Introduction to Algorithms

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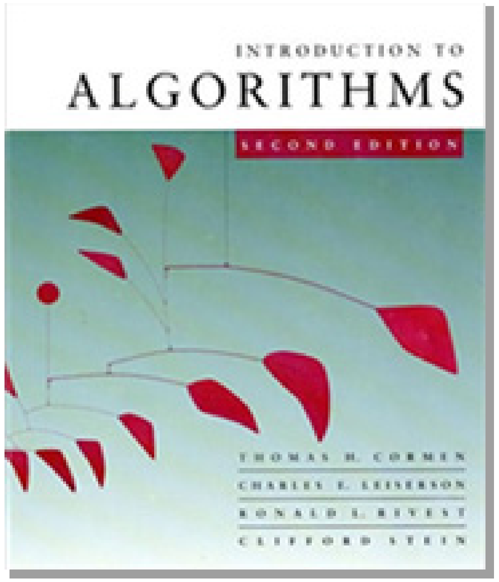
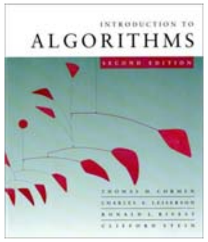


LECTURE 12

Skip Lists

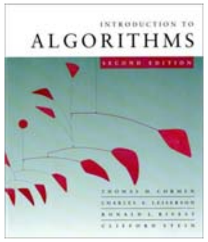
- Data structure
- Randomized insertion
- With-high-probability bound
- Analysis
- Coin flipping

Prof. Erik D. Demaine



Skip lists

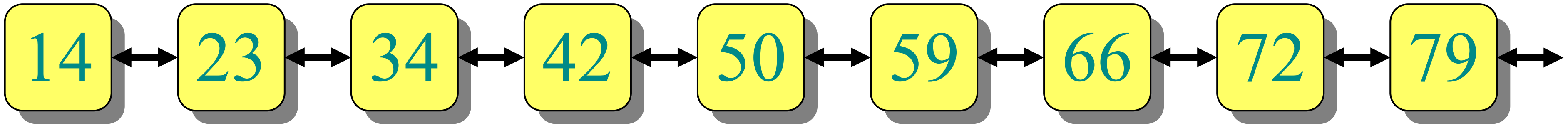
- Simple randomized dynamic search structure
  - Invented by William Pugh in 1989
  - Easy to implement
- Maintains a dynamic set of n elements in $O(\lg n)$ time per operation in expectation and *with high probability*
  - Strong guarantee on tail of distribution of $T(n)$
  - $O(\lg n)$ "almost always"

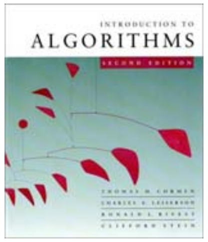


One linked list

Start from simplest data structure:
(sorted) linked list

- Searches take $\Theta(n)$ time in worst case
- How can we speed up searches?

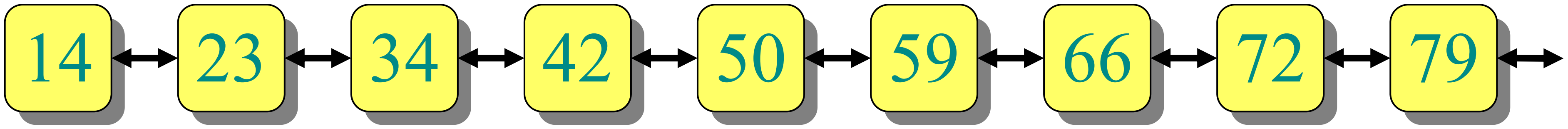


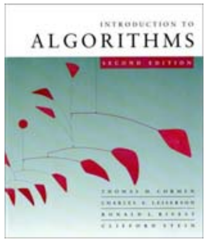


Two linked lists

Suppose we had *two* sorted linked lists (on subsets of the elements)

- Each element can appear in one or both lists
- How can we speed up searches?

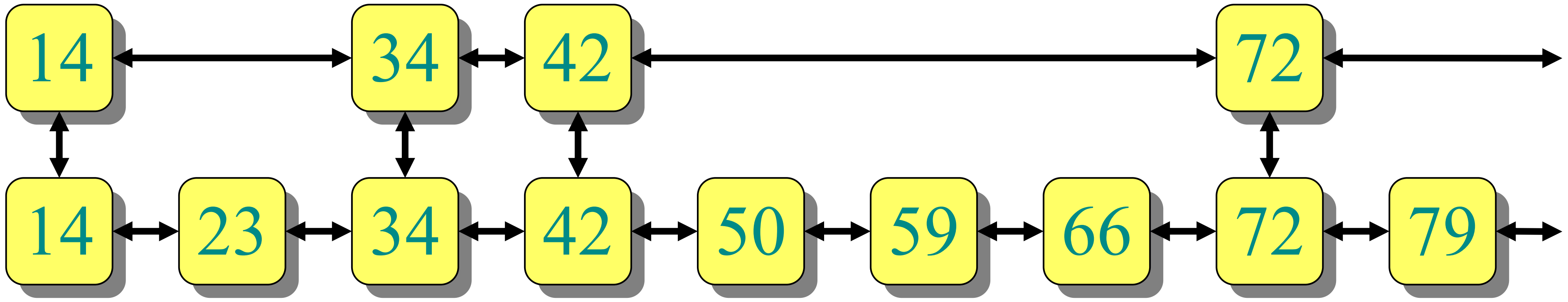


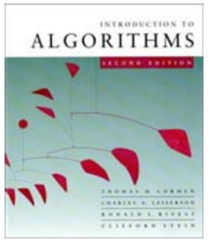


Two linked lists as a subway

IDEA: Express and local subway lines (à la New York City 7th Avenue Line)

