

This suggests, but does not show, that the assumptions are inconsistent with the modified axioms. With the modified axioms, it is conceivable that some probability 'gets lost' in going to the infinite union.

We apologize for assigning this problem, since it does not provide a good example of the need for countable additivity in the axioms. Showing this need correctly appears to require measure theory.

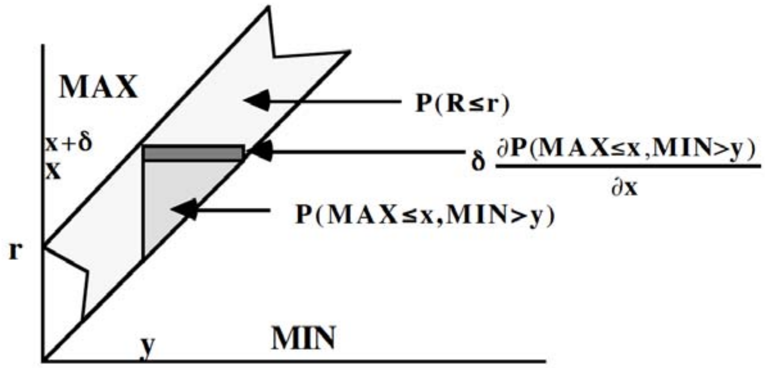
Solution to Exercise 1.9:

a) The maximum of n IID random variables is less than or equal to x if and only if (iff) each of the individual random variables is less than or equal to x. Since $F_X(x)$ is the probability that any given one of the n variables is less than or equal to x, $[F_X(x)]^n$ is the probability that each is less than or equal to x. Thus, letting $MAX = \max[X_1, X_2, \ldots, X_3]$

$$F_{MAX}(x) = [F_X(x)]^n$$

b) Similarly, the minimum, $MIN = \min[X_1, X_2, ..., X_n]$, is greater than x iff each variable is greater than x. Thus, $1 - F_{MIN}(x) = [1 - F_X(x)]^n$.

c) Let $R = MAX - MIN$. (See figure)



We start by looking at the joint probability of MAX and MIN by the same technique used in a) and b). In particular, $MAX \leq x$ and $MIN > y$ iff $y < X_i \leq x$ for $1 \leq i \leq n$. Thus, $\Pr(MAX \leq x, MIN > y) = [F_X(x) - F_Y(y)]^n$. From the figure, we see that

$$\Pr(R \leq r) = \int \frac{\partial \Pr(MAX \leq x, MIN > y)}{\partial x}\Big|_{y=x-r} \mathrm{d}x$$

Assuming that X has a density, this simplifies to $\int n f_x(x)[F_X(x) - F_X(x-r)]^{n-1}\,\mathrm{d}x$

Solution to Exercise 1.13:

a) Since X_1 and X_2 are identically distributed, $\Pr(X_1 < X_2) = \Pr(X_2 < X_1)$. Since $\Pr(X_1 = X_2) = 0$, $\Pr(X_1 < X_2) = 1/2$.

b) Again, using the symmetry between X_1, X_2, and $\ldots X_n$:

$$\Pr(X_1 < X_n, X_2 < X_n, \cdots, X_{n-1} < X_n) = 1/n$$

.

c) Let's define $\mathbb{I}_n = 1$ iff X_n is a record-to-date. Then the expected number of record-to-date that occur over the first m trials is:

$$
\begin{aligned}
\mathbb{E}[\sum_{n=1}^{m} \mathbb{I}_n] &= \sum_{n=1}^{m} \mathbb{E}[\mathbb{I}_n] \\
&= \sum_{n=1}^{m} \Pr(\mathbb{I}_n = 1) \\
&= \sum_{n=1}^{m} \frac{1}{n}
\end{aligned}
$$

The expected number is infinite as $m \to \infty$.

Solution to Exercise 1.20:

a) Suppose $Z = (X+Y)_{mod2}$. That is $Z = 0$ if $X = Y$ and $Z = 1$ if $X \neq Y$. Then, if X and Y are independent, there are 4 joint sample points, $(X = 0, Y = 0, Z = 0)$, $(X = 0, Y = 1, Z = 1)$, $(X = 1, Y = 0, Z = 1)$ and $(X = 1, Y = 1, Z = 0)$ each of which have probability 1/4. All other combinations have probability 0. Each pair of random variable is independent, but the set of three is not.

b) For the example above, the product XYZ is zero with probability 1, and thus $E[XYZ] = 0$. On the other hand $E[X]E[Y]E[Z] = (1/2)^3$. Thus pairwise independence is not sufficient for joint independence of all the variables.

Solution to Exercise 1.26:

The algebraic proof of this is straightforward:

$$F_Y(y) = \Pr(Y \leq y) = \Pr(F_X(X) \leq y)$$

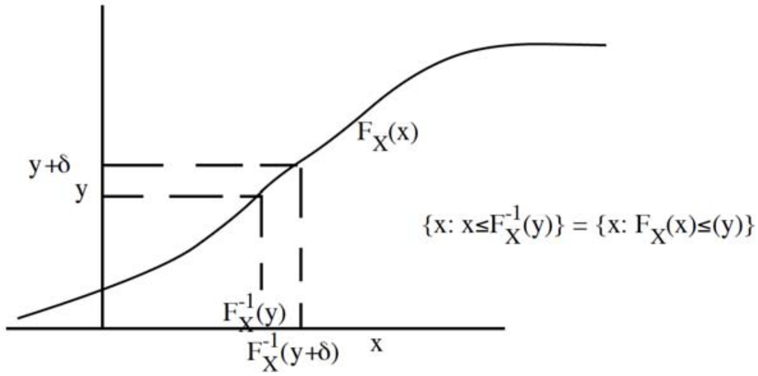
Note that the set of x satisfying $F_X(x) \leq y$, i.e., $\{x : F_X(x) \leq y\}$ is the same as the set of x for which $x \leq F_X^{-1}(y)$. (see figure). Thus

$$\Pr(F_X(X) \leq y) = \Pr(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y \tag{1}$$

If $F_X(x)$ is not strictly increasing in x, we define $F_X^{-1}(y)$ as the largest x for which $F_X(x) \leq y$ so that equation 1 is still satisfied. Since $F_Y(y) = y$, the density of Y is given by

$$f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & else \end{cases}$$

To see what is happening more clearly, consider $\Pr(y \leq Y \leq y + \delta)$ (see figure). The event $y \leq Y \leq y + \delta$ is the same as the event $F_X^{-1}(x) < x \leq F_X^{-1}(x + \delta)$. But from the figure, this event has probability $(y+\delta)-y=\delta$, meaning again that Y is uniformly distributed over $(0,1]$.



Note that the result does not hold for a discrete random variable. For example, if X is a Bernouli random variable with $P_X(1)=p$ and $P_X(0)=1-p$, then

$$F_X(x)=\begin{cases} 0 & x<0 \\ p & 0\leq x<1 \\ 1 & 1\leq x \end{cases}$$

From this we can see that the random variable $Y=F_X(x)$ can only take the values p and 1 (since X can only take on 0 and 1), so Y couldn't possibly be uniformly distributed between 0 and 1.