- Express line connects a few of the stations
- Local line connects all stations
- Links between lines at common stations

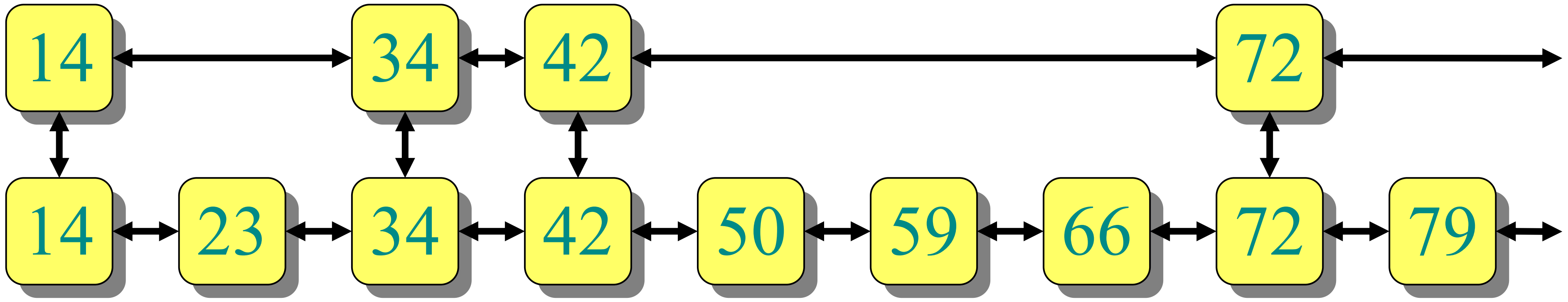


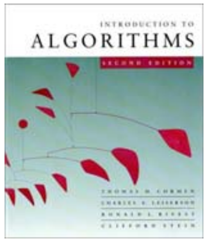


Searching in two linked lists

SEARCH(x):

- Walk right in top linked list (L_1) until going right would go too far
- Walk down to bottom linked list (L_2)
- Walk right in L_2 until element found (or not)

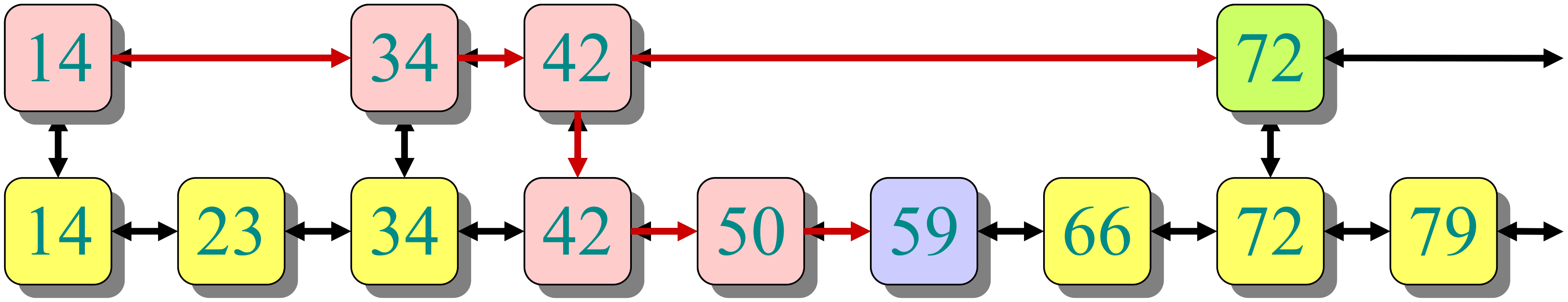


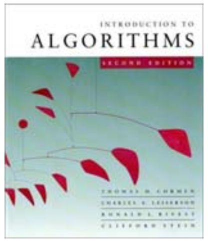


Searching in two linked lists

EXAMPLE: SEARCH(59)

Too far:
$59 < 72$

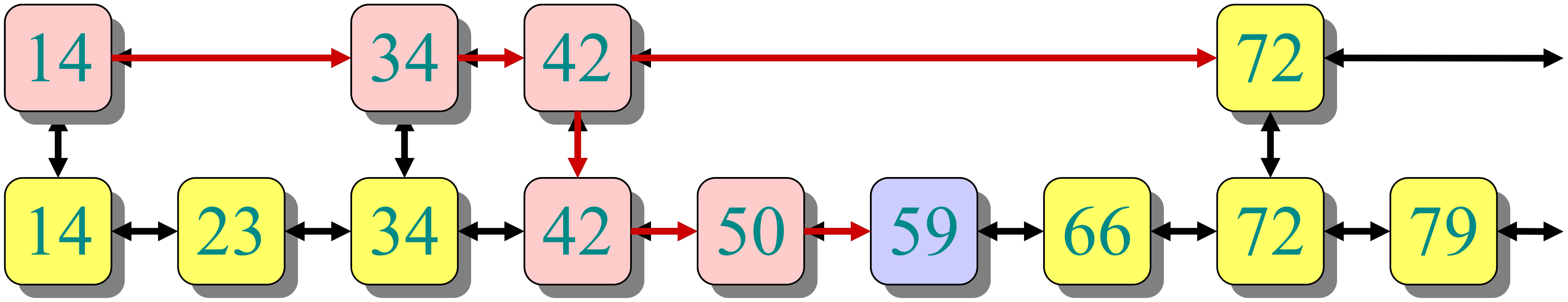


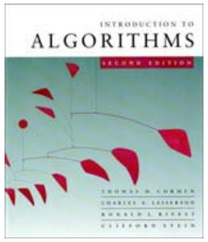


Design of two linked lists

QUESTION: Which nodes should be in L_1?

- In a subway, the “popular stations”
- Here we care about *worst-case performance*
- **Best approach:** Evenly space the nodes in L_1
- But *how many nodes* should be in L_1?

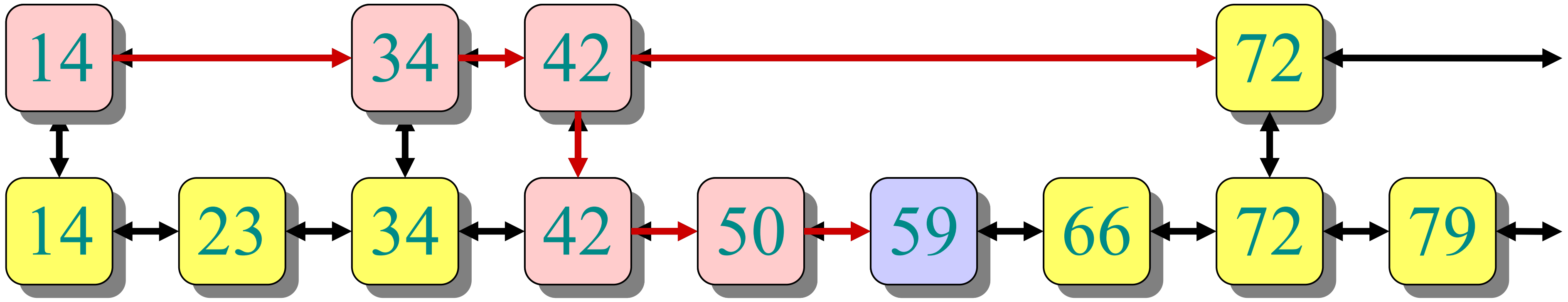


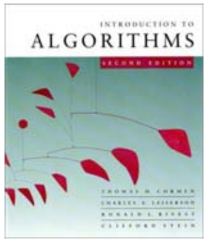


Analysis of two linked lists

ANALYSIS:

- Search cost is roughly $|L_1| + \dfrac{|L_2|}{|L_1|}$
- Minimized (up to constant factors) when terms are equal
- $|L_1|^2 = |L_2| = n \Rightarrow |L_1| = \sqrt{n}$



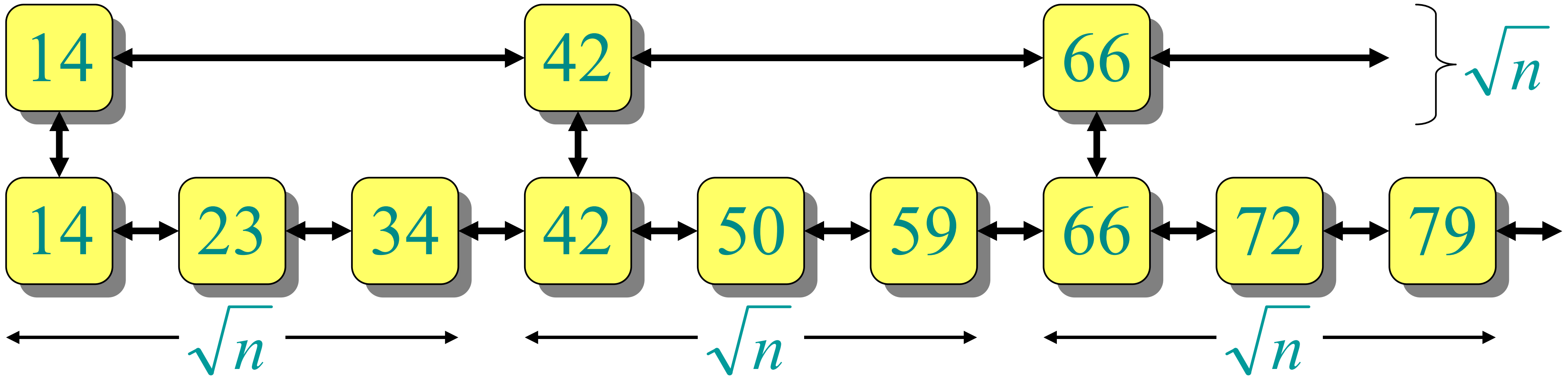


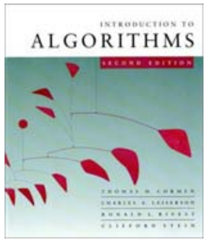
Analysis of two linked lists

ANALYSIS:

- $|L_1| = \sqrt{n}$, $|L_2| = n$
- Search cost is roughly

$$|L_1| + \frac{|L_2|}{|L_1|} = \sqrt{n} + \frac{n}{\sqrt{n}} = 2\sqrt{n}$$

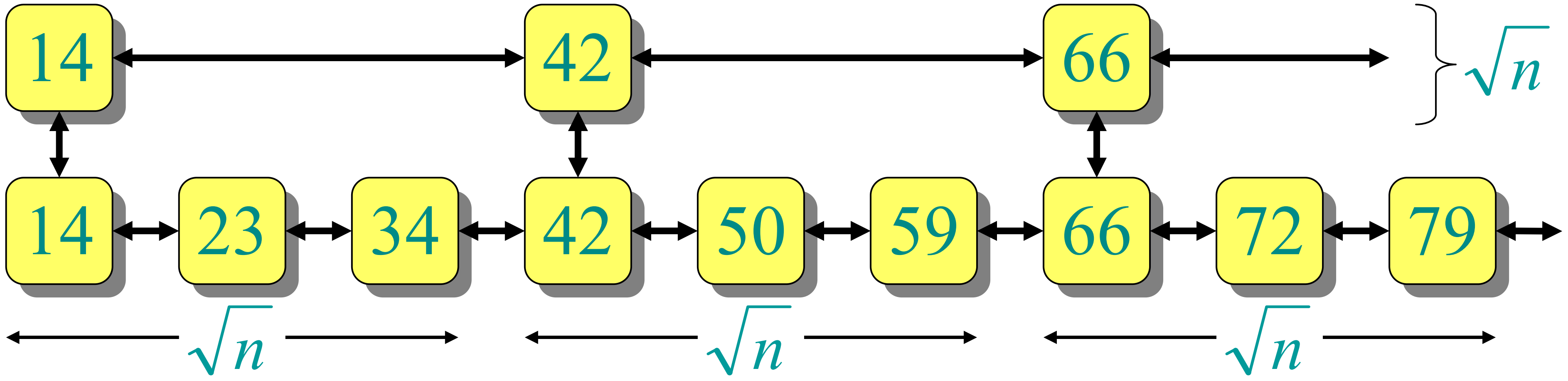


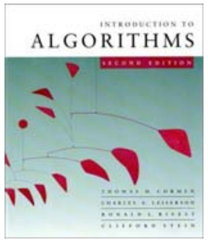


More linked lists

What if we had more sorted linked lists?

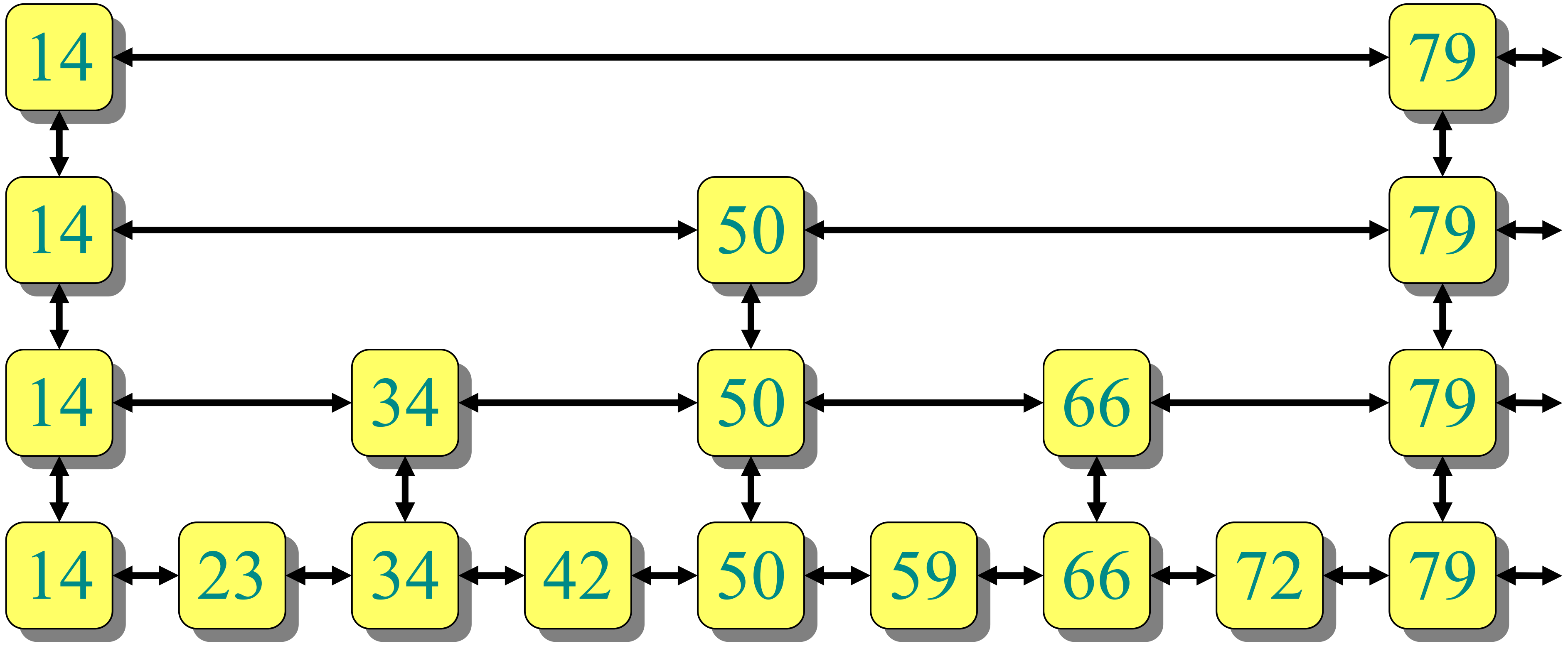
- 2 sorted lists $\Rightarrow 2 \cdot \sqrt{n}$
- 3 sorted lists $\Rightarrow 3 \cdot \sqrt[3]{n}$
- k sorted lists $\Rightarrow k \cdot \sqrt[k]{n}$
- $\lg n$ sorted lists $\Rightarrow \lg n \cdot \sqrt[\lg n]{n} = 2 \lg n$

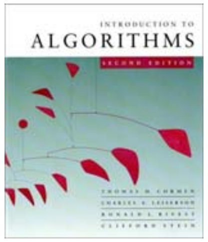




lg *n* linked lists

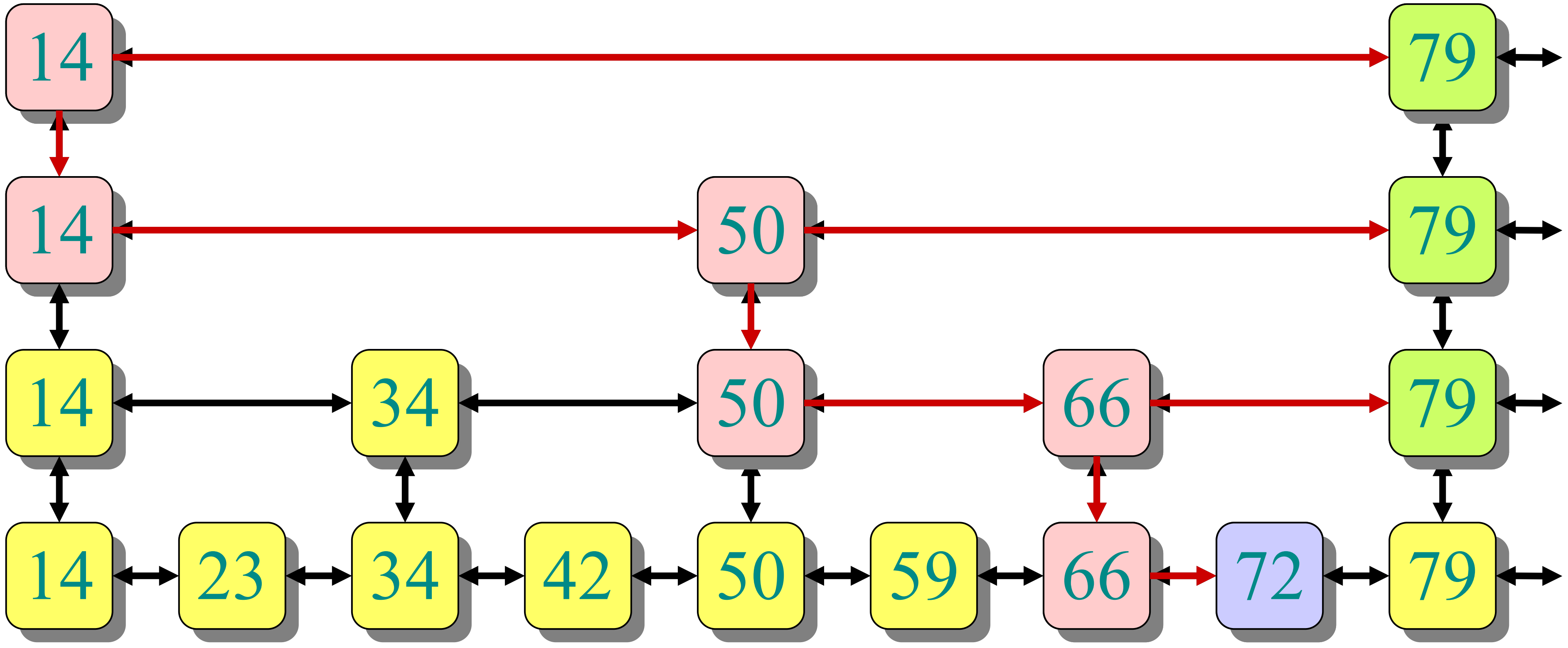
lg *n* sorted linked lists are like a binary tree
(in fact, level-linked B$^+$-tree; see Problem Set 5)

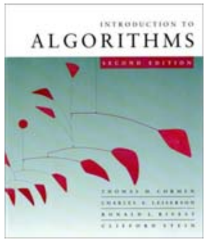




Searching in lg *n* linked lists

EXAMPLE: SEARCH(72)

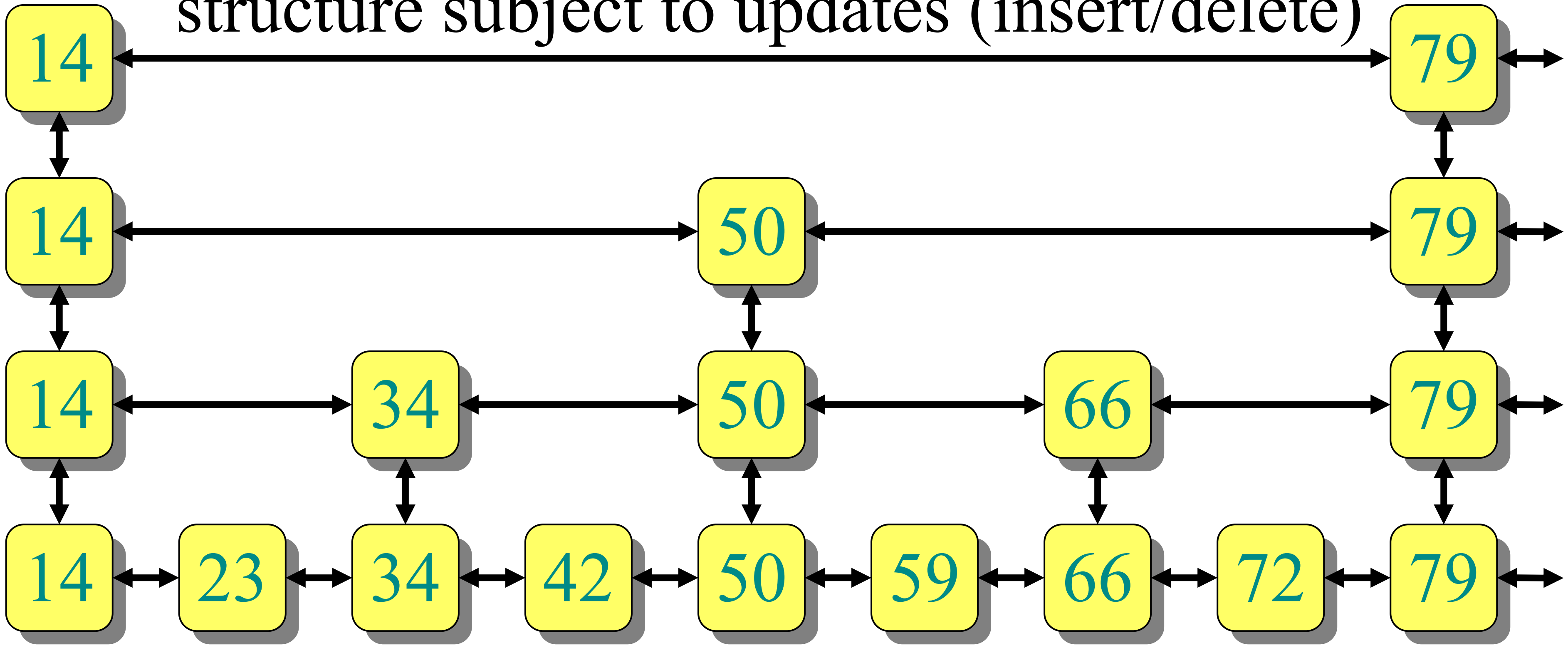


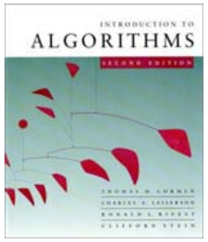


Skip lists

Ideal skip list is this lg *n* linked list structure

Skip list data structure maintains roughly this structure subject to updates (insert/delete)





INSERT(x)

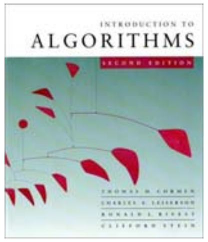
To insert an element x into a skip list:

- SEARCH(x) to see where x fits in bottom list
- Always insert into bottom list

INVARIANT: Bottom list contains all elements

- Insert into some of the lists above…

QUESTION: To which other lists should we add x?



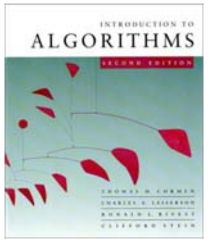
INSERT(x)

QUESTION: To which other lists should we add x?

IDEA: Flip a (fair) coin; if HEADS,
promote x to next level up and flip again

- Probability of promotion to next level = 1/2
- On average:
  - 1/2 of the elements promoted 0 levels
  - 1/4 of the elements promoted 1 level
  - 1/8 of the elements promoted 2 levels
  - etc.

Approx. balanced ?

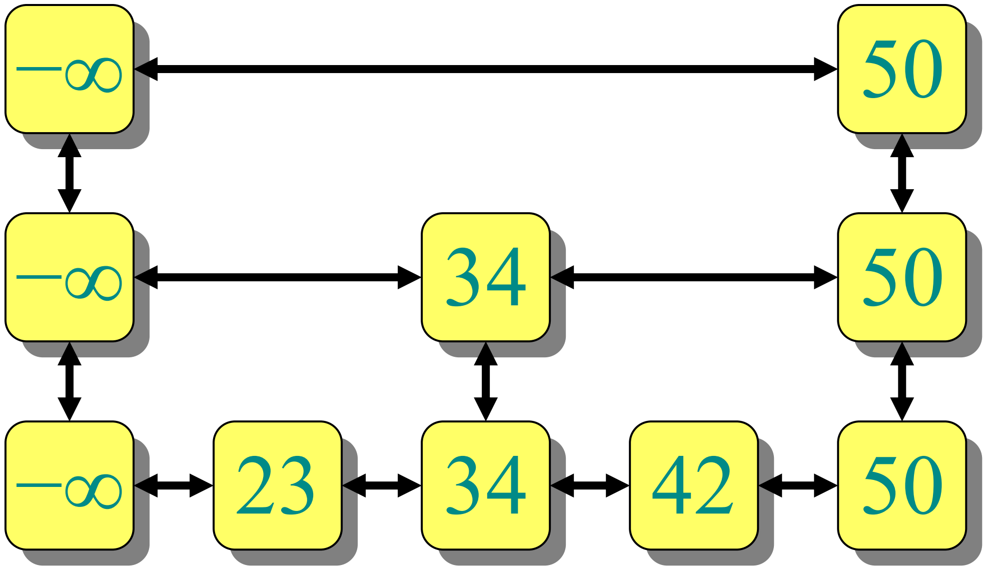


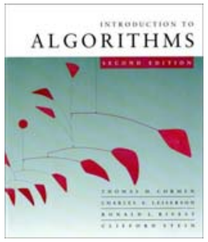
Example of skip list

EXERCISE: Try building a skip list from scratch by repeated insertion using a real coin

Small change:

- Add special $-\infty$ value to *every* list $\Rightarrow$ can search with the same algorithm

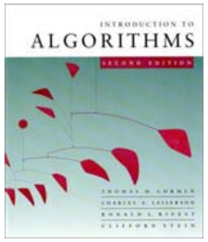

$$\begin{tikzcd}
-\infty \arrow[rrrr, leftrightarrow] \arrow[d, leftrightarrow] & & & & 50 \arrow[d, leftrightarrow] \\
-\infty \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & 34 \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & 50 \arrow[d, leftrightarrow] \\
-\infty \arrow[r, leftrightarrow] & 23 \arrow[r, leftrightarrow] & 34 \arrow[r, leftrightarrow] & 42 \arrow[r, leftrightarrow] & 50
\end{tikzcd}$$



Skip lists

A ***skip list*** is the result of insertions (and deletions) from an initially empty structure (containing just $-\infty$)

- INSERT(x) uses random coin flips to decide promotion level
- DELETE(x) removes x from all lists containing it



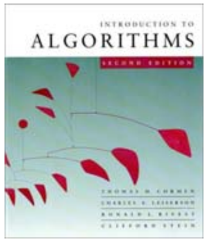
Skip lists

A ***skip list*** is the result of insertions (and deletions) from an initially empty structure (containing just $-\infty$)

- INSERT(x) uses random coin flips to decide promotion level
- DELETE(x) removes x from all lists containing it

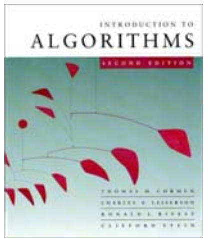
How good are skip lists? (speed/balance)

- **INTUITIVELY:** Pretty good on average
- **CLAIM:** Really, really good, almost always



With-high-probability theorem

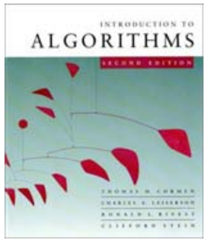
THEOREM: *With high probability*, every search in an n-element skip list costs $O(\lg n)$



With-high-probability theorem

THEOREM: *With high probability*, every search in a skip list costs $O(\lg n)$

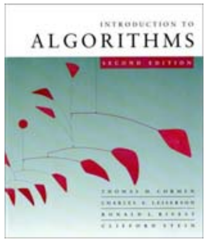
- **INFORMALLY:** Event E occurs ***with high probability*** (***w.h.p.***) if, for any $\alpha \geq 1$, there is an appropriate choice of constants for which E occurs with probability at least $1 - O(1/n^\alpha)$
  - In fact, constant in $O(\lg n)$ depends on α
- **FORMALLY:** Parameterized event E_α occurs ***with high probability*** if, for any $\alpha \geq 1$, there is an appropriate choice of constants for which E_α occurs with probability at least $1 - c_\alpha/n^\alpha$



With-high-probability theorem

THEOREM: With high probability, every search in a skip list costs $O(\lg n)$

- **INFORMALLY:** Event E occurs ***with high probability*** (***w.h.p.***) if, for any $\alpha \geq 1$, there is an appropriate choice of constants for which E occurs with probability at least $1 - O(1/n^\alpha)$
- **IDEA:** Can make ***error probability*** $O(1/n^\alpha)$ very small by setting α large, e.g., 100
- Almost certainly, bound remains true for entire execution of polynomial-time algorithm



Boole's inequality / union bound

Recall:

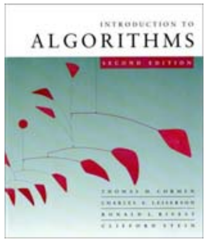
BOOLE'S INEQUALITY / UNION BOUND:

For any random events $E_1, E_2, \ldots, E_k$,

$$\Pr\{E_1 \cup E_2 \cup \ldots \cup E_k\} \le \Pr\{E_1\} + \Pr\{E_2\} + \ldots + \Pr\{E_k\}$$

Application to with-high-probability events:

If $k = n^{O(1)}$, and each E_i occurs with high probability, then so does $E_1 \cap E_2 \cap \ldots \cap E_k$



Analysis Warmup

LEMMA: With high probability, n-element skip list has $O(\lg n)$ levels

PROOF:

- Error probability for having at most $c \lg n$ levels

  $= \Pr\{$more than $c \lg n$ levels$\}$

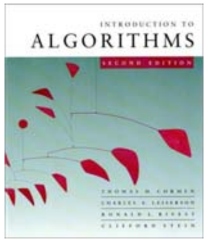
  $\leq n \cdot \Pr\{$element x promoted at least $c \lg n$ times$\}$

  (by Boole's Inequality)

  $= n \cdot (1/2^{c \lg n})$

  $= n \cdot (1/n^c)$

  $= 1/n^{c-1}$

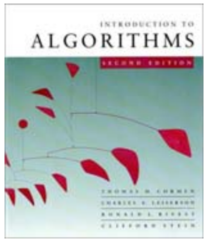


Analysis Warmup

LEMMA: With high probability, n-element skip list has $O(\lg n)$ levels

PROOF:

- Error probability for having at most $c \lg n$ levels $\le 1/n^{c-1}$
- This probability is ***polynomially small***, i.e., at most n^{α} for $\alpha = c - 1$.
- We can make α arbitrarily large by choosing the constant c in the $O(\lg n)$ bound accordingly. $\square$

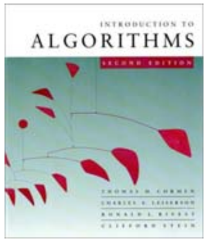


Proof of theorem

THEOREM: With high probability, every search in an n-element skip list costs $O(\lg n)$

COOL IDEA: Analyze search backwards—leaf to root

- Search starts [ends] at leaf (node in bottom level)
- At each node visited:
  - If node wasn't promoted higher (got TAILS here), then we go [came from] left
  - If node was promoted higher (got HEADS here), then we go [came from] up
- Search stops [starts] at the root (or $-\infty$)



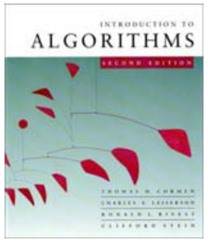
Proof of theorem

THEOREM: With high probability, every search in an n-element skip list costs $O(\lg n)$

COOL IDEA: Analyze search backwards—leaf to root

PROOF:

- Search makes "up" and "left" moves until it reaches the root (or $-\infty$)
- Number of "up" moves $<$ number of levels $\leq c \lg n$ w.h.p. *(Lemma)*
- $\Rightarrow$ w.h.p., number of moves is at most the number of times we need to flip a coin to get $c \lg n$ HEADS



Coin flipping analysis

CLAIM: Number of coin flips until $c \lg n$ HEADS $= \Theta(\lg n)$ with high probability

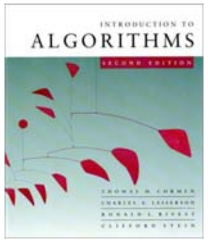
PROOF:

Obviously $\Omega(\lg n)$: at least $c \lg n$

Prove $O(\lg n)$ "by example":

- Say we make $10\, c \lg n$ flips
- When are there at least $c \lg n$ HEADS?

(Later generalize to arbitrary values of 10)



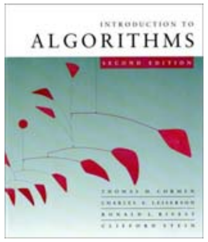
Coin flipping analysis

CLAIM: Number of coin flips until $c \lg n$ HEADS $= \Theta(\lg n)$ with high probability

PROOF:

- $\Pr\{\text{exactly } c \lg n \text{ HEADS}\} = \underbrace{\binom{10c \lg n}{c \lg n}}_{\text{orders}} \cdot \underbrace{\left(\frac{1}{2}\right)^{c \lg n}}_{\text{HEADS}} \cdot \underbrace{\left(\frac{1}{2}\right)^{9c \lg n}}_{\text{TAILS}}$

- $\Pr\{\text{at most } c \lg n \text{ HEADS}\} \le \underbrace{\binom{10c \lg n}{c \lg n}}_{\text{overestimate on orders}} \cdot \underbrace{\left(\frac{1}{2}\right)^{9c \lg n}}_{\text{TAILS}}$



Coin flipping analysis (cont'd)

- Recall bounds on $\binom{y}{x}$: $\left(\frac{y}{x}\right)^x \le \binom{y}{x} \le \left(e\frac{y}{x}\right)^x$
- $\Pr\{\text{at most } c \lg n \text{ HEADS}\} \le \binom{10c\lg n}{c\lg n} \cdot \left(\frac{1}{2}\right)^{9c\lg n}$

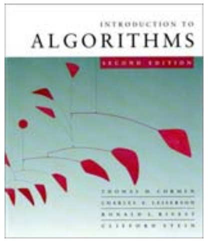
$$\le \left(e\frac{10c\lg n}{c\lg n}\right)^{c\lg n} \cdot \left(\frac{1}{2}\right)^{9c\lg n}$$

$$= (10e)^{c\lg n} 2^{-9c\lg n}$$

$$= 2^{\lg(10e)\cdot c\lg n} 2^{-9c\lg n}$$

$$= 2^{[\lg(10e)-9]\cdot c\lg n}$$

$$= 1/n^{\alpha} \text{ for } \alpha = [9 - \lg(10e)]\cdot c$$



Coin flipping analysis (cont'd)

- $\Pr\{\text{at most } c \lg n \text{ HEADS}\} \le 1/n^{\alpha}$ for $\alpha = [9 - \lg(10e)]c$
- **KEY PROPERTY:** $\alpha \to \infty$ as $10 \to \infty$, for any c
- So set 10, i.e., constant in $O(\lg n)$ bound, large enough to meet desired α □

This completes the proof of the coin-flipping claim and the proof of the theorem.